Preface

Toeplitz matrices emerge in plenty of applications and have been extensively studied for about a century. The literature on them is immense and ranges from thousands of articles in periodicals to huge monographs. This does not imply that there is nothing left to say on the topic. To the contrary, Toeplitz matrices are an active field of research with many facets, and the amount of material gathered only in the last decade would easily fill several volumes.

The present book lives within its limitations: to *banded* Toeplitz matrices on the one hand and to the *spectral properties* of such matrices on the other. As a third limitation, we consider large matrices only, and most of the results are actually *asymptotics*.

When speaking of banded Toeplitz matrices, we have in mind an $n \times n$ Toeplitz matrix of bandwidth $2r + 1$, and we silently assume that n is large in comparison with $2r + 1$. A Toeplitz matrix is completely specified by the (complex) numbers that constitute its first row and its first column. The function on the complex unit circle whose Fourier coefficients are just these numbers is referred to as the symbol of the matrix. In the case of Toeplitz band matrices, the symbol is a Laurent polynomial. Thus, we need not struggle with piecewise continuous or oscillating symbols, which arise in many applications, but "only" with Laurent polynomials. This circumstance simplifies part of the investigation. On the other hand, Laurent polynomials cause questions that are different from those one encounters in connection with more general symbols. Eventually, Toeplitz band matrices form their own realm in the world of Toeplitz matrices.

We understand spectral properties in a broad sense. Of course, we study such problems as the evolution of the eigenvalues of banded $n \times n$ Toeplitz matrices as n goes to infinity. The pioneering result in this direction was already proved by Schmidt and Spitzer in 1960, and every worker in the field has a personal copy of the Schmidt/Spitzer paper. Here we cite a full proof of this result for the first time in the monographical literature. This proof is Schmidt and Spitzer's original proof with several simplifications and improvements introduced by Hirschman and Widom.

We regard the singular values of a matrix as its most important spectral characteristics after the eigenvalues and pseudoeigenvalues; hence, we pay due attention to the asymptotic behavior of the singular values as the matrix dimension increases to infinity. Clearly, questions about the norm, the norm of the inverse, and the condition numbers of a matrix are questions about the extreme singular values.

Normal Toeplitz matrices raise specific problems, and these will be discussed. However, typically a Toeplitz matrix is nonnormal; hence, pseudospectra tell us more about Toeplitz matrices than spectra. Accordingly, we embark on pseudospectra of Toeplitz matrices and on related issues, such as the transient behavior of powers of large Toeplitz matrices.

Finally, the book contains some very recent results on the spectral behavior of Toeplitz matrices under certain structured perturbations. These results are far from what one wants to know about Toeplitz matrices with randomly perturbed main diagonal, but they are beautiful, they point in a good direction, and they have the potential to stimulate further research.

As already stated, the majority of the results describe the asymptotic behavior as the matrix dimension n goes to infinity. Many questions considered here can be easily answered by a few MATLAB commands if the matrix dimension is moderate, say in the low hundreds. We try to deliver answers in the case where n is really large and the computer quits. Part of the results are equipped with estimates of the convergence speed, which provides the user at least with a vague feeling for as to whether one can invoke the result for n in the hundreds. And, most importantly, several problems of this book are motivated by applications in statistical physics, where n is around 10^8, the cube root of the Avogadro number 10^{23}, and, for such astronomic values of n, asymptotic formulas are the only chance to describe and to understand something.

In summary, the book provides several pieces of information about the eigenvalues, singular values, determinants, norms, norms of inverses, (unstructured and structured) condition numbers, (unstructured and structured) pseudospectra, transient behavior, eigenvectors and pseudomodes, and spectral phenomena caused by perturbations of large Toeplitz band matrices. The selection of the material represents our taste and is to some extent determined by subjects we have worked on ourselves, and we think we can tell the community something about. Naturally, numerous problems are left open. Moreover, various important topics, such as fast inversion of Toeplitz matrices or fast solution of Toeplitz systems, are not touched at all. These topics are the business of other books (see, for example, [157] and [177]). However, the material of the present book is certainly useful and in many cases even indispensable when dealing with such practical problems as the effective solution of a large banded Toeplitz system.

The book is intended as an introductory text to some advanced topics. We assume that the reader is familiar with the basics of real and complex analysis, linear algebra, and functional analysis. Almost all results are accompanied by full proofs.

A baby version of this book was published under the title *Toeplitz Matrices, Asymptotic Linear Algebra, and Functional Analysis* by Hindustan Book Agency, New Delhi, and Birkhäuser, Basel, in 2000.

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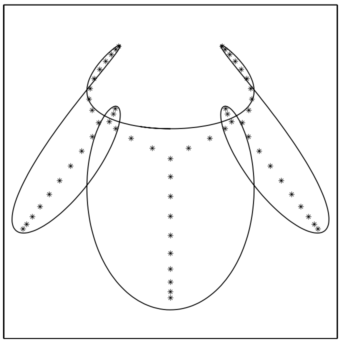
Tragically, Olga Grudskaya died in a car accident in February 2004. We have lost an exceptional woman, a wonderful friend, and an irreplaceable colleague. Her early death leaves an emptiness that can never be filled. In late 2003, she began working on the illustrations for this book with great enthusiasm. She could not accomplish her visions. We were left with the drafts of her illustrations and included some of them. They provide us with an idea of the beauty that would have emerged if she would have been able to complete her work. May this book keep the memory of our irretrievable Olga.

Chemnitz and Mexico City, spring 2005 The authors



Chapter 1
Infinite Matrices

When studying large finite matrices, it is natural to look also at their infinite counterparts. The spectral phenomena of the latter are sometimes easier to understand than those of the former. The question whether properties of infinite Toeplitz matrices mimic the corresponding properties of their large finite sections is very delicate and is, in a sense, the topic of this book.

We regard infinite Toeplitz matrices as operators on ℓ^p. This chapter is concerned with some basic properties of these operators, including boundedness, norms, invertibility and inverses, spectrum, eigenvalues, and eigenvectors. Wiener-Hopf factorization provides us with a fairly effective tool for the inversion of infinite (but not of finite) Toeplitz matrices. We also embark on some of the problems that are specific for selfadjoint operators.

1.1 Toeplitz and Hankel Matrices

An *infinite Toeplitz matrix* is a matrix of the form

$$(a_{j-k})_{j,k=0}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.1}$$

Such matrices are characterized by the property of being constant along the parallels to the main diagonal. Clearly, the matrix (1.1) is completely determined by its entries in the first row and first column, that is, by the sequence

$$\{a_k\}_{k=-\infty}^{\infty} = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\}. \tag{1.2}$$

Throughout this book we assume that the a_k's are complex numbers.

The matrix (1.1) is a band matrix if and only if at most finitely many of the numbers in (1.2) are nonzero. Although our subject is Toeplitz band matrices, it is also necessary to study Toeplitz matrices which are not band matrices. For example, the inverse of the band matrix

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots \\ 0 & 1 & -\frac{1}{2} & 0 & \dots \\ 0 & 0 & 1 & -\frac{1}{2} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

is the Toeplitz matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \dots \\ 0 & 1 & \frac{1}{2} & \frac{1}{2^2} & \dots \\ 0 & 0 & 1 & \frac{1}{2} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

and this is not a band matrix.

There is another type of matrix that arises when working with Toeplitz matrices. These are the Hankel matrices. An *infinite Hankel matrix* has the form

$$(a_{j+k+1})_{j,k=0}^{\infty} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & \dots & \dots \\ a_3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.3}$$

Notice that (1.3) is completely given by only the numbers with positive indices in (1.2). Obviously, if the sequence (1.2) has finite support, then the matrix (1.3) contains only finitely many nonzero entries.

1.2 Boundedness

The Wiener algebra. Let $\mathbf{T} := \{t \in \mathbf{C} : |t| = 1\}$ be the complex unit circle. The *Wiener algebra W* is defined as the set of all functions $a : \mathbf{T} \to \mathbf{C}$ with absolutely convergent Fourier series - that is, as the collection of all functions $a : \mathbf{T} \to \mathbf{C}$ which can be represented in the form

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n \ (t \in \mathbf{T}) \quad \text{with} \quad \|a\|_W := \sum_{n=-\infty}^{\infty} |a_n| < \infty. \tag{1.4}$$

Notice that instead of (1.4) we could also write

$$a(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \ (e^{i\theta} \in \mathbf{T}) \quad \text{with} \quad \|a\|_W := \sum_{n=-\infty}^{\infty} |a_n| < \infty. \tag{1.5}$$

The numbers a_n are the *Fourier coefficients* of a, and they can be computed by the formula

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-in\theta} \, d\theta. \tag{1.6}$$

Sometimes it will be convenient to identify a function $a : \mathbf{T} \to \mathbf{C}$ with the function $\theta \mapsto a(e^{i\theta})$; the latter function may be thought of as being given on $[0, 2\pi)$, $(-\pi, \pi]$, or even on all of the real line $\mathbf{R}$. Clearly, functions in W are continuous on $\mathbf{T}$ and, when regarded as functions on $\mathbf{R}$, they are 2π-periodic continuous functions.

Now let $a \in W$ and let $\{a_n\}_{n=-\infty}^{\infty}$ be the sequence of the Fourier coefficients of a. We denote by $T(a)$ and $H(a)$ the matrices (1.1) and (1.3), respectively:

$$T(a) := \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad H(a) := \begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & \dots & \dots \\ a_3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The matrix $T(a)$ is called the infinite Toeplitz matrix generated by a, while a is referred to as the *symbol* of the matrix $T(a)$. Note that if $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, then there is exactly one $a \in W$ such that (1.6) holds for all n. On the other hand, although $H(a)$ is uniquely determined by a, it is only the numbers a_n with $n \geq 1$ that can be recovered from the matrix $H(a)$. In other words: $H(a) = H(b)$ if and only if $a_n = b_n$ for all $n \geq 1$.

Infinite matrices as operators. We let $\ell^p := \ell^p(\mathbf{Z}_+)$ $(1 \leq p \leq \infty)$ stand for the usual Banach spaces of complex-valued sequences $\{x_n\}_{n=0}^{\infty}$: for $1 \leq p < \infty$,

$$x = \{x_n\}_{n=0}^{\infty} \in \ell^p \iff \|x\|_p^p := \sum_{n=0}^{\infty} |x_n|^p < \infty,$$

and for $p = \infty$,

$$x = \{x_n\}_{n=0}^{\infty} \in \ell^\infty \iff \|x\|_\infty := \sup_{n \geq 0} |x_n| < \infty.$$

An infinite matrix $A = (a_{jk})_{j,k=0}^{\infty}$ is said to *induce a bounded operator* on ℓ^p if there is a constant $M \in (0, \infty)$ such that for every $x = \{x_n\}_{n=0}^{\infty} \in \ell^p$ the inequality

$$\sum_{j=0}^{\infty} \Big| \sum_{k=0}^{\infty} a_{jk} x_k \Big|^p \leq M^p \sum_{k=0}^{\infty} |x_k|^p \tag{1.7}$$

holds; we remark that (1.7) includes the requirement that the series

$$y_j = \sum_{k=0}^{\infty} a_{jk} x_k \ (j \geq 0) \quad \text{and} \quad \sum_{j=0}^{\infty} |y_j|^p$$

are convergent. If $A = (a_{jk})_{j,k=0}^{\infty}$ induces a bounded operator on ℓ^p, we can simply think of A as *being* a bounded operator on ℓ^p which, after writing the elements of ℓ^p as column vectors, acts by the rule

$$y = Ax \quad \text{with} \quad \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}.$$

If A induces a bounded operator on ℓ^p, then there is a smallest M for which (1.7) is true for all $x \in \ell^p$. This number M is the norm of A, and it is denoted by $\|A\|_p$:

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p = 1} \|Ax\|_p.$$

If A does not induce a bounded operator on ℓ^p, we put $\|A\|_p = \infty$.

Let $\mathbf{Z}$ be the set of all integers. For $n \in \mathbf{Z}$, define $\chi_n \in W$ by

$$\chi_n(t) = t^n \quad (t \in \mathbf{T}).$$

The matrix $T(\chi_n)$ has units on a single parallel to the main diagonal and zeros elsewhere. Obviously, for $n \geq 0$,

$$T(\chi_n)x = \{\underbrace{0, \dots, 0}_{n}, x_0, x_1, \dots\}, \quad T(\chi_{-n})x = \{x_n, x_{n+1}, \dots\}. \tag{1.8}$$

Similarly, $H(\chi_n)$ is the zero matrix for $n \leq 0$ and is a matrix with units on a single "antidiagonal" and zeros elsewhere for $n \geq 1$:

$$H(\chi_n)x = \{x_{n-1}, x_{n-2}, \dots, x_0, 0, 0, \dots\} \quad \text{for} \quad n \geq 1. \tag{1.9}$$

Proposition 1.1. *If $a \in W$, then $T(a)$ and $H(a)$ induce bounded operators on the space ℓ^p $(1 \leq p \leq \infty)$ and*

$$\|T(a)\|_p \leq \|a\|_W, \quad \|H(a)\|_p \leq \|a\|_W.$$

Proof. If a is given by (1.4), then

$$T(a) = \sum_{n=-\infty}^{\infty} a_n T(\chi_n), \quad H(a) = \sum_{n=1}^{\infty} a_n H(\chi_n),$$

and from (1.8) and (1.9) we infer that $\|T(\chi_n)\|_p = 1$ for all n and $\|H(\chi_n)\|_p = 1$ for $n \geq 1$, whence

$$\|T(a)\|_p \leq \sum_{n=-\infty}^{\infty} |a_n|, \quad \|H(a)\|_p \leq \sum_{n=1}^{\infty} |a_n|. \qquad \square$$

By virtue of Proposition 1.1, we can regard $T(a)$ and $H(a)$ as bounded linear operators on ℓ^p. For Hankel operators, we can say even much more.

Proposition 1.2. *If $a \in W$, then $H(a)$ is compact on ℓ^p $(1 \leq p \leq \infty)$.*

Proof. Write a in the form (1.4) and put

$$(S_N a)(t) := \sum_{n=-N}^{N} a_n t^n \quad (t \in \mathbf{T}).$$

The operator $H(S_N a)$ is given by the matrix

$$H(S_N a) = \begin{pmatrix} a_1 & \dots & a_N & 0 & \dots \\ \vdots & & \vdots & \vdots & \\ a_N & \dots & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & \dots \\ \vdots & & \vdots & \vdots & \end{pmatrix}$$

and is therefore a finite rank operator. From Proposition 1.1 we infer that

$$\begin{aligned} \|H(a) - H(S_N a)\|_p &= \|H(a - S_N a)\|_p \\ &\le \|a - S_N a\|_W = \sum_{|n| \ge N} |a_j| = o(1) \quad \text{as} \quad N \to \infty. \end{aligned}$$

Therefore, $H(a)$ is a uniform limit of finite rank operators. This implies that $H(a)$ is compact. □

1.3 Products

It is easily seen that W is a Banach algebra with pointwise algebraic operations and the norm $\|\cdot\|_W$, i.e., $(W, \|\cdot\|_W)$ is a Banach space, and if $a, b \in W$, then $ab \in W$ and $\|ab\|_W \le \|a\|_W \|b\|_W$.

Given $a \in W$, we define the function $\tilde{a}$ by $\tilde{a}(t) := a(1/t)$ ($t \in \mathbf{T}$). Clearly, $\tilde{a}$ also belongs to W. Since

$$a(t) = \sum a_n t^n \quad \Longrightarrow \quad \tilde{a}(t) = \sum a_{-n} t^n,$$

we see that $T(\tilde{a})$ and $H(\tilde{a})$ are the matrices

$$T(\tilde{a}) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \dots \\ a_{-2} & a_{-1} & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad H(\tilde{a}) = \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \dots \\ a_{-2} & a_{-3} & \dots & \dots \\ a_{-3} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Thus, $T(\tilde{a})$ is simply the transpose of $T(a)$, but $H(\tilde{a})$ has nothing to do with $H(a)$.

Proposition 1.3. *If $a, b \in W$ then $T(ab) = T(a)T(b) + H(a)H(\tilde{b})$.*

Proof. The jk entry of $T(ab)$ is

$$(ab)_{j-k} = \sum_{m+n=j-k} a_m b_n = \sum_{\ell=-\infty}^{\infty} a_{j+\ell}\, b_{-k-\ell},$$

the jk entry of $T(a)T(b)$ equals

$$(a_j\, a_{j-1} \dots) \begin{pmatrix} b_{-k} \\ b_{-k+1} \\ \vdots \end{pmatrix} = \sum_{\ell=-\infty}^{0} a_{j+\ell}\, b_{-k-\ell},$$

and the jk entry of $H(a)H(\tilde{b})$ is equal to

$$(a_{j+1}\, a_{j+2}\, \ldots) \begin{pmatrix} b_{-k-1} \\ b_{-k-2} \\ \vdots \end{pmatrix} = \sum_{\ell=1}^{\infty} a_{j+\ell}\, b_{-k-\ell}. \qquad \square$$

Moral: The product of two infinite Toeplitz matrices is in general not a Toeplitz matrix, but it is always a Toeplitz matrix minus the product of two Hankel matrices. The previous proposition indicates the role played by Hankel matrices in the theory of Toeplitz matrices.

We now introduce two important subalgebras W_+ and W_- of W. Let W_+ and W_- stand for the set of all functions $a \in W$ which are of the form

$$a(t) = \sum_{n=0}^{\infty} a_n t \ (t \in \mathbf{T}) \quad \text{and} \quad a(t) = \sum_{n=-\infty}^{0} a_n t^n \ (t \in \mathbf{T}),$$

respectively. Equivalently, for $a \in W$ we have

$$a \in W_+ \Longleftrightarrow H(\tilde{a}) = 0 \Longleftrightarrow T(a) \quad \text{is lower-triangular},$$
$$a \in W_- \Longleftrightarrow H(a) = 0 \Longleftrightarrow T(a) \quad \text{is upper-triangular}.$$

Proposition 1.4. *If $a_- \in W_-$, $b \in W$, $a_+ \in W_+$, then*

$$T(a_- b a_+) = T(a_-)T(b)T(a_+).$$

Proof. Since $H(a_-) = H(\tilde{a}_+) = 0$, we deduce from Propostion 1.3 that

$$\begin{aligned} T(a_- b a_+) &= T(a_-)T(ba_+) + H(a_-)H(\widetilde{b\tilde{a}_+}) \\ &= T(a_-)T(ba_+) = T(a_-)T(b)T(a_+) + T(a_-)H(b)H(\tilde{a}_+) \\ &= T(a_-)T(b)T(a_+). \qquad \square \end{aligned}$$

1.4 Wiener-Hopf Factorization

In what follows, we have to work with a few important subsets of the Wiener algebra: GW, $\exp W$, $GW_\pm$, $\exp W_\pm$.

Wiener's theorem. We let GW stand for the group of the invertible elements of the algebra W. Thus, $a \in GW$ if and only if $a \in W$ and if there is a $b \in W$ such that $a(t)b(t) = 1$ for all $t \in \mathbf{T}$. Clearly, a function $a \in GW$ cannot have zeros on $\mathbf{T}$. The following famous theorem by Wiener says that the converse is also true.

Theorem 1.5. $GW = \{a \in W : a(t) \neq 0$ *for all* $t \in \mathbf{T}\}$.

The winding number. The set $\exp W$ is defined as the collection of all $a \in W$ which have a logarithm in W, that is, which are of the form $a = e^b$ with $b \in W$. To characterize $\exp W$, we need the notion of the winding number. If $a : \mathbf{T} \to \mathbf{C} \setminus \{0\}$ is a continuous function, then $a(t)$ traces out a continuous and closed curve in $\mathbf{C} \setminus \{0\}$ as t moves once around the counterclockwise oriented unit circle. The number of times this curve surrounds the origin counterclockwise is called the *winding number* of a and is denoted by wind a. Another (equivalent) definition is as follows. Every continuous function $a : \mathbf{T} \to \mathbf{C} \setminus \{0\}$ can be written in the form $a(e^{i\theta}) = |a(e^{i\theta})|e^{ic(\theta)}$ ($e^{i\theta} \in \mathbf{T}$), where $c : [0, 2\pi) \to \mathbf{R}$ is a continuous function. The number

$$\frac{1}{2\pi}\Big(c(2\pi - 0) - c(0 + 0)\Big)$$

is an integer which is independent of the particular choice of c. This integer is wind a.

Theorem 1.6. $\exp W = \{a \in GW : \text{ wind } a = 0\}$.

Analytic Wiener functions. In Section 1.3, we introduced the algebras $W_\pm$. We denote by $GW_\pm$ the functions $a_\pm \in W_\pm$ for which there exist $b_\pm \in W_\pm$ such that $a_\pm(t)b_\pm(t) = 1$ for all $t \in \mathbf{T}$, and we let $\exp W_\pm$ stand for the functions $a_\pm \in GW_\pm$ which can be represented in the form $a_\pm = e^{b_\pm}$ with $b_\pm \in W_\pm$. Notice that $GW_\pm$ is a proper subset of $W_\pm \cap GW$: for example, if $a_+ \in W_+$ is given by $a_+(t) = t$, then $1/a_+(t) = t^{-1}$ is a function in W_-.

Let $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk. Every function $a_+ \in W_+$ can be extended to an analytic function in $\mathbf{D}$ by the formula

$$a_+(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbf{D}),$$

where $\{a_n\}_{n=0}^{\infty}$ is the sequence of the Fourier coefficients of a. Analogously, every function $a_- \in W_-$ admits analytic continuation to $\{z \in \mathbf{C} : |z| > 1\} \cup \{\infty\}$ via

$$a_-(z) = \sum_{n=0}^{\infty} a_{-n} z^{-n} \quad (1 < |z| \le \infty).$$

Theorem 1.7. *We have*

$$\begin{aligned} &GW_+ = \{a \in W : a(z) \neq 0 \textit{ for all } |z| \le 1\}, \\ &GW_- = \{a \in W : a(z) \neq 0 \textit{ for all } |z| \ge 1 \textit{ and for } z = \infty\}, \\ &\exp W_+ = GW_+, \quad \exp W_- = GW_-. \end{aligned}$$

Theorems 1.5 to 1.7 are standard results of the theory of commutative Banach algebras and are essentially equivalent to the facts that the maximal ideal spaces of W, W_+, W_- are $\mathbf{T}$, $\mathbf{D} \cup \mathbf{T}$, $(\mathbf{C} \cup \{\infty\}) \setminus \mathbf{D}$, respectively.

Theorem 1.8 (Wiener-Hopf factorization). *Let $a \in W$ and suppose that $a(t) \neq 0$ for all $t \in \mathbf{T}$ and that* wind $a = m$. *Then a can be written in the form*

$$a(t) = a_-(t)t^m a_+(t) \quad (t \in \mathbf{T}) \quad \textit{with} \quad a_\pm \in GW_\pm.$$

Proof. Recall that $\chi_m(t) = t^m$. We have wind $(a\chi_{-m})$ = wind a+wind $\chi_{-m} = m - m = 0$, whence $a\chi_{-m} = e^b$ with some $b \in W$ by Theorems 1.5 and 1.6. Let

$$b(t) = \sum_{n=-\infty}^{\infty} b_n t^n \quad (t \in \mathbf{T})$$

and put

$$b_-(t) = \sum_{n=-\infty}^{-1} b_n t^n, \quad b_+(t) = \sum_{n=0}^{\infty} b_n t^n.$$

It is obvious that $e^{b_\pm} \in GW_\pm$. The representation $a = e^{b_-}\chi_m e^{b_+}$ is the desired factorization. □

Laurent polynomials. These are the functions in the Wiener algebra with only finitely many nonzero Fourier coefficients. Thus, $b : \mathbf{T} \to \mathbf{C}$ is a *Laurent polynomial* if and only if b is of the form

$$b(t) = \sum_{j=-r}^{s} b_j t^j \quad (t \in \mathbf{T}), \tag{1.10}$$

where r and s are integers and $-r \le s$. We denote the set of all Laurent polynomials by $\mathcal{P}$ and we write $\mathcal{P}_{r,s}$ for the Laurent polynomials of the form (1.10). We also put $\mathcal{P}_s := \mathcal{P}_{s,s}$. Finally, we let $\mathcal{P}_s^+ := \mathcal{P}_{0,s-1}$ stand for the analytic polynomials of degree at most $s-1$ and we set $\mathcal{P}^+ = \cup_{s\ge 1} \mathcal{P}_s^+$.

Let us assume that $b \in \mathcal{P}_{r,s}$ is not identically zero and that $b_{-r} \ne 0$ and $b_s \ne 0$. We can write $b(t) = t^{-r}(b_{-r} + b_{-r+1}t + \cdots + b_s t^{r+s})$. If $b(t) \ne 0$ for $t \in \mathbf{T}$, we further have

$$b(t) = t^{-r} b_s \prod_{j=1}^{J}(t-\delta_j) \prod_{k=1}^{K}(t-\mu_k), \tag{1.11}$$

where $|\delta_j| < 1$ for all j and $|\mu_k| > 1$ for all k. Obviously, wind $(t-\delta_j) = 1$ and wind $(t-\mu_k) = 0$, whence

$$\text{wind } b = J - r; \tag{1.12}$$

that is, wind b is the number of zeros of b in $\mathbf{D}$ minus the number of poles of b in $\mathbf{D}$ (all counted according to their multiplicity). The factorization

$$b(t) = b_s \prod_{j=1}^{J}\left(1 - \frac{\delta_j}{t}\right) t^{J-r} \prod_{k=1}^{K}(t-\mu_k) \tag{1.13}$$

is a Wiener-Hopf factorization; notice that

$$\left(1 - \frac{\delta_j}{t}\right)^{-1} = 1 + \frac{\delta_j}{t} + \frac{\delta_j^2}{t^2} + \cdots \quad (t \in \mathbf{T}) \tag{1.14}$$

and

$$(t-\mu_k)^{-1} = -\frac{1}{\mu_k}\left(1 + \frac{t}{\mu_k} + \frac{t^2}{\mu_k^2} + \cdots\right) \quad (t \in \mathbf{T}) \tag{1.15}$$

are functions in W_- and W_+, respectively.

1.5 Spectra

Fredholm operators. Let X be a Banach space. We denote by $\mathcal{B}(X)$ and $\mathcal{K}(X)$ the bounded and compact linear operators on X, respectively. The *spectrum* of an operator $A \in \mathcal{B}(X)$ is the set

$$\text{sp } A = \{\lambda \in \mathbf{C} : \; A - \lambda I \quad \text{is not invertible}\}.$$

The operator valued function $\mathbf{C} \setminus \text{sp } A \to \mathcal{B}(H)$, $\lambda \mapsto (A - \lambda I)^{-1}$ is well defined and analytic. It is called the *resolvent* of A. An operator $A \in \mathcal{B}(X)$ is said to be *Fredholm* if it is invertible modulo compact operators, that is, if there is an operator $B \in \mathcal{B}(X)$ such that $AB - I$ and $BA - I$ are compact. We define the *essential spectrum* of $A \in \mathcal{B}(X)$ as the set

$$\text{sp}_{\text{ess}} A = \{\lambda \in \mathbf{C} : \; A - \lambda I \quad \text{is not Fredholm}\}.$$

Clearly, $\text{sp}_{\text{ess}} A \subset \text{sp } A$ and $\text{sp}_{\text{ess}} A$ is invariant under compact perturbations.

The *kernel* and the *image* (= range) of $A \in \mathcal{B}(X)$ are defined as usual:

$$\text{Ker } A = \{x \in X : \; Ax = 0\}, \quad \text{Im } A := A(X).$$

An operator $A \in \mathcal{B}(X)$ is said to be *normally solvable* if Im A is a closed subspace of X. In that case the *cokernel* of A is

$$\text{Coker } A = X/\text{Im } A.$$

One can show that $A \in \mathcal{B}(X)$ is Fredholm if and only if A is normally solvable and both Ker A and Coker A have finite dimensions. The *index* of a Fredholm operator $A \in \mathcal{B}(X)$ is the integer

$$\text{Ind } A := \dim \text{Ker } A - \dim \text{Coker } A.$$

Theorem 1.9. *Let $a \in W$. The operator $T(a)$ is Fredholm on ℓ^p $(1 \le p \le \infty)$ if and only if $a(t) \ne 0$ for all $t \in \mathbf{T}$. In that case* Ind $T(a) = -$wind a.

Proof. If a has no zeros on $\mathbf{T}$ and if the winding number of a is m, then $a = a_- \chi_m a_+$ with $a_\pm \in GW_\pm$ by virtue of Theorem 1.5. From Proposition 1.4 we infer that

$$T(a) = T(a_-)T(\chi_m)T(a_+),$$

and the same proposition tells us that $T(a_\pm)$ are invertible, the inverses being $T(a_\pm^{-1})$. From (1.8) we see that $T(\chi_m)$ has closed range and that

$$\dim \text{Ker } T(\chi_m) = \begin{cases} 0 & \text{if} \quad m \ge 0, \\ |m| & \text{if} \quad m < 0, \end{cases} \qquad \dim \text{Coker } T(\chi_m) = \begin{cases} m & \text{if} \quad m \ge 0, \\ 0 & \text{if} \quad m < 0, \end{cases}$$

which implies that $T(\chi_m)$ is Fredholm of index $-m$. Consequently, $T(a)$ is also Fredholm of index $-m$.

Conversely, suppose now that $T(a)$ is Fredholm and let m be the index. Contrary to what we want, let us assume that $a(t_0) = 0$ for some $t_0 \in \mathbf{T}$. We can then find $b, c \in GW$

such that $\|a-b\|_W$ and $\|a-c\|_W$ are as small as desired and such that $|\text{wind } b - \text{wind } c| = 1$. Since Fredholmness and index are stable under small perturbations, it follows that $T(b)$ and $T(c)$ are Fredholm and that $\text{Ind } T(b) = \text{Ind } T(c) = m$. However, from what was proved in the preceding paragraph and from the equality $|\text{wind } b - \text{wind } c| = 1$ we know that $|\text{Ind } T(b) - \text{Ind } T(c)| = 1$. This contradiction shows that a cannot have zeros on $\mathbf{T}$. □

Corollary 1.10. *If $a \in W$, then $\text{sp}_{\text{ess}}\, T(a) = a(\mathbf{T})$.*

Proof. Apply Theorem 1.9 to $a - \lambda$. □

Corollary 1.11. *Let $a \in W$. The operator $T(a)$ is invertible on ℓ^p $(1 \le p \le \infty)$ if and only if $a(t) \neq 0$ for all $t \in \mathbf{T}$ and* $\text{wind } a = 0$.

Proof. If $T(a)$ is invertible, then $T(a)$ is Fredholm of index zero and Theorem 1.9 shows that a has no zeros on $\mathbf{T}$ and that $\text{wind } a = 0$. If $a(t) \neq 0$ for $t \in \mathbf{T}$ and $\text{wind } a = 0$, then $a = a_- a_+$ with $a_\pm \in GW_\pm$ due to Theorem 1.5. From Proposition 1.4 we deduce that $T(a_+^{-1})T(a_-^{-1})$ is the inverse of the operator $T(a) = T(a_-)T(a_+)$. □

The following beautiful purely geometric description of the spectrum of a Toeplitz operator is illustrated by Figure 1.1.

Corollary 1.12. *If $a \in W$, then*

$$\text{sp } T(a) = a(\mathbf{T}) \cup \left\{\lambda \in \mathbf{C} \setminus a(\mathbf{T}) : \ \text{wind } (a - \lambda) \neq 0\right\}.$$

Proof. This is Corollary 1.11 with a replaced by $a - \lambda$. □

In Section 1.2, we observed that $H(a)$ is compact for every $a \in W$. The following result shows that the zero operator is the only compact Toeplitz operator.

Corollary 1.13. *If $a \in W$ and $T(a)$ is compact on ℓ^p $(1 \le p \le \infty)$, then a vanishes identically.*

Proof. If $T(a)$ is compact, then $\text{sp}_{\text{ess}}\, T(a) = \{0\}$, and Corollary 1.10 tells us that this can only happen if $a(\mathbf{T}) = \{0\}$. □

1.6 Norms

The cases $p = 1$ and $p = \infty$. It is well known that an infinite matrix $A = (a_{jk})_{j,k=0}^{\infty}$ induces a bounded operator on ℓ^1 and ℓ^∞, respectively, if and only if

$$\sup_{k \ge 1} \sum_{j=1}^{\infty} |a_{jk}| < \infty \quad \text{and} \quad \sup_{j \ge 1} \sum_{k=1}^{\infty} |a_{jk}| < \infty,$$

in which case

$$\|A\|_1 = \sup_{k \ge 1} \sum_{j=1}^{\infty} |a_{jk}| \quad \text{and} \quad \|A\|_\infty = \sup_{j \ge 1} \sum_{k=1}^{\infty} |a_{jk}|. \tag{1.16}$$

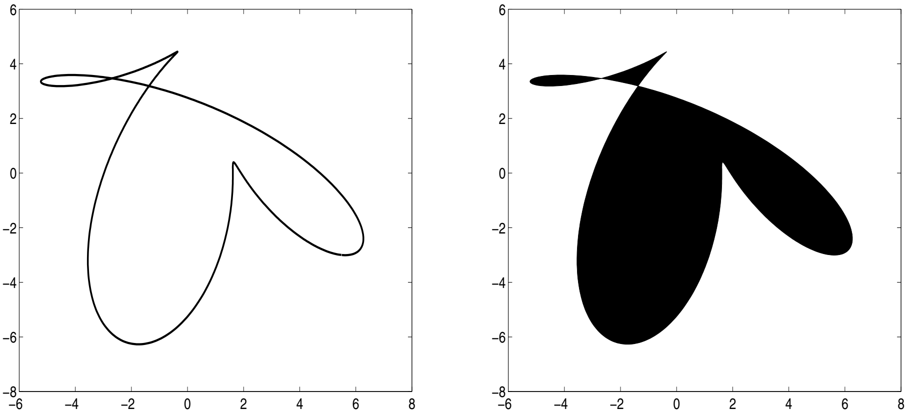


Figure 1.1. *The essential spectrum* $\mathrm{sp}_{\mathrm{ess}}\, T(a) = a(\mathbf{T})$ *on the left and the spectrum* $\mathrm{sp}\ T(a)$ *on the right.*

This easily implies the following.

Theorem 1.14. *If* $a \in W$ *then* $\|T(a)\|_1 = \|T(a)\|_\infty = \|a\|_W$. $\square$

The case $p = 2$. Let $L^2 := L^2(\mathbf{T})$ be the usual Lebesgue space of complex-valued functions on $\mathbf{T}$ with the norm

$$\|f\|_2 := \left(\int_{\mathbf{T}} |f(t)|^2 \frac{|dt|}{2\pi} \right)^{1/2} = \left(\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2}.$$

The set $H^2 := H^2(\mathbf{T}) := \{f \in L^2 : f_n = 0 \text{ for } n < 0\}$ is a closed subspace of L^2 and is referred to as the *Hardy space* of L^2. Let $P : L^2 \to H^2$ be the orthogonal projection. Thus, if $f \in L^2$ is given by

$$f(t) = \sum_{n=-\infty}^{\infty} f_n t^n \quad (t \in \mathbf{T}),$$

then

$$(Pf)(t) = \sum_{n=0}^{\infty} f_n t^n \quad (t \in \mathbf{T}).$$

The map

$$\Phi : H^2 \to \ell^2, \quad f \mapsto \{f_n\}_{n=0}^{\infty} \tag{1.17}$$

is a unitary operator of H^2 onto ℓ^2. It is not difficult to check that if $a \in W$, then $\Phi^{-1} T(a) \Phi$ is the operator

$$\Phi^{-1} T(a) \Phi : H^2 \to H^2, \quad f \mapsto P(af), \tag{1.18}$$

where $(af)(t) := a(t)f(t)$. The observation (1.18) is in fact the key to the theory of Toeplitz operators on ℓ^2. We here confine ourselves to the following consequence of (1.18).

Theorem 1.15. *If $a \in W$ then $\|T(a)\|_2 = \|a\|_\infty$, where $\|a\|_\infty := \max_{t\in\mathbf{T}} |a(t)|$.*

Proof. If $f \in H^2$, then $\|\Phi^{-1}T(a)\Phi f\|_2 = \|P(af)\|_2 \le \|af\|_2 \le \|a\|_\infty \|f\|_2$, whence $\|T(a)\|_2 = \|\Phi^{-1}T(a)\Phi\|_2 \le \|a\|_\infty$. On the other hand, Corollary 1.12 implies that the *spectral radius*

$$\operatorname{rad} T(a) = \max\left\{|\lambda| : \lambda \in \operatorname{sp}\, T(a)\right\}$$

is equal to $\|a\|_\infty$. Because $\operatorname{rad} T(a) \le \|T(a)\|_2$, it follows that $\|a\|_\infty \le \|T(a)\|_2$. □

Other values of p. This case is more delicate, but one has at least two-sided estimates.

Proposition 1.16. *If $a \in W$ and $1 \le p \le \infty$, then $\|a\|_\infty \le \|T(a)\|_p \le \|a\|_W$.*

Proof. The inequality $\|T(a)\|_p \le \|a\|_W$ results from Proposition 1.1, and the inequality $\|a\|_\infty \le \|T(a)\|_p$ is a consequence of Corollary 1.12 together with the estimate $\|a\|_\infty = \operatorname{rad} T(a) \le \|T(a)\|_p$. □

1.7 Inverses

Let b be a Laurent polynomial of the form (1.10). Suppose $b(t) \neq 0$ for $t \in \mathbf{T}$ and $\operatorname{wind} b = 0$. From Section 1.4 we know that b can be written in the form $b = b_- b_+$ with

$$b_-(t) = \prod_{j=1}^{r}\left(1 - \frac{\delta_j}{t}\right), \quad b_+(t) = b_s \prod_{k=1}^{s}(t - \mu_k), \tag{1.19}$$

where $\delta := \max(|\delta_1|, \dots, |\delta_r|) < 1$ and $\mu := \min(|\mu_1|, \dots, |\mu_s|) > 1$. When proving Corollary 1.11, we observed that

$$T^{-1}(b) = T(b_+^{-1})T(b_-^{-1}). \tag{1.20}$$

From (1.19) we see that

$$b_-^{-1}(t) = \prod_{j=1}^{r}\left(1 + \frac{\delta_j}{t} + \frac{\delta_j^2}{t^2} + \cdots\right) =: \sum_{m=0}^{\infty} \frac{c_m}{t^m},$$

$$b_+^{-1}(t) = \frac{1}{b_s}\prod_{k=1}^{s}\left(-\frac{1}{\mu_k}\right)\prod_{k=1}^{s}\left(1 + \frac{t}{\mu_k} + \frac{t^2}{\mu_k^2} + \cdots\right) =: \sum_{m=0}^{\infty} d_m t^m.$$

With the coefficients c_m and d_m, formula (1.20) takes the form

$$T^{-1}(b) = \begin{pmatrix} d_0 & & & \\ d_1 & d_0 & & \\ d_2 & d_1 & d_0 & \\ \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} c_0 & c_1 & c_2 & \dots \\ & c_0 & c_1 & \dots \\ & & c_0 & \dots \\ & & & \dots \end{pmatrix}. \tag{1.21}$$

Proposition 1.3 and (1.20) imply that

$$T^{-1}(b) = T(b^{-1}) - H(b_+^{-1})H(\widetilde{b}_-^{-1}). \tag{1.22}$$

Representation (1.20) gives us $T^{-1}(b)$ as the product of the lower triangular matrix $T(b_+^{-1})$ and the upper triangular matrix $T(b_-^{-1})$, while (1.22) shows that $T^{-1}(b)$ is the difference of the (in general full) Toeplitz matrix $T(b^{-1})$ and the product $H(b_+^{-1})H(\widetilde{b}_-^{-1})$ of two Hankel matrices.

Let α be any number satisfying

$$0 < \alpha < \min\left(\log\frac{1}{\delta}, \log\mu\right). \tag{1.23}$$

Lemma 1.17. *For every $n \ge 0$,*

$$|\left(b_-^{-1}\right)_{-n}| \le \left(\min_{|z|=\delta+\varepsilon} |b_-(z)|\right)^{-1} (\delta+\varepsilon)^n \quad (\varepsilon > 0), \tag{1.24}$$

$$|\left(b_+^{-1}\right)_n| \le \left(\min_{|z|=\mu-\varepsilon} |b_+(z)|\right)^{-1} (\mu-\varepsilon)^n \quad (0 < \varepsilon < \mu). \tag{1.25}$$

Consequently, $(b_-^{-1})_{-n}$ and $(b_+^{-1})_n$ are $O(e^{-\alpha n})$ as $n \to \infty$.

Proof. Since $1/b_-(z)$ is analytic for $|z| > \delta$, we get

$$(b_-^{-1})_{-n} = \frac{1}{2\pi i}\int_{|z|=1} \frac{z^{n-1}dz}{b_-(z)} = \frac{1}{2\pi i}\int_{|z|=\delta+\varepsilon} \frac{z^{n-1}dz}{b_-(z)}$$

and hence

$$|(b_-^{-1})_{-n}| \le \frac{1}{2\pi}\left(\min_{|z|=\delta+\varepsilon} |b_-(z)|\right)^{-1} (\delta+\varepsilon)^{n-1} 2\pi(\delta+\varepsilon),$$

which is (1.24). Estimate (1.25) can be verified analogously. □

Proposition 1.18. *For the j, k entry of $T^{-1}(b)$ we have the estimate*

$$\left[T^{-1}(b)\right]_{j,k} = \left(b^{-1}\right)_{j-k} + O\left(e^{-\alpha(j+k)}\right).$$

Proof. From (1.22) we see that

$$\left|\left[T^{-1}(b)\right]_{j,k} - (b^{-1})_{j-k}\right| = \left|\sum_{\ell=1}^{\infty} (b_+^{-1})_{j+\ell}(b_-^{-1})_{-k-\ell}\right|$$
$$\le \left(\sum_{\ell=1}^{\infty} |(b_+^{-1})_{j+\ell}|^2 \sum_{\ell=1}^{\infty} |(b_-^{-1})_{-k-\ell}|^2\right)^{1/2}, \tag{1.26}$$

and Lemma 1.17 implies that (1.26) is

$$O\left(\left(\sum_{\ell=1}^{\infty} e^{-2\alpha(j+\ell)}\right)^{1/2}\right) O\left(\left(\sum_{\ell=1}^{\infty} e^{-2\alpha(k+\ell)}\right)^{1/2}\right) = O(e^{-\alpha j})\, O(e^{-\alpha k}). \qquad \square$$

Given two sequences $x = \{x_k\}$ and $y = \{y_k\}$, we set $(x, y) = \sum x_k \overline{y}_k$. The j, k entry of $T^{-1}(b)$ is just $(T^{-1}(b)x, y)$ for $x = e_k$ and $y = e_j$, where $\{e_n\}$ is the standard basis of ℓ^2. The following useful observation evaluates $(T^{-1}(b)x, y)$ at another interesting pair (x, y). For $z \in \mathbf{D}$, define $w_z \in \ell^2$ by $(w_z)_n = z^n$ $(n \geq 0)$.

Proposition 1.19. *Let $b = b_- b_+$ with $b_\pm$ given by* (1.19). *Then for $\alpha, \beta \in \mathbf{D}$,*

$$(T^{-1}(b)w_\alpha, w_\beta) = \frac{1}{b_s} \frac{1}{1 - \alpha\overline{\beta}} \prod_{j=1}^{r} \frac{1}{1 - \delta_j \alpha} \prod_{k=1}^{s} \frac{1}{\overline{\beta} - \mu_k} \tag{1.27}$$

$$= \frac{1}{1 - \alpha\overline{\beta}} \frac{1}{b_-(1/\alpha) b_+(\overline{\beta})}. \tag{1.28}$$

Proof. We have

$$(T^{-1}(b)w_\alpha, w_\beta) = (T(b_+^{-1})T(b_-^{-1})w_\alpha, w_\beta) = (T(b_-^{-1})w_\alpha, T(\overline{b}_+^{-1})w_\beta).$$

Define $\chi_n(t) := t^n$ $(t \in \mathbf{T})$. It is easily seen that, for $|\delta| < 1$,

$$T^{-1}(1 - \delta\chi_{-1})w_\alpha = T(1 + \delta\chi_{-1} + \delta^2\chi_{-2} + \cdots)w_\alpha = \frac{1}{1 - \delta\alpha} w_\alpha,$$

whence

$$T(b_-^{-1})w_\alpha = \left(\prod_{j=1}^{r} \frac{1}{1 - \delta_j \alpha} \right) w_\alpha.$$

Analogously, if $|\mu| > 1$,

$$T^{-1}(\chi_{-1} - \overline{\mu})w_\beta = -\frac{1}{\overline{\mu}} T^{-1} \left(1 - \frac{1}{\overline{\mu}} \chi_{-1} \right) w_\beta = -\frac{1}{\overline{\mu}} \frac{1}{1 - \overline{\mu}^{-1}\beta} w_\beta = \frac{1}{\beta - \overline{\mu}} w_\beta,$$

which implies that

$$T(\overline{b}_+^{-1})w_\beta = \overline{b}_s^{-1} \left(\prod_{k=1}^{s} \frac{1}{\beta - \overline{\mu_k}} \right) w_\beta.$$

Consequently,

$$(T(b_-^{-1})w_\alpha, T(\overline{b}_+^{-1})w_\beta) = \frac{1}{b_s} \prod_{j=1}^{r} \frac{1}{1 - \delta_j \alpha} \prod_{k=1}^{s} \frac{1}{\overline{\beta} - \mu_k} (w_\alpha, w_\beta).$$

As $(w_\alpha, w_\beta) = 1/(1 - \alpha\overline{\beta})$, we arrive at (1.27). Clearly, (1.28) is nothing but another way of writing (1.27). □

1.8 Eigenvalues and Eigenvectors

Let b be a Laurent polynomial. In this section we study the problem of finding the $\lambda \in \operatorname{sp} T(b)$ for which there exist nonzero $x \in \ell^p$ such that $T(b)x = \lambda x$. These λ are called eigenvalues of $T(b)$ on ℓ^p, and the corresponding x's are referred to as eigenelements or eigenvectors (which sounds much better). Since $T(b) - \lambda I = T(b - \lambda)$, our problem is equivalent to the question of when a Toeplitz operator has a nontrivial kernel. Throughout this section we assume that b is not constant on the unit circle $\mathbf{T}$.

Outside the essential spectrum. For a point $\lambda \in \mathbf{C} \setminus b(\mathbf{T})$, we denote by $\operatorname{wind}(b, \lambda)$ the winding number of b about λ, that is, $\operatorname{wind}(b, \lambda) := \operatorname{wind}(b - \lambda)$. A sequence $\{x_n\}_{n=0}^{\infty}$ is said to be *exponentially decaying* if there are $C \in (0, \infty)$ and $\gamma \in (0, \infty)$ such that $|x_n| \le Ce^{-\gamma n}$ for all $n \ge 0$.

Proposition 1.20. *Let $1 \le p \le \infty$. A point $\lambda \notin b(\mathbf{T})$ is an eigenvalue of $T(b)$ as an operator on ℓ^p if and only if* $\operatorname{wind}(b, \lambda) = -m < 0$*, in which case* $\operatorname{Ker}(T(b) - \lambda I)$ *has the dimension m and each eigenvector is exponentially decaying.*

Proof. From Theorem 1.8 (or simply from (1.13)) we get a Wiener-Hopf factorization $b(t) - \lambda = b_-(t)t^{-m}b_+(t)$. Proposition 1.4 implies that $T(b - \lambda)$ decomposes into the product $T(b_-)T(\chi_{-m})T(b_+)$ and that the operators $T(b_\pm)$ are invertible, the inverses being $T(b_\pm^{-1})$. Thus, $x \in \operatorname{Ker} T(b - \lambda)$ if and only if $T(\chi_{-m})T(b_+)x = 0$. If $m \le 0$, this is equivalent to the equation $T(b_+)x = 0$ and hence to the equality $x = 0$. So let $m > 0$. We denote by $e_j \in \ell^p$ the sequence given by $(e_j)_k = 1$ for $k = j$ and $(e_j)_k = 0$ for $k \ne j$. Clearly, $T(\chi_{-m})T(b_+)x = 0$ if and only if $T(b_+)x$ belongs to the linear hull $\operatorname{lin}\{e_0, \dots, e_{m-1}\}$ of $e_0, \dots, e_{m-1}$. Consequently,

$$\operatorname{Ker} T(b - \lambda) = \operatorname{lin}\{T(b_+^{-1})e_0, \dots, T(b_+^{-1})e_{m-1}\}.$$

This shows that $\dim \operatorname{Ker} T(b-\lambda) = m$, and from Lemma 1.17 we deduce that the sequences in $\operatorname{Ker} T(b - \lambda)$ are exponentially decaying. □

Inside the essential spectrum. Things are a little bit more complicated for points $\lambda \in b(\mathbf{T})$. In that case $b - \lambda$ has zeros on $\mathbf{T}$. For $\tau \in \mathbf{T}$, we define the function ξ_τ by

$$\xi_\tau(t) = 1 - \frac{\tau}{t} \quad (t \in \mathbf{T}).$$

Notice that ξ_τ has a single zero on $\mathbf{T}$ and that $T(\xi_\tau)$ is the upper triangular matrix

$$T(\xi_\tau) = \begin{pmatrix} 1 & -\tau & 0 & \dots \\ 0 & 1 & -\tau & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Lemma 1.21. *Let $\tau_1, \dots, \tau_\ell$ be distinct points on $\mathbf{T}$ and let $\alpha_1, \dots, \alpha_\ell$ be positive integers. Then*

$$\operatorname{Ker} T\left(\xi_{\tau_1}^{\alpha_1} \dots \xi_{\tau_\ell}^{\alpha_\ell}\right) = \{0\} \;\; on \;\; \ell^p \quad (1 \le p < \infty) \tag{1.29}$$

and

$$\operatorname{Ker} T\left(\xi_{\tau_1}^{\alpha_1} \dots \xi_{\tau_\ell}^{\alpha_\ell}\right) = \operatorname{lin}\{w_{\tau_1}, \dots, w_{\tau_\ell}\} \text{ on } \ell^\infty, \tag{1.30}$$

where $(w_\tau)_n := 1/\tau^n$.

Proof. Put $\xi = \xi_{\tau_1}^{\alpha_1} \dots \xi_{\tau_\ell}^{\alpha_\ell}$ and write

$$\xi(t) = a_0 + a_1 \frac{1}{t} + a_2 \frac{1}{t^2} + \dots + a_N \frac{1}{t^N}.$$

The equation $T(\xi)x = 0$ is the difference equation

$$a_0 x_n + a_1 x_{n+1} + \dots + a_N x_{n+N} = 0 \quad (n \ge 0),$$

which is satisfied if and only if

$$x_n = \sum_{k=0}^{\alpha_1 - 1} \gamma_k^{(1)} \frac{n^k}{\tau_1^n} + \dots + \sum_{k=0}^{\alpha_\ell - 1} \gamma_k^{(\ell)} \frac{n^k}{\tau_\ell^n} \tag{1.31}$$

with complex numbers $\gamma_k^{(j)}$. The sequence given by (1.31) belongs to ℓ^p $(1 \le p < \infty)$ if and only if it is identically zero, which proves (1.29), and it is in ℓ^∞ if and only if it is of the form

$$x_n = \gamma_0^{(1)} \frac{1}{\tau_1^n} + \dots + \gamma_0^{(\ell)} \frac{1}{\tau_\ell^n},$$

which gives (1.30). □

Given $\lambda \in b(\mathbf{T})$, we denote the distinct zeros of $b - \lambda$ on $\mathbf{T}$ by $\tau_1, \dots, \tau_\ell$ and their multiplicities by $\alpha_1, \dots, \alpha_\ell$. We extract the zeros by "anti-analytic" linear factors, that is, we write $b - \lambda$ in the form

$$b(t) - \lambda = \prod_{j=1}^{\ell} \left(1 - \frac{\tau_j}{t}\right)^{\alpha_j} c(t), \tag{1.32}$$

where $c(t) \ne 0$ for $t \in \mathbf{T}$.

Proposition 1.22. *Let* $1 \le p < \infty$. *Suppose* $\lambda \in b(\mathbf{T})$ *and write* $b - \lambda$ *in the form* (1.32). *Then* λ *is an eigenvalue of the operator* $T(b)$ *on* ℓ^p *if and only if* $\operatorname{wind} c = -m < 0$, *in which case* $\operatorname{Ker}(T(b) - \lambda I)$ *is of the dimension* m *and all eigenvectors are exponentially decaying.*

Proof. By Proposition 1.4,

$$T(b - \lambda) = \prod_{j=1}^{\ell} [T(\xi_{\tau_j})]^{\alpha_j} T(c). \tag{1.33}$$

From (1.29) we see that $\operatorname{Ker} T(b-\lambda) = \operatorname{Ker} T(c)$, and Proposition 1.20 therefore gives the assertion. □

We now turn to the case $p = \infty$. A sequence $\{x_n\}_{n=0}^{\infty}$ is called *extended* if

$$\limsup_{n\to\infty} |x_n| > 0.$$

Proposition 1.23. *Let $\lambda \in b(\mathbf{T})$ and write $b-\lambda$ in the form (1.32). Then λ is an eigenvalue of $T(b)$ on ℓ^∞ if and only if* wind $c = -m < \ell$. *In that case the dimension of* $\operatorname{Ker}(T(b)-\lambda I)$ *is $m+\ell$. There is a basis in* $\operatorname{Ker}(T(b)-\lambda I)$ *whose elements enjoy the following properties:*

(a) *if $m > 0$, then m elements of the basis decay exponentially and ℓ elements have zeros in the first m places and are extended;*

(b) *if $m \le 0$, then all the $\ell - |m|$ elements of the basis are extended.*

Proof. Combining (1.30) and (1.33), we see that $T(b-\lambda)x = 0$ if and only if there are complex numbers γ_j such that

$$T(c)x = \sum_{j=1}^{\ell} \gamma_j w_{\tau_j}. \tag{1.34}$$

Let $c = c_-\chi_{-m}c_+$ be a Wiener-Hopf factorization of c. It can be readily checked that $T(c_-^{-1})w_\tau = c_-^{-1}(1/\tau)w_\tau$. Thus, setting $\delta_j = \gamma_j c_-^{-1}(1/\tau_j)$, we can rewrite (1.34) in the form

$$T(\chi_{-m})T(c_+)x = \sum_{j=1}^{\ell} \delta_j w_{\tau_j}. \tag{1.35}$$

If $m \ge 0$, then (1.35) holds if and only if

$$T(c_+)x \in \operatorname{lin}\{e_0, \dots, e_{m-1}, T(\chi_m)w_{\tau_1}, \dots, T(\chi_m)w_{\tau_\ell}\},$$

which is equivalent to the requirement that x be in

$$\operatorname{lin}\{T(c_+^{-1})e_0, \dots, T(c_+^{-1})e_{m-1}, T(\chi_m)T(c_+^{-1})w_{\tau_1}, \dots, T(\chi_m)T(c_+^{-1})w_{\tau_\ell}\}.$$

The sequences $T(c_+^{-1})e_j$ decay exponentially (Lemma 1.17), and since

$$[T(c_+^{-1})w_\tau]_n = \frac{1}{\tau^n}\sum_{k=0}^{n}(c_+^{-1})_k \tau^k,$$

it follows that the sequences $T(c_+^{-1})w_{\tau_j}$ are extended. This completes the proof in the case $m \ge 0$.

Let now $m < 0$. In that case (1.35) is satisfied if and only if

$$\sum_{j=1}^{\ell} \delta_j (1/\tau_j)^n = 0 \ \text{ for } \ n = 0, 1, \dots, |m|-1 \tag{1.36}$$

and

$$x = \sum_{j=1}^{\ell} \delta_j T(c_+^{-1}) T(\chi_{-|m|}) w_{\tau_j}. \tag{1.37}$$

Equations (1.36) are a Vandermonde system for the δ_j's. If $|m| \ge \ell$, then (1.36) has only the trivial solution. If $|m| < \ell$, we can choose $\delta_1, \dots, \delta_{\ell-|m|}$ arbitrarily. The numbers $\delta_{\ell-|m|+1}, \dots, \delta_\ell$ are then uniquely determined. This shows that the set of all x of the form (1.37) has the dimension $\ell - |m|$ and that all nonzero x in this set are extended. □

In geometric terms, the winding number of the function c in (1.32) can be determined as follows. Choose a number $\varrho > 1$ and consider the function b_ϱ defined by $b_\varrho(t) = b(\varrho t)$ ($t \in \mathbf{T}$). If ϱ is sufficiently close to 1, then $b_\varrho - \lambda$ has no zeros on $\mathbf{T}$ and hence $\operatorname{wind}(b_\varrho, \lambda)$ is well defined.

Proposition 1.24. *We have* $\operatorname{wind} c = \lim_{\varrho \to 1+0} \operatorname{wind}(b_\varrho, \lambda)$.

Proof. Let $c_\varrho(t) := c(\varrho t)$. From (1.32) we obtain $b_\varrho(t) - \lambda = \prod_{j=1}^{\ell} (1 - \frac{\tau_j}{\varrho t})^{\alpha_j} c_\varrho(t)$, whence $\operatorname{wind}(b_\varrho, \lambda) = \operatorname{wind} c_\varrho$. Since $\operatorname{wind} c_\varrho$ converges to $\operatorname{wind} c$ as $\varrho \to 1$, we arrive at the assertion. □

Frequently, the following observation is very useful.

Proposition 1.25. *Let* $\lambda \in b(\mathbf{T})$ *and suppose* Ω *is a connected component of* $\mathbf{C} \setminus b(\mathbf{T})$ *whose boundary contains* λ. *If* $\operatorname{wind}(b, z) = \varkappa$ *for* $z \in \Omega$, *then* $\operatorname{wind} c \ge \varkappa$.

Proof. For $z \in \Omega$, we have

$$b(t) - z = b_s t^{-r} \prod_{j=1}^{r+\varkappa} (t - \delta_j(z)) \prod_{k=1}^{s-\varkappa} (t - \mu_k(z))$$

with $|\delta_j(z)| < 1$ and $|\mu_k(z)| > 1$. Now let $z \in \Omega$ approach $\lambda \in \partial\Omega$. The some of the $\delta_j(z)$, say $\delta_1(z), \dots, \delta_m(z)$, move onto the unit circle, while the remaining $\delta_j(z)$ stay in the open unit disk. Analogously, some of the $\mu_k(z)$, say $\mu_1(z), \dots, \mu_\ell(z)$, attain modulus 1, whereas the remaining $\mu_k(z)$ keep modulus greater than 1. We can write $b(t) - \lambda$ as

$$b_s t^{-r} \prod_{j=1}^{m} (t - \delta_j(\lambda)) \prod_{j=m+1}^{r+\varkappa} (t - \delta_j(\lambda)) \prod_{k=1}^{\ell} (t - \mu_k(\lambda)) \prod_{k=\ell+1}^{s-\varkappa} (t - \mu_k(\lambda)).$$

The zeros of $b_\varrho - \lambda$ are $\delta_j(\lambda)/\varrho$ and $\mu_k(\lambda)/\varrho$. If $\varrho > 1$, then $|\delta_j(\lambda)| < 1$ for all j. If, in addition, $\varrho > 1$ is sufficiently close to 1, then $|\mu_k(\lambda)/\varrho| > 1$ for $k = \ell + 1, \dots, s - \varkappa$ and $|\mu_k(\lambda)/\varrho| < 1$ for $k = 1, \dots, \ell$. The result is that

$$\lim_{\varrho \to 1+0} \operatorname{wind}(b_\varrho - \lambda) = -r + (r + \varkappa) + \ell = \varkappa + \ell \ge \varkappa.$$

From Proposition 1.24 we so obtain that $\operatorname{wind} c \ge \varkappa$. □

Corollary 1.26. *If λ lies on the boundary of a connected component Ω of $\mathbf{C} \setminus b(\mathbf{T})$ such that* wind $(b, z) \geq 0$ *for $z \in \Omega$, then λ cannot be an eigenvalue of $T(b)$ on ℓ^p $(1 \leq p < \infty)$.*

Proof. Immediate from Propositions 1.22 and 1.25. □

Corollary 1.27. *If b is a real-valued Laurent polynomial, then $T(b)$ as an operator on ℓ^p $(1 \leq p < \infty)$ has no eigenvalues.*

Proof. This is a straightforward consequence of Corollary 1.26. Here is an alternative proof. Let $\lambda \in b(\mathbf{T})$ and write $b(t) - \lambda = b_r t^{-r} \prod_{j=1}^{2r}(t - z_j)$. Since $b(t) - \lambda$ is real valued, passage to the complex conjugate gives $b(t) - \lambda = b_r t^{-r} \prod_{j=1}^{2r}(t - 1/\overline{z}_j)$. Thus, if $b - \lambda$ has $\ell \geq 1$ distinct zeros $\tau_1, \ldots, \tau_\ell$ on $\mathbf{T}$ and $n \geq 0$ distinct zeros $\mu_1, \ldots, \mu_n$ of modulus greater than 1, then

$$b(t) - \lambda = b_r t^{-r} \prod_{j=1}^{n}[(t - \mu_j)(t - 1/\overline{\mu}_j)]^{\beta_j} \prod_{j=1}^{\ell}(t - \tau_j)^{\alpha_j} = \prod_{j=1}^{\ell}\left(1 - \frac{\tau_j}{t}\right)^{\alpha_j} c(t)$$

with

$$c(t) = b_r t^{-r} t^{\alpha_1 + \cdots + \alpha_\ell} \prod_{j=1}^{n}[(t - \mu_j)(t - 1/\overline{\mu}_j)]^{\beta_j}.$$

Clearly, wind $c = -r + \alpha_1 + \cdots + \alpha_\ell + \beta_1 + \cdots + \beta_n$. Since $\alpha_1 + \cdots + \alpha_\ell + 2\beta_1 + \cdots + 2\beta_n = 2r$ and $\alpha_j \geq 1$ for all j, we get $\beta_1 + \cdots + \beta_n < r$ and thus wind $c = r - \beta_1 - \cdots - \beta_n > 0$. The assertion now follows from Proposition 1.22. □

Example 1.28. We remark that Corollary 1.27 is not true for $p = \infty$: the sequence $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$ is obviously in the kernel of the operator

$$T(\chi_{-1} + \chi_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

For this operator, things are as follows. The symbol is $b(t) = t^{-1} + t$ and hence sp $T(b) = b(\mathbf{T}) = [-2, 2]$. For $\lambda \in [-2, 2]$,

$$b(t) - \lambda = \left(1 - \frac{\tau_1(\lambda)}{t}\right)\left(1 - \frac{\tau_2(\lambda)}{t}\right) t,$$

where $\tau_1(\lambda), \tau_2(\lambda) \in \mathbf{T}$ are given by

$$\tau_{1,2}(\lambda) = \frac{\lambda}{2} \pm i\sqrt{1 - \frac{\lambda^2}{4}}.$$

Thus, if $\lambda \in (-2, 2)$, then Proposition 1.23 (with wind $c = 1$ and $\ell = 2$) implies that $T(b) - \lambda I$ has a one-dimensional kernel in ℓ^∞ whose nonzero elements are extended, and if $\lambda \in \{-2, 2\}$, then Proposition 1.23 (with wind $c = 1$ and $\ell = 1$) shows that the kernel of $T(b) - \lambda I$ in ℓ^∞ is trivial. □

Example 1.29. Let $b(t) = (1 + 1/t)^3$. The image $b(\mathbf{T})$ is the solid curve in the left picture of Figure 1.2; this curve is traced out in the clockwise direction. The curve intersects itself at the point -1. We see that $\mathbf{C} \setminus b(\mathbf{T})$ has two bounded connected components Ω_1 and Ω_2 with wind $(b, \lambda) = -1$ for $\lambda \in \Omega_1$ and wind $(b, \lambda) = -2$ for $\lambda \in \Omega_2$. Thus, $\operatorname{sp} T(b) = \Omega_1 \cup \Omega_2 \cup b(\mathbf{T})$.

By Proposition 1.20, the points in $\Omega_1 \cup \Omega_2$ are eigenvalues. Looking at Figure 1.2 and using Proposition 1.24, we see that wind $c = -1$ for the points on the two small open arcs of $b(\mathbf{T})$ that join 0 and -1. Thus, by virtue of Propositions 1.22 and 1.23, these points are also eigenvalues. The points of $b(\mathbf{T})$ which are boundary points of the unbounded connected component of $\mathbf{C} \setminus b(\mathbf{T})$, including the point -1, are not eigenvalues if $1 \le p < \infty$ (Corollary 1.26), but they are eigenvalues if $p = \infty$, because wind $c = 0$ (Propositions 1.23 and 1.24). Finally, for $\lambda = 0$ we have $\ell = 1$, and the right picture of Figure 1.2 reveals that wind $c = 0$. Consequently, $\lambda = 0$ is not an eigenvalue if $1 \le p < \infty$ (Proposition 1.22) and is an eigenvalue if $p = \infty$ (Proposition 1.23). □

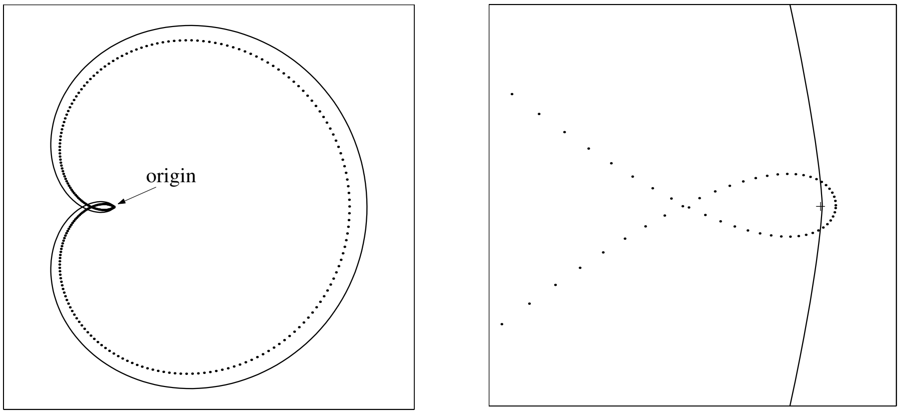


Figure 1.2. *The curves $b(\mathbf{T})$ (solid) and $b_\varrho(\mathbf{T})$ with $\varrho = 1.05$ (dotted). Both curves are traced out clockwise. The right picture is a close-up (with a magnification about 4300) of the left picture in a neighborhood of the origin, which is marked by +.*

Remark 1.30. Let b be a real-valued Laurent polynomial and suppose λ is a point in $b(\mathbf{T})$. We know from Corollary 1.27 that $\operatorname{Ker} T(b - \lambda) = \{0\}$ in ℓ^p $(1 \le p < \infty)$. This implies that if $1 < p < \infty$, then the range $\operatorname{Im} T(b - \lambda)$ is not closed but dense in ℓ^p. In other words, $T(b)$ has no residual spectrum on ℓ^p for $1 < p < \infty$. However, the polynomial $b = \chi_{-1} + \chi_1$ is an example of a symbol for which $\operatorname{Ker} T(b) \ne \{0\}$ in ℓ^∞ and thus $\operatorname{Im} T(b)$ is not dense in ℓ^1. Consequently, there are b's such that $T(b)$ has a residual spectrum on ℓ^1.

1.9 Selfadjoint Operators

We now consider Toeplitz operators on the space ℓ^2. Obviously, $T(b)$ is selfadjoint if and only if $\overline{b}_n = b_{-n}$ for all n, that is, if and only if b is real valued. Thus, let

$$b(e^{ix}) = \sum_{k=-s}^{s} b_k e^{ikx} = b_0 + \sum_{n=1}^{s} (a_n \cos nx + c_n \sin nx),$$

where b_0, a_n, c_n are real numbers.

The resolution of the identity. Let A be a bounded selfadjoint operator on the space ℓ^2. Then the operator $f(A)$ is well defined for every bounded Borel function f on $\mathbf{R}$. For $\lambda \in \mathbf{R}$, put $E(\lambda) = \chi_{(-\infty,\lambda]}(A)$, where $\chi_{(-\infty,\lambda]}$ is the characteristic function of $(-\infty, \lambda]$. The family $\{E(\lambda)\}_{\lambda\in\mathbf{R}}$ is called the *resolution of the identity* for A. Stone's formula states that

$$\begin{aligned}&\frac{1}{2}(E(\lambda+0) + E(\lambda-0))x \\ &\quad = \lim_{\varepsilon\to 0+0} \frac{1}{2\pi i} \int_{-\infty}^{\lambda} \left((A - (\lambda + i\varepsilon)I)^{-1} - (A - (\lambda - i\varepsilon)I)^{-1}\right) x \, d\lambda \end{aligned} \qquad (1.38)$$

for every $x \in \ell^2$. Let $\ell^2_{\mathrm{pp}}, \ell^2_{\mathrm{ac}}, \ell^2_{\mathrm{sing}}$ denote the set of all $x \in \ell^2$ for which the measure $d_x(\lambda) := d(E(\lambda)x, x)$ is a pure point measure, is absolutely continuous with respect to Lebesgue measure, and is singular continuous with respect to Lebesgue measure, respectively. The sets $\ell^2_{\mathrm{pp}}, \ell^2_{\mathrm{ac}}, \ell^2_{\mathrm{sing}}$ are closed subspaces of ℓ^2 whose orthogonal sum is all of ℓ^2. Moreover, each of the spaces $\ell^2_{\mathrm{pp}}, \ell^2_{\mathrm{ac}}, \ell^2_{\mathrm{sing}}$ is an invariant subspace of A. The spectra of the restrictions $A|\ell^2_{\mathrm{ac}}$ and $A|\ell^2_{\mathrm{sing}}$ are referred to as the *absolutely continuous spectrum* and the *singular continuous spectrum* of A, respectively. The *point spectrum* of A is defined as the set of the eigenvalues of A (and *not* as the spectrum of the restriction $A|\ell^2_{\mathrm{pp}}$). It is well known that the spectrum of A is the union of the absolutely continuous spectrum, the singular continuous spectrum, and the closure of the point spectrum.

The spectrum of $T(b)$ is the line segment $b(\mathbf{T}) =: [m, M]$. Corollary 1.27 tells us that the point spectrum of $T(b)$ is empty unless b is a constant. As the following theorem shows, the singular continuous spectrum is also empty.

Theorem 1.31 (Rosenblum). *The spectrum of a Toeplitz operator generated by a real-valued nonconstant Laurent polynomial is purely absolutely continuous.*

Proof. Let b be a real-valued nonconstant Laurent polynomial. We may without loss of generality assume that the highest coefficient b_s is 1. Fix $\lambda \in \mathbf{R}$ and $\varepsilon > 0$, and put $z = \lambda + i\varepsilon$. As in Section 1.4, we can write

$$b(t) - z = \prod \left(1 - \frac{\delta_j}{t}\right) \prod (t - \mu_j),$$

where $|\delta_j| = |\delta_j(z)| < 1$ and $|\mu_j| = |\mu_j(z)| > 1$. Passing to the complex conjugate, we get

$$b(t) - \overline{z} = \prod \left(1 - \frac{1}{\overline{\mu}_j t}\right) \prod \left(t - \frac{1}{\delta_j}\right).$$

Proposition 1.19 now implies that, for $\alpha \in \mathbf{D}$,

$$(T^{-1}(b-z)w_\alpha, w_\alpha) = \frac{1}{f(z)} \frac{1}{1-|\alpha|^2}, \quad (T^{-1}(b-\overline{z})w_\alpha, w_\alpha) = \frac{1}{\overline{f(z)}} \frac{1}{1-|\alpha|^2},$$

where $f(z) := \prod(1-\delta_j\alpha)\prod(\overline{\alpha}-\mu_j)$. We have

$$\left|\frac{1}{f(z)} - \frac{1}{\overline{f(z)}}\right| = 2\left|\frac{\operatorname{Im} f(z)}{|f(z)|^2}\right| \le \frac{2}{|f(z)|} = \frac{2}{\prod|1-\delta_j\alpha|\prod|\overline{\alpha}-\mu_j|} \le \frac{2}{(1-|\alpha|)^{2s}},$$

because $|1-\delta_j\alpha| \ge 1-|\alpha|$ and $|\overline{\alpha}-\mu_j| \ge 1-|\alpha|$ for all j. Since $E(\lambda-0) = E(\lambda+0)$, formula (1.38) gives

$$|(E(\lambda_2)w_\alpha, w_\alpha) - (E(\lambda_1)w_\alpha, w_\alpha)| \le \frac{1}{2\pi}\int_{\lambda_1}^{\lambda_2} \frac{2}{(1-|\alpha|)^{2s}}\, d\lambda,$$

which shows that the function $\lambda \mapsto (E(\lambda)w_\alpha, w_\alpha)$ is absolutely continuous for each $\alpha \in \mathbf{D}$. It follows that $w_\alpha \in \ell^2_{\rm ac}$ for each $\alpha \in \mathbf{D}$, and as the linear hull of the set $\{w_\alpha\}_{\alpha\in\mathbf{D}}$ is dense in ℓ^2, we arrive at the conclusion that $\ell^2_{\rm ac} = \ell^2$, which is the assertion. □

The problem of diagonalizing selfadjoint bounded Toeplitz operators is solved. More or less explicit formulas can be found in [227], [228], [229], and [288]. We here confine ourselves to a few simple observations.

Chebyshev polynomials. We denote by $\{U_n\}_{n=0}^\infty$ the normalized Chebyshev polynomials of the second kind:

$$U_n(\cos\theta) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(n+1)\theta}{\sin\theta}.$$

The polynomials $\{U_n\}_{n=0}^\infty$ constitute an orthonormal basis in the Hilbert space $L^2((-1,1), \sqrt{1-\lambda^2}) =: L^2(\sigma)$,

$$\int_{-1}^{1} U_j(\lambda)U_k(\lambda)\sqrt{1-\lambda^2}\, d\lambda = \delta_{jk},$$

and they satisfy the identities

$$\lambda U_n(\lambda) = \frac{1}{2}\,U_{n+1}(\lambda) + \frac{1}{2}\,U_{n-1}(\lambda), \quad U_{-1}(\lambda) := 0. \tag{1.39}$$

For $\alpha \in \mathbf{T}$, we define $V_\alpha : \ell^2 \to L^2(\sigma)$ by

$$(V_\alpha x)(\lambda) = \sum_{n=0}^{\infty} x_n \alpha^n U_n(\lambda), \quad \lambda \in (-1,1).$$

Clearly, V_α is unitary and $V_\alpha^{-1} : L^2(\sigma) \to \ell^2$ acts by the rule

$$(V_\alpha^{-1} f)_n = \frac{1}{\alpha^n} \int_{-1}^{1} f(\lambda) U_n(\lambda) \sqrt{1-\lambda^2}\, d\lambda, \quad n \ge 0.$$

We denote by $M_{f(\lambda)}$ the operator of multiplication by the function $f(\lambda)$ on $L^2(\sigma)$.

Proposition 1.32. *Let*

$$b(e^{ix}) = \overline{b}_1 e^{-ix} + b_0 + b_1 e^{ix} = b_0 + a_1 \cos x + c_1 \sin x$$

be a real valued trinomial. Put

$$\alpha = \sqrt{\frac{\overline{b}_1}{b_1}} = \sqrt{\frac{a_1 + ic_1}{a_1 - ic_1}}, \quad \beta = 2|b_1| = \frac{1}{2}\sqrt{a_1^2 + c_1^2}.$$

Then $T(b) = V_\alpha^{-1} M_{b_0 + \beta\lambda} V_\alpha$.

Proof. Using (1.39) and the orthonormality of the polynomials U_n we obtain

$$\begin{aligned}
(V_\alpha^{-1} M_\lambda V_\alpha x)_n &= \frac{1}{\alpha^n} \int_{-1}^{1} \lambda (V_\alpha x)(\lambda) U_n(\lambda) \sqrt{1-\lambda^2}\, d\lambda \\
&= \frac{1}{\alpha^n} \sum_{k=0}^{\infty} x_k \alpha^k \int_{-1}^{1} \lambda U_k(\lambda) U_n(\lambda) \sqrt{1-\lambda^2}\, d\lambda \\
&= \frac{1}{\alpha^n} \sum_{k=0}^{\infty} x_k \alpha^k \int_{-1}^{1} \left(\frac{1}{2} U_k(\lambda) U_{n+1}(\lambda) + \frac{1}{2} U_k(\lambda) U_{n-1}(\lambda) \right) \sqrt{1-\lambda^2}\, d\lambda \\
&= \frac{1}{\alpha^n} \left(x_{n+1} \frac{\alpha^{n+1}}{2} + x_{n-1} \frac{\alpha^{n-1}}{2} \right) = \frac{\alpha}{2} x_{n+1} + \frac{1}{2\alpha} x_{n-1},
\end{aligned}$$

where $x_{-1} := 0$. Equivalently,

$$V_\alpha^{-1} M_\lambda V_\alpha = T\left(\frac{\alpha}{2} \chi_{-1} + \frac{1}{2\alpha} \chi_1 \right).$$

This implies that $V_\alpha^{-1} M_{b_0 + 2|b_1|\lambda} V_\alpha$ is the Toeplitz operator with the symbol

$$b_0 + \sqrt{\frac{\overline{b}_1}{b_1}}\, |b_1| \chi_{-1} + \sqrt{\frac{b_1}{\overline{b}_1}}\, |b_1| \chi_1 = b_0 + \overline{b}_1 \chi_{-1} + b_1 \chi_1. \qquad \square$$

In particular,

$$T(\cos x) := T\left(\frac{1}{2} \chi_{-1} + \frac{1}{2} \chi_1 \right) = V_1^{-1} M_\lambda V_1, \tag{1.40}$$

$$T(\sin x) := T\left(\frac{i}{2} \chi_1 - \frac{i}{2} \chi_{-1} \right) = V_i^{-1} M_\lambda V_i. \tag{1.41}$$

Diagonalization of symmetric and skewsymmetric Toeplitz matrices. The polynomials

$$g(x) = b_0 + \sum_{n=1}^{s} a_n \cos nx \quad \text{and} \quad u(x) = \sum_{n=1}^{s} c_n \sin nx$$

generate symmetric ($A^\top = A$) and skewsymmetric ($A^\top = -A$) Toeplitz matrices, respectively. From identities (1.39) we infer that

$$\lambda^2 U_n(\lambda) = \frac{1}{4}U_{n+2}(\lambda) + \frac{1}{2}U_n(\lambda) + \frac{1}{4}U_{n-2}(\lambda),$$
$$\lambda^3 U_n(\lambda) = \frac{1}{8}U_{n+3}(\lambda) + \frac{3}{8}U_{n+1}(\lambda) + \frac{3}{8}U_{n-1}(\lambda) + \frac{1}{8}U_{n-3}(\lambda),$$

and so on, where $U_{-2}(\lambda) = U_{-3}(\lambda) = \cdots = 0$. Consequently, as in the proof of Proposition 1.32,

$$T\left(\frac{1}{2} + \frac{1}{4}\cos 2x\right) = T\left(\frac{1}{4}\chi_{-2} + \frac{1}{2}\chi_0 + \frac{1}{4}\chi_2\right) = V_1^{-1} M_{\lambda^2} V_1,$$
$$T\left(\frac{3}{8}\cos x + \frac{1}{8}\cos 3x\right) = T\left(\frac{1}{8}\chi_{-3} + \frac{3}{8}\chi_{-1} + \frac{3}{8}\chi_1 + \frac{1}{8}\chi_3\right) = V_1^{-1} M_{\lambda^3} V_1,$$

etc. This shows that we can find coefficients $\gamma_0, \gamma_1, \ldots, \gamma_s$ such that

$$T\left(b_0 + \sum_{n=1}^{s} a_n \cos nx\right) = V_1^{-1} M_{\gamma_0 + \gamma_1\lambda + \cdots + \gamma_s \lambda^s} V_1.$$

One can diagonalize the skewsymmetric matrices $T(u)$ analogously.

Resolution of the identity for Toeplitz operators. Let A be a bounded selfadjoint operator on ℓ^2 and suppose we have a unitary operator V such that VAV^{-1} is multiplication by λ on $L^2(\sigma) := L^2((-1,1), \sqrt{1-\lambda^2})$. Then the resolution of the identity for A can be computed from the formula $E(\lambda) = VM_{\chi_{(-\infty,\lambda]}}V^{-1}$. Clearly, we can think of $E(\lambda)$ as an infinite matrix $(E_{jk}(\lambda))_{j,k=0}^{\infty}$.

Proposition 1.33. *The resolution of the identity for* $T(\cos x) = T\left(\frac{1}{2}\chi_{-1} + \frac{1}{2}\chi_1\right)$ *is given by* $E(\lambda) = 0$ *for* $\lambda \in (-\infty, -1]$, $E(\lambda) = I$ *for* $\lambda \in [1, \infty)$, *and*

$$E_{jk}(\lambda) = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{\sin(j+k+2)\theta}{j+k+2} - \dfrac{\sin(j-k)\theta}{j-k}\right) & \text{if} \quad j \neq k \\ \dfrac{1}{\pi}\left(\dfrac{\sin(2j+2)\theta}{2j+2} - \theta + \pi\right) & \text{if} \quad j = k \end{cases}$$

with $\theta = \arccos\lambda$ *for* $\lambda \in (-1, 1)$.

Proof. It suffices to consider λ in $(-1, 1)$. Let e_n be the nth element of the standard basis of ℓ^2. By virtue of (1.40),

$$\begin{aligned} E_{jk}(\lambda) &= (E(\lambda)e_k, e_j) = (V_1^{-1} M_{\chi(-\infty,\lambda)} V_1 e_k, e_j) \\ &= (M_{\chi(-\infty,\lambda)} V_1 e_k, V_1 e_j) = (M_{\chi(-\infty,\lambda)} U_k, U_j) \\ &= \int_{-1}^{\lambda} U_k(\mu) U_j(\mu) \sqrt{1-\mu^2}\, d\mu \\ &= \frac{2}{\pi} \int_{\theta}^{\pi} \frac{\sin(k+1)\varphi}{\sin\varphi} \frac{\sin(j+1)\varphi}{\sin\varphi} \sin^2\varphi\, d\varphi \\ &= \frac{1}{\pi} \int_{\theta}^{\pi} [\cos(j-k)\varphi - \cos(j+k+2)\varphi]\, d\varphi, \end{aligned}$$

which implies the assertion. □

From (1.40) we also deduce that if f is any continuous function on $[-1, 1]$, then the j, k entry of $f(T(\cos x))$ is

$$\begin{aligned} [f(T(\cos x))]_{jk} &= (V_1^{-1} M_{f(\lambda)} V_1 e_k, e_j) = (M_{f(\lambda)} U_k, U_j) \\ &= \int_{-1}^{1} f(\lambda) U_k(\lambda) U_j(\lambda) \sqrt{1-\lambda^2}\, d\lambda \\ &= \frac{2}{\pi} \int_{\theta}^{\pi} f(\cos\theta) \frac{\sin(k+1)\theta}{\sin\theta} \frac{\sin(j+1)\theta}{\sin\theta} \sin^2\theta\, d\theta \\ &= \frac{1}{\pi} \int_{\theta}^{\pi} f(\cos\theta)[\cos(j-k)\theta - \cos(j+k+2)\theta]\, d\theta. \end{aligned}$$

For example, the nonnegative square root of

$$T(2-2\cos x) = \begin{pmatrix} 2 & -1 & 0 & \dots \\ -1 & 2 & -1 & \dots \\ 0 & -1 & 2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

has j, k entry

$$\begin{aligned} &\frac{1}{\pi} \int_{\theta}^{\pi} \sqrt{2-2\cos\theta}\, [\cos(j-k)\theta - \cos(j+k+2)\theta]\, d\theta \\ &= \frac{2}{\pi} \int_{\theta}^{\pi} \left(\sin\frac{\theta}{2}\right) [\cos(j-k)\theta - \cos(j+k+2)\theta]\, d\theta \\ &= \frac{1}{\pi} \int_{-1}^{1} \left[\sin\left(j-k+\frac{1}{2}\right)\theta - \sin\left(j-k-\frac{1}{2}\right)\theta \right. \\ &\qquad \left. - \sin\left(j+k+2+\frac{1}{2}\right)\theta + \sin\left(j+k+2-\frac{1}{2}\right)\theta\right] d\theta \\ &= \frac{1}{\pi}\left(\frac{1}{j-k+\frac{1}{2}} - \frac{1}{j-k-\frac{1}{2}} + \frac{1}{j+k+2+\frac{1}{2}} - \frac{1}{j+k+2-\frac{1}{2}}\right), \end{aligned}$$

and this equals

$$\frac{4}{\pi}\left(\frac{1}{4(j+k+2)^2+1} - \frac{1}{4(j-k)^2+1}\right).$$

Exercises

1. (a) Find a function $a \in L^\infty(\mathbf{T})$ whose Fourier coefficients a_n ($n \in \mathbf{Z}$) are just $a_n = 1/(n+1/2)$.

   (b) Show that the infinite Toeplitz matrix

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} & \dots \\ \frac{1}{2} & 1 & -\frac{1}{2} & \dots \\ \frac{1}{3} & \frac{1}{2} & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

   induces a bounded operator on ℓ^2 but not on ℓ^1.

   (c) Show that the infinite Toeplitz matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ \frac{1}{2} & 1 & \frac{1}{2} & \dots \\ \frac{1}{3} & \frac{1}{2} & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

   does not generate a bounded operator on ℓ^2.

   (d) Show that the infinite upper-triangular Toeplitz matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ & 1 & \frac{1}{2} & \dots \\ & & 1 & \dots \\ & & & \dots \end{pmatrix}$$

   does not define a bounded operator on ℓ^2 but that the infinite Hankel matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ \frac{1}{2} & \frac{1}{3} & \dots & \dots \\ \frac{1}{3} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

   is the matrix of a bounded operator on ℓ^2.

2. Prove that $H(ab) = H(a)\,T(\tilde{b}) + T(a)\,H(b)$ for all $a, b \in W$.

3. Find a Wiener-Hopf factorization of $6t - 41 + 31t^{-1} - 6t^{-2}$.

4. Let $b_1, \dots, b_m \in \mathcal{P}^+$ have no common zero on $\mathbf{T}$. Prove that there are $c_1, \dots, c_m \in W$ such that $T(c_1)T(b_1) + \cdots + T(c_m)T(b_m) = I$. Can one choose the $c_1, \dots, c_m$ as rational functions without poles on $\mathbf{T}$?

5. Let $b(t) = 1 + 2t + \gamma t^3$. Show that there is no $\gamma \in \mathbf{C}$ for which $T(b)$ is invertible.

6. Let

$$b(t) = \det \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & t & t^2 & t^3 & t^4 \end{pmatrix}.$$

Show that b has no zeros on $\mathbf{T}$ and that wind $b = 4$. Try to prove the analogue of this if the 5×5 determinant is replaced by an $n \times n$ determinant in the obvious way.

7. Let $b(t) = 4 + \sum_{j=-5}^{5} t^j$. Show that $T(b)$ is invertible.

8. Let b be a Laurent polynomial and $1 \le p \le \infty$. Show that $T(b) : \ell^p \to \ell^p$ has closed range if and only if either b is identically zero or b has no zeros on $\mathbf{T}$.

9. Let $b_n(t) = 1 + \frac{1}{2}(t + t^{-1}) + \frac{1}{3}(t^2 + t^{-2}) + \cdots + \frac{1}{n}(t^n + t^{-n})$. Show that

$$2 \log n + 0.1544 \le \|T(b_n)\|_4 \le 2 \log n + 0.1545$$

for all sufficiently large n.

10. Prove that $\|T(a) + K\|_p \ge \|T(a)\|_p$ for every $a \in W$ and every compact operator on ℓ^p ($1 \le p \le \infty$). Deduce that the zero operator is the only compact Toeplitz operator.

11. Prove that $\|T^n(a)\|_2 = \|T(a^n)\|_2$ for every $a \in W$.

12. Show that there exist Laurent polynomials b such that $\|H(b)\|_2 < \|b\|_\infty$ and $\|H(b)\|_1 < \|b\|_W$.

13. For $b = \sum_j b_j \chi_j \in \mathcal{P}$, define

$$S_n b = \sum_{|j| \le n-1} b_j \chi_j, \quad \sigma_n b = \frac{1}{n}(S_1 b + \cdots + S_n b).$$

Prove that always $\|T(\sigma_n b)\|_2 \le \|T(b)\|_2$ but that there exist b and n such that $\|T(S_n b)\|_2 > 100\,\|T(b)\|_2$.

14. Show that if $a \in W$ and $T(a)$ is a unitary operator on ℓ^2, then a is a unimodular constant.

15. Let $B = (b_{jk})_{j,k=1}^\infty$ with $b_{23} = b_{32} = -1$ and $b_{jk} = 0$ otherwise. Show that the operator $T(\chi_{-1} + \chi_1) + B \in \mathcal{B}(\ell^p)$ ($1 \le p \le \infty$) has eigenvalues in $(-2, 2)$.

16. Let $A = T(\chi_{-1} + \chi_1) + \operatorname{diag}(v_j)_{j=1}^\infty$. Show that if $v_j = o(1/j)$, then $A \in \mathcal{B}(\ell^2)$ has at most finitely many eigenvalues in each segment $[\alpha, \beta] \subset (-2, 2)$ and that if $v_j = o(1/j^{1+\varepsilon})$ with some $\varepsilon > 0$, then the only possible eigenvalues of $A \in \mathcal{B}(\ell^2)$ are -2 and 2.

17. Show that the positive square root of $T(2+2\cos x)$ is the Toeplitz-plus-Hankel matrix

$$\frac{4}{\pi}\left(\frac{(-1)^{j-k+1}}{(2j-2k-2)(2j-2k+1)}+\frac{(-1)^{j+k+2}}{(2j+2k+3)(2j+2k+5)}\right)_{j,k=0}^{\infty}.$$

Notes

In his 1911 paper [267], Otto Toeplitz considered doubly infinite matrices of the form $(a_{j-k})_{j=-\infty}^{\infty}$ and proved that the spectrum of the corresponding operator on $\ell^2(\mathbf{Z})$ is just the curve

$$\left\{\sum_{k=-\infty}^{\infty} a_k t^k : t \in \mathbf{T}\right\}.$$

The matrices $L(a) := (a_{j-k})_{j=-\infty}^{\infty}$ are nowadays called Laurent matrices. In a footnote of [267], Toeplitz established that the simply infinite matrix $(a_{j-k})_{j=0}^{\infty}$ induces a bounded operator on $\ell^2(\mathbf{Z}_+)$ if *and only if* the doubly infinite matrix $(a_{j-k})_{j=-\infty}^{\infty}$ generates a bounded operator on $\ell^2(\mathbf{Z})$. This is why the matrices $(a_{j-k})_{j=0}^{\infty}$ now bear his name.

The material of Sections 1.1 to 1.7 is standard. The books [71] and [130] may serve as introductions to the basic phenomena in connection with infinite Toeplitz matrices. A nice source is also [150]. In [25], infinite systems with a banded Toeplitz matrix $T(a)$ are treated with the tools of the theory of difference equations; in this book, we find formulas for the entries of the inverses in terms of the zeros of $a(z)$ ($z \in \mathbf{C}$) and solvability criteria in the spaces of sequences $x = \{x_n\}_{n=1}^{\infty}$ subject to the condition $x_n = O(\varrho^n)$. Advanced topics in the theory of infinite Toeplitz matrices (= Toeplitz operators) are treated in the monographs [70], [103], [195], [196]. The standard texts on Hankel matrices are [196], [201], [204], [213].

Full proofs of Theorems 1.5, 1.6, 1.7 can be found in [103] or [230], for example. Theorem 1.8 as it is stated is due to Mark Krein [184]. The method of Wiener-Hopf factorization was introduced by N. Wiener and E. Hopf in 1931. What we call Wiener-Hopf factorization has its origin in the work of Gakhov [123], although the basic idea (in the case of vanishing winding number) was already employed by Plemelj [205]. Mark Krein [184] was the first to understand the operator theoretic essence and the Banach algebraic background of Wiener-Hopf factorization and to present the method in a crystal-clear manner.

The results of Section 1.5 are also due to Krein [184]. However, it had been known a long time before that $T(a)$ is Fredholm of index $-\text{wind}\, a$ whenever a has no zeros on $\mathbf{T}$; this insight is more or less explicit in works by F. Noether, S. G. Mikhlin, N. I. Muskhelishvili, F. D. Gakhov, V. V. Ivanov, A. P. Calderón, F. Spitzer, H. Widom, A. Devinatz, G. Fichera, and certainly others. Moreover, in 1952, Israel Gohberg [128] had already proved that $T(a)$ is Fredholm if *and only if* a has no zeros on $\mathbf{T}$. From this result it is only a small step (from the present-day understanding of the matter) to the formula $\text{Ind}\; T(a) = -\text{wind}\; a$.

Section 1.8 is based on known results of [129], [130], [184].

Rosenblum's papers [226], [227], [228] are the classics on selfadjoint Toeplitz operators. The monograph [229] contains very readable material on the topic. In these works one can also find precise references to previous work on selfadjoint Toeplitz operators. For example, in [229] it is pointed out that the diagonalization (1.40) was carried out by Hilbert (1912) and Hellinger (1941). Proposition 1.19 is from [81] and [226] and Theorem 1.31 was established in [226]. The results around Proposition 1.32 are special cases of more general results in [227], [228]. Part of Rosenblum's theory was simplified and generalized by Vreugdenhil [288]. We took Proposition 1.33 and the example following after it from [288].

Exercises 5 and 6 are from [208]. Exercises 15 and 16 are results of the papers [182], [183]. Actually, these two papers are devoted to the following more general problem: If λ is not an eigenvalue for $T(b) \in \mathcal{B}(\ell^p)$, for which perturbations $B \in \mathcal{B}(\ell^p)$ is λ not an eigenvalue of $T(b)+B$? In [183] it is in particular proved that if $B = (b_{jk})_{j,k=1}^{\infty}$ is such that $b_{jk} = 0$ for $j > k$ and $(j^{1+\varepsilon} b_{jk})_{j,k=1}^{\infty}$ induces a bounded operator on ℓ^p $(1 \le p < \infty)$, then the interval $(-2, 2)$ contains no eigevalues of $T(\chi_{-1} + \chi_1) + B$. As Exercise 15 shows, the requirement that B be upper-triangular is essential. A solution to Exercise 17 is in [288].