

Example 1.6.1. We now show that the backward-time backward-space scheme (1.6.2) is stable when a is positive and λ is any positive number. This shows that Theorem 1.6.2 does not extend to implicit schemes.

We first write the scheme (1.6.2) as

$$(1 + a\lambda)v_m^{n+1} = v_m^n + a\lambda v_{m-1}^{n+1}.$$

If we take the square of both sides, we obtain

$$\begin{aligned} (1 + a\lambda)^2 |v_m^{n+1}|^2 &\leq |v_m^n|^2 + 2a\lambda |v_m^n| |v_{m-1}^{n+1}| + (a\lambda)^2 |v_{m-1}^{n+1}|^2 \\ &\leq (1 + a\lambda) |v_m^n|^2 + (a\lambda + (a\lambda)^2) |v_{m-1}^{n+1}|^2. \end{aligned}$$

Taking the sum over all values of m , we obtain

$$(1 + a\lambda)^2 \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq (1 + a\lambda) \sum_{m=-\infty}^{\infty} |v_m^n|^2 + (a\lambda + (a\lambda)^2) \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2.$$

Subtracting the last expression on the right-hand side from the left-hand side gives the estimate

$$\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq \sum_{m=-\infty}^{\infty} |v_m^n|^2,$$

showing that the scheme is stable for every value of λ when a is positive. \square

We point out that even though we can choose λ arbitrarily large for scheme (1.6.2) and still have a stable scheme, the solution will not be accurate unless λ is restricted to reasonable values. We discuss the accuracy of solutions in Chapter 3, and in Section 5.2 we show that there are advantages to choosing $|a\lambda|$ small.

Exercises

1.6.1. Show that the following modified Lax–Friedrichs scheme for the one-way wave equation, $u_t + au_x = f$, given by

$$v_m^{n+1} = \frac{1}{2} (v_{m+1}^n + v_{m-1}^n) - \frac{a\lambda}{1 + (a\lambda)^2} (v_{m+1}^n - v_{m-1}^n) + kf_m^n$$

is stable for all values of λ . Discuss the relation of this explicit and unconditionally stable scheme to Theorem 1.6.2.

1.6.2. Modify the proof of Theorem 1.6.1 to cover the leapfrog scheme.

1.6.3. Show that schemes of the form

$$\alpha v_{m+1}^{n+1} + \beta v_{m-1}^{n+1} = v_m^n$$

are stable if $|\alpha| - |\beta|$ is greater than or equal to 1. Conclude that the reverse Lax–Friedrichs scheme,

$$\frac{\frac{1}{2} (v_{m+1}^{n+1} + v_{m-1}^{n+1}) - v_m^n}{k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0,$$

is stable if $|a\lambda|$ is greater than or equal to 1.