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From "Approximation of Large-Scale Dynamical Systems" by Athanasios C. Antoulas.
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Foreword

Lectori Salutem.

This book deals with important problems and results on the interface between three central areas of (applied) mathematics: linear algebra, numerical analysis, and system theory. Whereas the first two of these fields have had a very fruitful symbiosis for a long time in the form of numerical linear algebra, the third area has remained somewhat distant from all this, until recently. The book in front of you is an example of this rapprochement.

At center stage in this book, we find the problem of model reduction. This problem can be formulated as follows. Given a linear time-invariant input/output system defined by the convolution integral, a transfer function, a state space representation, or their discrete-time counterparts, approximate this system by a simpler system.

As all approximation questions, this problem can be viewed as a trade-off between complexity and misfit. We are looking for a system of minimal complexity that approximates the original one (optimally) with a maximal allowed misfit, or, conversely, we are looking for a system that approximates the original one with minimal misfit within the class of systems with maximal admissible complexity.

A number of key questions arise:

1. What is meant by the complexity of a linear time-invariant system?
2. How is the complexity computed from the impulse response matrix, the transfer function, or from another system representation?
3. What is meant by the misfit between two systems?
4. How is this misfit computed?
5. What are the algorithms that compute optimal approximants?

The complexity issue leads to the theory of state space representations, commonly called realization theory. This theory, originally championed in the work of Kalman, is one of the most beautiful and useful parts of system theory.

The misfit issue leads to an in-depth discussion of matrix, operator, and system norms. Two important system norms emerge: the $L_2$-induced, or $H_\infty$-norm, and the Hankel-norm. There are numerous inequalities relating matrix norms, and some of these extend to system norms. Perhaps the most remarkable of these relations is the inequality that bounds the $H_\infty$-norm of a system by twice the sum (without repetition) of its Hankel singular values.

From "Approximation of Large-Scale Dynamical Systems" by Athanasios C. Antoulas.
The issue of system approximation centers around the singular values of the associated Hankel operator. Two effective approximation algorithms, both related to these Hankel singular values, are the following:

(i) Approximation by balancing,

(ii) AAK model reduction.

Model reduction by balancing is based on a very elegant method of finding a state space representation of a system with state components that are, so to speak, equally controllable as observable. This reduction method is heuristic, but it is shown that the resulting reduced system has very good properties. AAK model reduction is based on a remarkable result of Arov, Adamjan, and Krein, three Russian mathematicians who proved that a Hankel operator can be approximated equally well within the class of Hankel operators as in the class of general linear operators. While neither balancing nor AAK offers optimal reductions in the all-important $H_\infty$-norm, very nice inequalities bounding the $H_\infty$ approximation error can be derived.

Unfortunately, these singular value oriented methods are, computationally, rather complex, requiring of the order of $n^3$ operations, where $n$ denotes the dimension of the state space of the system to be approximated. Together with accuracy considerations, this makes these methods applicable to systems of only modest dimension (a few hundred variables).

This book also introduces a second set of approximation methods based on moment matching. In system theory language, this moment matching is a generalization of the well-known partial realization problem. These methods can be iteratively implemented using standard algorithms from numerical linear algebra, namely, the Krylov iterative methods. These schemes were originally developed for computing eigenvalues and eigenvectors, but can be applied to model reduction via moment matching. Typically, these methods require only of the order of $n^2$ operations. Their disadvantage, however, is that stability of the reduced model is not guaranteed, and there is no known global error bound.

This brings us to the last part of the book, which aims at combining the singular value decomposition (SVD) based methods and the Krylov methods into what are called SVD-Krylov methods.

The SVD-based approach can be extended to nonlinear systems. The resulting method is known as POD (proper orthogonal decomposition) and is widely used by the PDE community.

The scope of this book is the complete theory of primarily linear system approximation. Special attention is paid to numerical aspects, simulation questions, and practical applications. It is hard to overestimate the importance of the theory presented in this book. I believe that its impact (for example, for numerical simulation of PDEs) has not yet been fully achieved. The mathematical ideas underlying the interplay of the SVD and linear system theory are of the most refined mathematical ideas in the field of system theory.

The book in front of you is unique in its coverage and promises to be a stimulating experience to everyone interested in mathematics and its relevance to practical problems.

Jan C. Willems
Leuven, May 3, 2003

From "Approximation of Large-Scale Dynamical Systems" by Athanasios C. Antoulas.

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Preface

In today's technological world, physical and artificial processes are mainly described by mathematical models, which can be used for simulation or control. These processes are dynamical systems, as their future behavior depends on their past evolution. The weather and very large scale integration (VLSI) circuits are examples, the former physical and the latter artificial. In simulation (control) one seeks to predict (modify) the system behavior; however, simulation of the full model is often not feasible, necessitating simplification of it. Due to limited computational, accuracy, and storage capabilities, system approximation—the development of simplified models that capture the main features of the original dynamical systems—evolved. Simplified models are used in place of original complex models and result in simulation (control) with reduced computational complexity. This book deals with what may be called the curse of complexity, by addressing the approximation of dynamical systems described by a finite set of differential or difference equations together with a finite set of algebraic equations. Our goal is to present approximation methods related to the singular value decomposition (SVD), to Krylov or moment matching methods, and to combinations thereof, referred to as SVD-Krylov methods.

Part I addresses the above in more detail. Part II is devoted to a review of the necessary mathematical and system theoretic prerequisites. In particular, norms of vectors and (finite) matrices are introduced in Chapter 3, together with a detailed discussion of the SVD of matrices. The approximation problem in the induced 2-norm and its solution given by the Schmidt–Eckart–Young–Mirsy theorem are tackled next. This result is generalized to linear dynamical systems in Chapter 8, which covers Hankel-norm approximation. Elements of numerical linear algebra are also presented in Chapter 3. Chapter 4 presents some basic concepts from linear system theory. Its first section discusses the external description of linear systems in terms of convolution integrals or convolution sums. The section following treats the internal description of linear systems. This is a representation in terms of first-order ordinary differential or difference equations, depending on whether we are dealing with continuous- or discrete-time systems. The associated structural concepts of reachability and observability are analyzed. Gramians, which are important tools for system approximation, are introduced in this chapter and their properties are explored. The last section of Chapter 4 is concerned with the relationship between internal and external descriptions, which is known as the realization problem. Finally, aspects of the more general problem of rational interpolation are displayed.

Chapter 5 introduces various norms of linear systems that are essential for system approximation and for the quantification of approximation errors. The Hankel operator is introduced, and its eigenvalues and singular values, together with those of the convolution operator.
operator, are computed. This leads to the concept of Hankel singular values and of the 8-
norm of a system. After a more general discussion on induced norms of linear systems, we
turn our attention to a brief review of system stability, followed by a discussion of 2-systems
and all-pass 2-systems, which play an important role in Hankel-norm system approximation.
The concept of dissipativity, which generalizes that of stability from autonomous systems
to systems with external influences, is introduced; the special cases of bounded real and
positive real systems are briefly explored. Chapter 6 is devoted to the study of two linear
matrix equations, the Sylvester equation and the closely related Lyapunov equation. These
equations are central to SVD-based methods, and various solution methods (from complex
integration to the Cayley–Hamilton theorem to the sign function method) are discussed.
Next, the inertia theorem for the Lyapunov equation is investigated. The chapter concludes
with numerically reliable algorithms for the solution of these equations.

Following this presentation of the preparatory material, Part III commences with the
exposition of the first class of approximation methods, namely, SVD-based approximation
methods. Chapter 7 is devoted to approximation by balanced truncation. The ingredients
are Lyapunov equations and the Hankel singular values. The main result is followed by
a canonical form that can be applied to balanced systems. The last part of the chapter is
involved with special types of balancing, including bounded real, positive real, frequency
weighted balancing, which lead to methods for approximating unstable systems.

Chapter 8 presents the theory of optimal and suboptimal approximation in the induced
2-norm of the Hankel operator, which can be viewed as a refinement of approximation
by balanced truncation. The final section of this chapter is devoted to the exposition of
a polynomial approach that offers new insights into and connections between balancing
and Hankel-norm approximation. Part III concludes with a chapter dedicated to special
topics in SVD-based approximation methods. In this context, a brief description of the
proper orthogonal decomposition method is given in section 9.1; its relation with balanced
truncation is also mentioned. Approximation by modal truncation is discussed next. The
latter part of the chapter is dedicated to a study of the decay rates of the Hankel singular
values, which is of importance in predicting how well a given system can be approximated
by a low-order system.

Part IV is concerned with Krylov-based approximation methods. These methods have
their roots in numerical linear algebra and address the problem of providing good estimates
for a few eigenvalues of a big matrix. Consequently, Chapter 10 gives an account of Lanc-
zos and Arnoldi methods as they apply to eigenvalue problems. Chapter 11 discusses the
application of these methods to system approximation. Krylov methods lead to approxi-
mants by matching moments. These methods turn out to be numerically efficient, although
they lack other important properties, such as quantification of the approximation error. The
connection with rational interpolation is discussed in some detail.

The final section, Part V, is concerned with the connections between SVD-based and
Krylov-based approximation methods. In particular, a method that involves least squares
combines attributes of both approaches. Furthermore, two iterative methods are presented
that provide approximate solutions to Lyapunov equations and therefore can be used to
obtain reduced-order systems that are approximately balanced. In Chapter 13, aspects of
the approximation methods presented earlier are illustrated by means of numerical exper-
iments, by applying them to various systems. Algorithms are compared in terms of both
approximation error and computational effort. The book concludes with a chapter on pro-

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jections, computational complexity, software, and open problems, followed by a collection
of exercises (Chapter 15) appropriate for classroom use.

At this stage we would like to point to several related books: Obinata and Anderson [252] (model reduction for control system design, in particular, controller reduction); Gawronksi [136] (model reduction of flexible structures); Fortuna, Nunnari, and Gallo [115] (model reduction with applications in electrical engineering); Datta [91] (comprehensive treatment of numerical issues in systems and control); Berkooz, Holmes, and Lumley [63] (dynamical systems described by partial differential equations); Banks [42] (control and estimation for distributed parameter systems); and Zhou et al. [370], [371] (comprehensive treatment of systems and control with emphasis on robustness issues). See also the special issue of *Control Systems Magazine* [347] on numerical awareness in control as well the collection of reprints [262] on numerical linear algebra and control. Insights into aspects mentioned earlier but not discussed further in this book can be found in the book by Skelton, Grigoriadis, and Iwasaki [299] (controller complexity) and in the surveys by Tempo and Dabbene [323] (randomized algorithms) and Blondel and Tsitsiklis [66] (complexity of algorithms in system theory).

There are numerous individuals without whose help this book would not have been completed. First, I would like to thank Jan Willems for his friendship and inspiration for more than a quarter-century. I would also like to thank Paul van Dooren for his advice on the book while I was visiting Louvain-la-Neuve—but I am even more thankful to him for introducing me to Dan Sorensen. Dan took it on himself to teach me numerical linear algebra, and not some watered-down version but the real deal. Meeting Dan was an event that added a new dimension to my research. Dan is also acknowledged for contributing part of the section on the decay rates. Many thanks go to Mark Embree for numerous discussions on pseudospectra and for his very careful reading and substantial comments on several chapters. The next recipient of my gratitude is Paolo Rapisarda, who was always willing to read critically and provide invaluable advice over extensive portions of the book. I would also like to thank Angelika Bunse-Gerstner, Peter Benner, and Caroline Boss for their comments. Next, my acknowledgements go to Yutaka Yamamoto, longtime friend and colleague from our years as graduate students under R. E. Kalman. Thanks also go to Brian Anderson, Alessandro Astolfi, Siep Weiland, Roberto Tempo, Matthias Heinikenschloss, Yunkai Zhou, Michael Hinze, and Stefan Volkwein for reading and commenting on various parts of the book. I would also like to thank the anonymous referees and several students who commented on the book at various stages. Special thanks go to Serkan Gugercin, who contributed in many ways over the last 3 years and to whom most numerical experiments and figures, as well as part of the last two chapters, are due. Finally, I would like to thank the editors at SIAM for their professional and efficient handling of this project.

_Athanasios C. Antoulas_
How to Use This Book

• A first course in model reduction would consist of the following sections:
  – Chapter 3, sections 3.1, 3.2.1–3.2.5, 3.3
  – Chapter 4, sections 4.1, 4.2, 4.3
  – Chapter 5, sections 5.1, 5.2, 5.3, 5.4, 5.5, 5.8.1, 5.8.2
  – Chapter 6
  – Chapter 7, sections 7.1, 7.2, 7.3
  – Chapter 10, sections 10.1, 10.3, 10.4
  – Chapter 11, sections 11.1, 11.2

• Prerequisites

  The most important prerequisite is familiarity with linear algebra. Knowledge of elementary system theory and numerical analysis is desirable.

  The target readership consists of graduate students and researchers in the fields of system and control theory, numerical analysis, theory of partial differential equations and computational fluid dynamics and anyone interested in model reduction.

• Sections omitted at first reading

  Sections marked with an asterisk contain material that can be omitted at first reading.
• Notation

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<tr>
<td>$\mathbb{R}$</td>
<td>real numbers (positive, negative)</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>complex numbers (with positive, negative real part)</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>integers (positive, negative)</td>
</tr>
<tr>
<td>$\lambda_{\text{max}}(M)$</td>
<td>largest (in magnitude) eigenvalue of $M \in \mathbb{R}^{n \times n}$</td>
</tr>
<tr>
<td>$\delta_{\text{max}}(M)$</td>
<td>largest (in magnitude) diagonal entry of $M \in \mathbb{R}^{n \times n}$</td>
</tr>
<tr>
<td>$\text{adj} M$</td>
<td>adjoint (matrix of cofactors) of $M \in \mathbb{R}^{n \times n}$</td>
</tr>
<tr>
<td>$\chi_M(s)$</td>
<td>characteristic polynomial of $M \in \mathbb{R}^{n \times n}$</td>
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<tr>
<td>$\text{rank } M$</td>
<td>rank of $M \in \mathbb{R}^{n \times m}$</td>
</tr>
<tr>
<td>$\sigma_i(M)$</td>
<td>$i$th singular value of $M \in \mathbb{R}^{n \times m}$</td>
</tr>
<tr>
<td>$\kappa(M)$</td>
<td>condition number of $M \in \mathbb{R}^{n \times m}$</td>
</tr>
<tr>
<td>$M^*$</td>
<td>transpose if $M \in \mathbb{R}^{n \times m}$</td>
</tr>
<tr>
<td>$M^\dagger$</td>
<td>complex conjugate transpose if $M \in \mathbb{C}^{n \times m}$</td>
</tr>
<tr>
<td>$M(:,j)$</td>
<td>$j$th column of $M \in \mathbb{R}^{n \times m}$</td>
</tr>
<tr>
<td>$M(i, :)$</td>
<td>$i$th row of $M \in \mathbb{R}^{n \times m}$</td>
</tr>
<tr>
<td>$\text{spec}(M)$</td>
<td>spectrum (set of eigenvalues) of $M \in \mathbb{R}^{n \times n}$</td>
</tr>
<tr>
<td>$\text{spec}_\epsilon(M)$</td>
<td>$\epsilon$-pseudo-spectrum of $M \in \mathbb{R}^{n \times n}$</td>
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<td>$\text{in}(A)$</td>
<td>inertia of $A \in \mathbb{R}^{n \times n}$ (6.14)</td>
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<td>general dynamical system (1.1)</td>
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<td>$U, X, Y$</td>
<td>input, state, output spaces (1.2)</td>
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<td>linear system matrices (1.4), (4.13)</td>
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<td>$\Sigma = \begin{pmatrix} A &amp; B \ &amp; \end{pmatrix}$</td>
<td>linear system with missing $C$ and $D$ section 4.2.1</td>
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<td>$\hat{\Sigma} = \begin{pmatrix} \hat{A} &amp; \hat{B} \ \hat{C} &amp; \end{pmatrix}$</td>
<td>system matrices of linear reduced-order system (1.8)</td>
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<td>$\mathcal{S}$</td>
<td>convolution operator (discrete and continuous time) (4.16)</td>
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<td>$\mathcal{S}^\dagger$</td>
<td>adjoint of convolution operator</td>
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<td>$|x|_p$</td>
<td>$p$-norm of $x \in \mathbb{R}^n$ (3.2)</td>
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<td>$|A|_{p,q}$</td>
<td>$(p, q)$-induced norm of $A$ (3.4)</td>
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<td>$f(x)$</td>
<td>floating point representation of $x \in \mathbb{R}$ (3.21)</td>
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<td>$e^M$</td>
<td>matrix exponential (4.16)</td>
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<td>reachability matrix</td>
<td>(4.25)</td>
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<tr>
<td>$\mathcal{R}_n(A, B)$</td>
<td>finite reachability matrix</td>
<td>(4.26)</td>
</tr>
<tr>
<td>$P(t)$</td>
<td>(finite) reachability gramian continuous, discrete time</td>
<td>(4.28), (4.29)</td>
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<td>$C(A, C)$</td>
<td>observability matrix</td>
<td>(4.25)</td>
</tr>
<tr>
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Part I

Introduction
Chapter 1

Introduction

In today’s technological world, physical as well as artificial processes are described mainly by mathematical models. These models can be used to simulate the behavior of the processes in question. Sometimes, they are also used to modify or control the processes’ behavior. The weather, on the one hand, and very large scale integration (VLSI) circuits, on the other, constitute examples of such processes, the former physical and the latter artificial. Furthermore, these are dynamical systems, as their future behavior depends on their past evolution. In this framework of mathematical models, there is an ever-increasing need for improved accuracy, which leads to models of high complexity.

The basic motivation for system approximation is the need for simplified models of dynamical systems, which capture the main features of the original complex model. This need arises from limited computational, accuracy, and storage capabilities. The simplified model is then used in place of the original complex model, for either simulation or control.

In the former case, simulation, one seeks to predict the system behavior. However, often simulation of the full model is not feasible. Consequently, an appropriate simplification of this model is necessary, resulting in simulation with reduced computational complexity. Prominent examples include weather prediction and air quality simulations. The complexity of models, measured in terms of the number of coupled first-order differential or difference equations, may reach the tens or hundreds of thousands. In particular, discretization in problems that arise from dynamical partial differential equations (PDEs) which evolve in three spatial dimensions can easily lead to 1 million equations. In such cases, reduced simulation models are essential for the quality and timeliness of the prediction. Other methods for accelerating the simulation time exist, like parallelization of the corresponding algorithm. (These aspects, however, are not addressed in this book.)

In the latter case, control, we seek to modify the system behavior to conform with certain desired performance specifications (e.g., we seek to control a CD player to decrease its sensitivity to disturbances (outside shocks)). Such modifications are achieved in the vast majority of cases by interconnecting the original system with a second dynamical system, called the controller. Generically, the complexity of the controller (the number of first-order differential or difference equations describing its behavior) is approximately the same as that of the system to be controlled. Hence, if the latter has high complexity, so
will the controller. This, however, has three potential problems: storage—it may be hard to implement a high-order controller on a chip; accuracy—due to computational considerations (ill-conditioning), it may be impossible to compute such a high-order controller with any degree of accuracy; computational speed—due to limited computational speed, the time needed to compute the parameters of such a controller may be prohibitively large. The design of reduced-order controllers is a challenging problem, aspects of which have been investigated at least for systems of not-too-high complexity (see [252]). This will not be addressed in what follows. (See section 14.2 on open problems.)

In a broader context, this book deals with what is called the curse of dimensionality or, paraphrasing, the curse of complexity. In the computer science community, efficient algorithms are those that can be executed in polynomial time, that is, algorithms whose execution time grows polynomially with the size of the problem.

Here are examples of problems that can be solved in polynomial time (the complexity of a generic problem is of the order $n^\gamma$, where $n$ is the size of the problem):

- set of linear equations, $\gamma = 3$;
- eigenvalue problem, $\gamma = 3$;
- linear matrix inequalities (LMI), $\gamma \approx 4 \ldots 6$.

On the other hand, the problems

- factorization of an integer into prime factors, and
- stabilization of a linear system with constant output feedback

can be solved in exponential but not polynomial time.

In our framework there are two additional constraints. First, the algorithm, besides being efficient in the sense mentioned above, must produce an answer in a given amount of time. This becomes problematic for sufficiently large complexities even if the underlying algorithm is polynomial in time. Second, the solution must be accurate enough, which is a problem given that numbers can be represented in a computer only with finite precision. We will see later on that a popular method for model reduction of dynamical systems, balanced truncation, requires of the order $n^3$ operations, where $n$ is the complexity of the system to be approximated. It will also follow that methods that reduce the number of operations to $k \cdot n^2$ or $k^2 \cdot n$, where $k$ is the complexity of the reduced model, represent considerable improvements. The reason is that with order $n^3$ operations one can deal with system complexities of a few hundred states, while with $k^2 \cdot n$, the complexity of the systems that can be dealt with climbs into the millions. It should be mentioned at this point that while the available computing power increases, this turns out to be a mixed blessing, since with increased computing power, the numerical errors increase as well.

There are numerous remedies against the curse of dimensionality. Randomized algorithms are one instance; the solution obtained, however, may fail to satisfy all constraints for some specified percentage of the cases. Other kinds of remedies mostly applicable to problems with polynomial time solution algorithms but very large complexity are parallelization methods, as mentioned earlier.

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1.1. Problem set-up

This book addresses the approximation of dynamical systems that are described by a finite set of differential or difference equations, together with a finite set of algebraic equations. Our goal is to present approximation methods related to the singular value decomposition (SVD), on one hand, and approximation methods related to Krylov or moment matching concepts, on the other. Roughly speaking, the former family preserves important properties of the original system, like stability, and in addition provides an explicit quantization of the approximation error. The latter family lacks these properties but leads to methods that can be implemented in a numerically much more efficient way. Thus, while the former family of methods can be applied to relatively low-dimensional systems (a few hundred states), the latter methods are applicable to problems whose complexity can be several orders of magnitude higher. The combination of these two basic approximation methods leads to a third one, which aims at merging their salient features and is referred to as SVD-Krylov-based approximation.

Finally, we present some thoughts that have guided our choice of topics concerning the dilemma linear versus nonlinear. The basic argument used is that real systems are nonlinear, and therefore methods addressing nonlinear system approximation should be primarily considered. However, we espouse the following arguments:

- All physical systems are locally linear; in applications, typically one linearizes around an operating point of interest. If the operating point cannot be fixed, linear time-varying models or piecewise linear models can be considered.

- Many physical laws, e.g., Newton’s second law, Maxwell’s equations, Kirchhoff’s voltage laws (KVL), Kirchhoff’s current laws (KCL), the diffusion equation, the wave equation, Schrödinger’s equation, and probability laws (Markov equations), are linear, and this linearity holds for large ranges of the operating conditions.

- Linear theory is rich and extensive and offers a coherent picture.

- Artificial systems are sometimes designed to be linear.

- There are attempts in developing a nonlinear approximation theory, but they remain mostly ad hoc.

This book is dedicated to the presentation of primarily linear theory.

1.1 Problem set-up

The broader framework of the problems to be investigated is shown in Figure 1.1. The starting point is a physical or artificial system together with measured data. The modeling phase consists of deriving a set of ordinary differential equations (ODEs) or partial differential equations (PDEs). In the latter case, the equations are typically discretized in the space variables leading to a system of ODEs. This system will be denoted by \( \Sigma \). The modeling phase consists of deriving a dynamical system \( \hat{\Sigma} \) by appropriately reducing the number of ODEs describing the system. \( \hat{\Sigma} \) is now used to simulate and possibly control \( \Sigma \). Sometimes the ODEs are discretized in time as well, yielding discrete-time dynamical systems.
Next, we will formalize the notion of a dynamical system $\Sigma$ (see Figure 1.2). First, the time axis $T$ is needed; we will assume for simplicity that $T = \mathbb{R}$, the real numbers; other choices are $\mathbb{R}_+$ ($\mathbb{R}_-$), the positive (negative) real numbers, or $\mathbb{Z}_+$, $\mathbb{Z}_-$, $\mathbb{Z}$, the positive, negative, and all integers. In what follows we will assume that $\Sigma$ consists of first-order ODEs together with the set of algebraic equations

$$\Sigma : \begin{cases} \frac{d}{dt}x = f(x, u), \\ y = g(x, u), \end{cases} \quad (1.1)$$

where

$$u \in U = \{u : T \to \mathbb{R}^m\}, \quad x \in X = \{x : T \to \mathbb{R}^n\}, \quad y \in Y = \{y : T \to \mathbb{R}^p\},$$
1.1. Problem set-up

are the input (excitation), an internal variable (usually the state), and the output (observation), respectively, while $f$, $g$ are vector-valued maps of appropriate dimensions. The complexity $n$ of such systems is measured by the number of internal variables involved (assumed finite); that is, $n$ is the size of $x = (x_1 \cdots x_n)^\ast$. Thus we will be dealing with dynamical systems that are finite-dimensional and described by a set of explicit first-order differential equations; the description is completed with a set of measurement or observation variables $y$. Also, within the same framework, systems whose behavior is discrete in time can be treated equally well. In this case $T = \mathbb{Z}$, the set of integers (or $\mathbb{Z}_+$, or $\mathbb{Z}_-$), and the first equation in (1.1) is replaced by the difference equation $x(t+1) = f(x(t), u(t))$. In what follows, however, we concentrate primarily on continuous-time systems. Often in practice, this explicit nonlinear system is linearized around some equilibrium trajectory (fixed point), with the resulting system being linear, parameter time-varying, and denoted by $\Sigma_{LPTV}$. Finally, if this trajectory happens to be stationary (independent of time), we obtain a linear, time-invariant, system, denoted by $\Sigma_{LTI}$:

$$\begin{align*}
\Sigma_{LPTV} : & \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \\
y(t) &= C(t)x(t) + D(t)u(t),
\end{align*}$$

$$\begin{align*}
\Sigma_{LTI} : & \quad \dot{x}(t) = Ax(t) + Bu(t), \\
y(t) &=Cx(t) + Du(t).
\end{align*}$$

A more general class of systems is obtained if we assume that the first equation in (1.1) is implicit in the derivative of $x$, that is, $F(\dot{x}, x, u) = 0$, for an appropriate vector-valued function $F$. Such systems are known as differential algebraic equation (DAE) systems. Besides its occasional mention (see, e.g., Remark 11.3.1), this more general class of systems will not be addressed. See the book by Brenan, Campbell, and Petzold [74] for an account of this class of systems.

**Problem statement.** Given $\Sigma = (f, g)$ with $u \in U, x \in X, y \in Y$, find

$$\hat{\Sigma} = (\hat{f}, \hat{g}), \quad u \in U, \quad y \in Y, \quad \text{and} \quad \hat{X} = \{\hat{x} : T \to \mathbb{R}^k\}, \quad \text{where} \quad k < n,$$

such that (some of or all) the following conditions are satisfied:

1. The approximation error is small—existence of global error bound
2. Stability and passivity are preserved
3. The procedure is computationally stable and efficient

**COND**

**Special case: linear dynamical systems.** If we consider linear, time-invariant dynamical systems $\Sigma_{LTI}$ as in (1.3), denoted by

$$\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m)},$$

1Given a vector or matrix with real entries, the superscript $\ast$ denotes its transpose. If the entries are complex, the same superscript denotes complex conjugation with transposition.

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the problem consists in approximating $\Sigma$ with
\[
\tilde{\Sigma} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \in \mathbb{R}^{(k+p)\times(k+m)}, \quad k < n,
\]
so that the above conditions are satisfied. Pictorially, we have the following:

\[ \Sigma: \begin{array}{ccc}
A & B \\
C & D
\end{array} \Rightarrow \begin{array}{ccc}
\hat{A} & \hat{B} \\
\hat{C} & \hat{D}
\end{array} : \tilde{\Sigma}. \]

That is, only the size of $A$ is reduced, while the number of columns of $B$ and the number of rows of $C$ remain unchanged.

One possible measure for judging how well $\tilde{\Sigma}$ approximates $\Sigma$ consists of comparing the outputs $y$ and $\hat{y}$ obtained by using the same excitation function $u$ on $\Sigma$ and $\tilde{\Sigma}$, respectively. We require, namely, that the size (norm) of the worst output error $y - \hat{y}$ be kept small or even minimized for all normalized inputs $u$. Assuming that $\Sigma$ is stable (that is, all eigenvalues of $A$ are in the left half of the complex plane), this measure of fit is known as the $H_\infty$-norm of the error. We will also make use of another norm for measuring approximation errors, the so-called $H_2$-norm. This turns out to be the norm of the impulse response of the error system. For details on norms of linear systems, see Chapter 5.

### 1.1.1 Approximation by projection

Projections constitute a unifying feature of the approximation methods discussed in what follows. This feature is equivalent to simple truncation in an appropriate basis. Consider the change of basis $T \in \mathbb{R}^{n\times n}$ in the state space $\tilde{x} = Tx$. We define the following quantities by partitioning $\tilde{x}$, $T$, and $T^{-1}$:

\[
\tilde{x} = \begin{pmatrix} \hat{x} \\ \tilde{x} \end{pmatrix}, \quad T^{-1} = [V \ T_1], \quad T = \begin{bmatrix} W^* \\ T_2^* \end{bmatrix}, \quad \text{where } \hat{x} \in \mathbb{R}^k, \ \tilde{x} \in \mathbb{R}^{n-k}, \ V, \ W \in \mathbb{R}^{n\times k}.
\]

Since $W^*V = I_k$, it follows that

\[
\Pi = VW^* \in \mathbb{R}^{n\times n}
\] (1.6)

is an oblique projection onto the $k$-dimensional subspace spanned by the columns $V$ along the kernel of $W^*$.

Substituting for $x$ in (1.1) we obtain
\[
\frac{d}{dt} \tilde{x} = T f(T^{-1}\tilde{x}, u) \quad \text{and} \quad y = g(T^{-1}\tilde{x}, u).
\]

Retaining the first $k$ differential equations leads to
\[
\frac{d}{dt} \hat{x} = W^*f(V\hat{x} + T_1\tilde{x}, u), \quad y = g(V\hat{x} + T_1\tilde{x}, u).
\]

These equations describe the evolution of the $k$-dimensional trajectory $\hat{x}$ in terms of $\tilde{x}$; notice that they are exact. The approximation occurs by neglecting the term $T_1\tilde{x}$. What results is a

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1.1. Problem set-up

![Flowchart of approximation methods and their interconnections.](image)

Figure 1.3. Flowchart of approximation methods and their interconnections.

A dynamical system that evolves in a $k$-dimensional subspace obtained by restricting the full state as follows: $\tilde{x} = W^*x$. The resulting approximant $\tilde{\Sigma}$ of $\Sigma$ is

\[
\tilde{\Sigma} : \begin{cases}
\frac{d}{dt} \tilde{x}(t) &= W^*f(V\tilde{x}(t), u(t)), \\
y(t) &= g(V\tilde{x}(t), u(t)).
\end{cases}
\]  

(1.7)

Consequently, if $\tilde{\Sigma}$ is to be a “good” approximation of $\Sigma$, the influence of the neglected term $T_1\tilde{x}$ must be “small” in some appropriate sense. In the linear time-invariant case the resulting approximant is

\[
\tilde{\Sigma} = \begin{pmatrix}
\hat{A} & \hat{B} \\
\hat{C} & \hat{D}
\end{pmatrix} = \begin{pmatrix}
W^*AV & W^*B \\
CV & D
\end{pmatrix}.
\]  

(1.8)
A flowchart of the contents of this book is given in Figure 1.3. Proper orthogonal decomposition (POD) and SVD are concepts that will be explained later.

1.2 Summary of contents

The purpose of this book is to describe certain families of approximation methods by projection. Three different methods are discussed for choosing the projection $\Pi$ (that is, the matrices $V$ and $W$) so that some or all conditions (COND) are satisfied: (I) SVD-based methods, which are well known in the systems and control community and have good system theoretic properties; (II) Krylov-based methods, which are well known in the numerical analysis community and, to a lesser degree, in the system theory community and have good numerical properties; and (III) SVD-Krylov-based methods, which seek to merge the best attributes of (I) and (II). Furthermore, the class of weighted SVD methods, which establishes a link between (I) and (II), will also be discussed. We refer to the Preface for a description of the book contents.
Chapter 2
Motivating Examples

In this section we describe various applications in which large-scale dynamical systems arise. There are two categories of examples: those in which simulation and/or prediction of future behavior is of interest (examples 1, 2, 3, 4, 5, 6 in Figure 2.1), and those in which simulation and control are of primary interest (examples 7, 8, 9, 10).

The examples given in Figure 2.1 are described briefly.

2.1 Passive devices

*High-frequency, submicron VLSI circuits.* The integrated circuit (IC) was introduced in the 1960s. Since then, the scaling trends in VLSI design are that (i) the decrease in feature size is greater than 10% per year; (ii) the increase in chip size is greater than 10% per year; and (iii) there is an increase in operating frequency which now reaches the gigahertz range. As a consequence, the chip *complexity* has been increasing by at least 50% each year. A comparison between the Intel® 4004 processor, released in 1971, and the Intel Pentium® IV processor, released in 2001, shows that the feature size has decreased from 10\(\mu\) to 0.18\(\mu\), the number of components has increased from 2300 to 42 million, and the speed has increased from 64 KHz to 2 GHz; in addition, the length of all interconnections in the Pentium IV totals approximately 2 km, and the components are arranged in seven layers.

These trends impact physical parameters due to the increase of interconnect length and interconnect resistance. Furthermore, capacitance and inductance effects influence the chip, and there is a decrease of metal width and dielectric thickness. The resulting chips are multilayered, and the passive parts thus correspond to three-dimensional resistor-inductor-capacitor (RLC) circuits.

The design phase of a VLSI circuit is followed by the physical verification phase, where potential design flaws are discovered. Simulations are thus required to verify that internal electromagnetic fields do not significantly delay or distort circuit signals. This requires the solution of Maxwell’s equations for three-dimensional circuits with interconnections.
of the order of several kilometers, and submicron scale geometric resolution. One method for deriving a model in this case is known as the partial element equivalent circuit (PEEC), which works by spatial discretization of Maxwell’s equations for three-dimensional geometries. The complexity of the resulting model reaches unmanageable proportions and can be anywhere from $n \approx 10^5$ to $10^6$. Therefore, reduced-order modeling is of great importance. We thus seek to generate models that are as simple as possible but are nevertheless capable of generating the actual chip behavior. For details, see van der Meijs [330]. See also [279] and [202].

There is a general-purpose simulation program for electric circuits known as SPICE (Simulation Program with Integrated Circuit Emphasis), which was developed in the 1970s at the University of California at Berkeley. This allows the following components: resistors, capacitors, inductors, independent sources, dependent sources, transmission lines, diodes, and transistors. SPICE can handle complexities of a few hundred such elements. It thus becomes inadequate for complexities of the order mentioned above.

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2.2. Weather prediction—data assimilation

2.2.1 North Sea wave surge forecast

Because part of The Netherlands is below sea level, it is important to monitor wave surges at river openings. In the case of such a surge, water barriers can be closed to prevent flooding. Since these rivers are in many cases important waterways, the barriers must stay closed only while the surge lasts. Furthermore, the warning has to come about 6 hours in advance.

The equations governing the evolution of the wave surge are in this case the shallow water equations, which are PDEs. In Figure 2.2, the horizontal and vertical axes indicate the number of discretization points, while the color bar on the right-hand side indicates the depth of the sea at various locations of interest (justifying the use of shallow water equations).

![Figure 2.2. Wave surge prediction problem: depth of the North Sea.](image)

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The discretization grid used in this case is shown in Figure 2.3. The problem in this case is not just prediction of the wave surge based on initial conditions. There are several locations where the wave surge is measured (see Figure 2.4). There are also locations where the movement of the sea currents is measured. The problem thus becomes data assimilation, as one wishes to predict the wave surge based on both the model and the provided measurements. This is achieved by means of a Kalman filter.

The finite element (FE) discretization of the shallow water equations yields approximately 60,000 equations, and the resulting computational time is several times the allowed limit of 6 hours. Therefore, reduced-order models are necessary. Figure 2.5 shows the error covariance of water level prediction in two cases: first, with wind disturbance and

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no additional measurements, and second, by assimilating the measurements from the eight locations. This problem has been studied by Verlaan and Heemink at Delft University of Technology in The Netherlands; for details see [348], [168].

2.2.2 Pacific storm tracking

The issue here is to determine the sensitivity of atmospheric equilibria to perturbations. In particular, we wish to determine the initial perturbation that produces the greatest perturbation growth over some specified interval of time. In [109], perturbations to the vorticity equation of a Couette flow are studied. These are governed by the Orr–Sommerfeld equation; assuming harmonic perturbations in the wind velocity of the form \( \Phi(x, y, t) = \phi(y, t)e^{ikx} \), we have

\[
\frac{\partial \phi(y, t)}{\partial t} = -i ky \frac{\partial^2 \phi(y, t)}{\partial y^2} + \frac{1}{Re} \left( \frac{\partial^2 \phi(y, t)}{\partial y^2} - k^2 \phi(y, t) \right)^2,
\]

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where Re denotes the Reynolds number. Discretization in $y$ yields the set of ODEs:

$$\frac{d\hat{\phi}(t)}{dt} = \hat{A}\hat{\phi}(t), \quad \hat{A} \in \mathbb{R}^{n \times n}.$$  

We assume that this system is influenced by perturbations; in particular, we assume that (i) random inputs are affecting all variables $\hat{\phi}_i$, and (ii) all these variables are measured (observed). The discretized system is thus a linear system having the same number of inputs $m$, state variables $n$, and outputs $p$:

$$\Sigma = \begin{pmatrix} \hat{A} & I_n \\ I_n & 0 \end{pmatrix} \Rightarrow m = p = n.$$  

Such models are used for storm tracking in the midlatitude Pacific. For data assimilation in this context, see [110].

### 2.2.3 America’s Cup

The America’s Cup is a race of sailing boats that takes place every 4 years. The 31st competition since 1848 took place in 2003 in New Zealand between Team NZ and the Swiss team Alinghi. Much technological know-how goes into the construction of the boats. Less well known, however, is that the contestants set up weather teams. These are meant to advise on the weather (in particular, the wind direction and speed) that will prevail during the approximately 8 hours of the race. It is important to be able to predict wind shifts. The goal is to choose the right sails and develop appropriate strategies for the race. The 2003 winner, the Alinghi team, set up a strong weather forecasting group, lead by Dr. J. Katzfey, an expert in weather forecasting at the Commonwealth Scientific and Industrial Research Organization in Australia. Furthermore, the Alinghi team set up eight weather stations, which provided data for the data assimilation part of the model. At 7:00 a.m. before each race, the weather team presented a weather prediction for the next 6 to 8 hours of sailing. This strategy turned out to be an important factor for the Alinghi team; in the third regatta, for instance, last-minute updates brought Alinghi 200 m ahead of the New Zealand boat, which proved decisive for the winners.

### 2.3 Air quality simulations—data assimilation

Air pollution was thought to be a local phenomenon until the 1970s and a regional one during the 1980s. At present it is recognized that air pollution extends from urban to regional and global scales. Emissions from fossil fuel combustion and biomass burning and the resulting photochemical production of tropospheric ozone and climate warming are considered global problems.

A current challenge to the atmospheric science community is to quantify the impact of human activities on global atmospheric photochemistry. This is achieved by means of air quality models (AQMs), which provide a mechanism for elucidating the underlying physical and chemical processes responsible for the formation, accumulation, transport, and removal of air pollutants. AQMs are designed to calculate concentrations of ozone and
other pollutants and their variations in space and time in response to particular emission inputs and for specified meteorological scenarios. The AQM is the only prognostic tool available to the policy-making community, i.e., a tool capable of quantitatively estimating future air quality outcomes for conditions or emissions different from those that have existed in the past. Because of this unique capability, AQMs have come to play a central role in determining how pollutant emissions should be managed to achieve air quality goals.

A variety of AQMs are being applied on urban, regional, and global scales. Many models share common features. In particular, AQMs are based on solving the same species conservation equations which describe the formation, transport, and fate of air pollutants:

$$\frac{\partial c_i}{\partial t} + \nabla \cdot (u c_i) - \nabla \cdot (K \nabla c_i) = R_i(c_1, c_2, \ldots, c_s) + S_i, \quad i = 1, \ldots, s.$$ 

Here, $c_i$ is the (averaged) concentration of species $i$; $u(x, t)$ is the wind velocity vector at location $x$ and time $t$; $K(x, t)$ is the turbulence diffusivity tensor; $R_i$ is the rate of concentration change of species $i$ by chemical reactions; $S_i(x, t)$ is the source/sink of species $i$; and $s$ is the number of predicted species. $R_i$ can also be a function of meteorological variables (e.g., temperature). The source/sink term $S_i$ can include emissions of a species as well as its loss due to various processes, such as dry deposition and rainout. Ensemble averaging is used to dispense with the need to capture the extremely small-scale fluctuations due to turbulence. With appropriate initial and boundary conditions, the system described by the above equation represents the continuum chemistry transport model (CTM). A CTM system is composed of four basic components: a chemical kinetic mechanism, a source emissions inventory, a description of pollutant transport and removal, and a set of numerical algorithms for integrating the governing equations. Differences between various AQMs stem from alternative choices made by their developers in characterizing these physical and chemical processes, procedures for their numerical solution, and the approach taken to adapt the model to the computational domain of interest.

After spatial discretization of the CTM, we obtain

$$\frac{d\mathbf{c}}{dt} = f(\mathbf{c}) = f_a(\mathbf{c}) + f_d(\mathbf{c}) + f_r(\mathbf{c}) + \mathbf{u}, \quad \mathbf{c}(t_0) = \mathbf{c}_0, \quad \mathbf{c} = [c_1, \ldots, c_s].$$

The grid in the $x$ and $y$ directions is $1 \text{ km} = 100$ points; in the $z$ direction it is $10 \text{ km} = 30$ points. This results in 300,000 equations.

Many measurements of pollutants exist. In the past few years, the MOPITT satellite was launched by NASA to take pictures of pollutant concentrations. Thus, measurements consist mainly of satellite images, and the problem becomes, once more, one of data assimilation.

For details, see [103], [246], [292], [293]. An important recent development in three-dimensional chemistry transport modeling is *MOZART* (Model of Ozone and Related Tracers); it has been developed in the framework of National Center for Atmospheric Research community climate model (CCM) [179].

### 2.4 Biological systems: Honeycomb vibrations

The honeybee dance language, in which foragers perform dances containing information about the distance and direction to a food source, is an example of symbolic communication.

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in nonprimates. Honeybees and human beings possess an abstract system of communication. It was noted by Aristotle that honeybees recruit nestmates and lead them to a food source. In the 1960s, K. von Frisch (1973 Nobel Prize winner in physiology and medicine) postulated that this recruitment takes place by means of the so-called waggle dance performed on the honeycomb. This dance consists of a looping figure eight movement with a central straight waggle run. During this dance, distance and direction information about the food source are transmitted. The possible mechanisms involved in this unique communication are mechanical and chemical; there are no optical signals involved given the darkness of the hive. The mechanical mechanism results from the vibration of the honeycomb, while the chemical one results by transmission of pollen or nectar.

During the waggle dance, the dancer bee waggles her body at 15 Hz and vibrates her wings intermittently at 200–300 Hz. The transmitted vibrations have an amplitude of about 1.4 µm. The question thus arises as to the ability of the bees to detect weak vibrations in a noisy environment (the hive).

Experimental measurements have shown that the vibrations occur only in the horizontal direction to the plane of the combs; furthermore, the comb seems to amplify vibrations in frequencies around 250 Hz. It has also been experimentally observed that the combs exhibit an impedance minimum to horizontal vibrations at 230–270 Hz. Most of the experimental investigations of this problem have been performed by Dr. J. Tautz at the Universität Würzburg.

The goal is therefore to find a model of the honeycomb that explains the phenomena observed and measured and that provides new insights into these phenomena. In particular, we would like to know to what extent a honeycomb is appropriate as a medium for the transmission of information through vibrations. For more details, see [102] and the more recent article [321].

2.5 Molecular dynamics

This example involves simulation in molecular dynamics, and in particular protein substate modeling and identification. The main tool used for this example is the SVD. For details see [275].

Proteins are dynamic entities primarily due to the thermal motion of their atoms. They have different states which are attained by thermal motion and determine their biological properties. Proteins can exist in a number of different conformational substates, a conformational substate being a collection of structures that are energetically degenerate. The thermal motion of the atoms drives the transitions between the different substates accessible to a protein. The distribution of these substates and their transitions are major factors in determining the biological properties of proteins. The equations describing the motion of proteins are of the type

$$\frac{d^2}{dt^2}x(t) = -\nabla \phi(x(t)),$$

where $\phi$ is a potential function, and $x \in \mathbb{R}^{3n}$, where $n$ is the number of atoms in the protein.

To find the most important protein configurations, an SVD of snapshots of $x$ is used. This method is known as POD and is described in section 9.1. This SVD provides a way of decomposing a molecular dynamics trajectory into fundamental modes of atomic motion. The left singular vectors describe the direction in which each atom prefers to
2.5. Molecular dynamics

Figure 2.6. Molecular dynamics: myoglobin (protein); heme: active site; histidine: part that opens and closes, catching oxygen molecules.

move. The right singular vectors provide temporal information; they are projections of the protein conformations onto these modes showing the protein motion in a generalized low-dimensional basis.

If an atom were constrained to move along only one of the left singular vectors, then its motion in Cartesian space can be projected onto this vector, giving a curve. The elements of the right singular vector can be thought of as scaling the size of the left singular vector to describe where the atom is at each time point or in each conformation.

Figure 2.6 depicts a protein called myoglobin (more precisely, F46V mutant bioglobin). The active site that catches oxygen molecules is called the heme (this is shown in a yellow closing and a white closed state); finally, oxygen molecules (shown in green) are captured by means of the active site, which is called histidine.

The first left singular vector of the (distal) histidine in this F46V mutant bioglobin describes more than 90% of the total motion of the histidine. This histidine also ranked second on the list of residues. Thus, (i) incorporating conformational substate information improves the refinement model; (ii) multiconformer refinement appears to be the better method for overall improvement of the model; and (iii) the SVD provides a powerful tool for visualizing complex high-dimensional systems in a low-dimensional space.

Heat capacity of molecular systems

The next application is concerned with the determination of the heat capacity of a molecular system. This involves the calculation of the following integral:

\[ C_v = \int_0^\infty f(\omega)g(\omega)d\omega, \]

where \( g(\omega)d\omega \) gives the number of vibrational frequencies in the interval \( (\omega, \omega + d\omega) \). Since the molecular system is discrete we have \( \sigma(\omega) = \int_0^\omega g(\tau)d\tau = \sum_{i=1}^n \mathbb{I}(\omega - \omega_i) \), where \( n \) is the number of particles in the molecular system and \( \mathbb{I} \) is the Heaviside step function. This requires all the fundamental frequencies of the system to be computed in advance. Instead, Gauss quadrature is used; this involves the Lanczos algorithm, where \( A \) is the Hessian of \( \phi \) and \( b \) is arbitrary. For details, see [364].

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2.6 International Space Station

The International Space Station (ISS) is a complex structure composed of many modules; these modules are contributed by NASA and other space agencies. The flex modes of each module are described in terms of $n \approx 10^3$ state variables. The goal is to develop controllers that are implemented onboard the space station. Consequently, controllers of low complexity are needed because of hardware, radiation, throughput, and testing issues. In Chapter 13, model reduction of the flex models for two specific modules, namely, the 1R (Russian service module) and the 12A (second left-side truss segment), are discussed. Figure 2.7 shows the frequency response (amplitude Bode plot) as more components are added to the space station. The complexity of the resulting model is reflected in the number of spikes present in the frequency response. These models were provided by Draper Laboratories in Houston. For details on the ISS and its assembly see http://spaceflight.nasa.gov/station/.

2.7 Vibration/acoustic systems

Consider a car windscreen subject to an acceleration load. The problem consists of computing the noise generated at points away from the window. The first step in solving this problem is to solve the PDE which describes the deformation of the windscreen of a specific material. The finite element discretization gives, in a specific case, 7564 nodes (three layers of 60-by-30 elements); the material is glass with Young modulus $7 \cdot 10^{10}$ N/m², density $2490$ kg/m³, and Poisson ratio 0.23. These parameters help determine the coefficients of the resulting FE model experimentally. Finally, the windscreen is subjected to a point force...
at some given point (node 1891), and the goal is to compute the displacement at the same point. The discretized problem in this particular case has a dimension of 22,692. Notice that the last two problems (the windscreen and the ISS) yield second-order equations:

\[ M \frac{d^2}{dt^2} x(t) + C \frac{d}{dt} x(t) + K x(t) = f(t), \]

where \( x \) is position, \( \frac{d}{dt} x \) is velocity of the windscreen at the grid points chosen, and \( M, C, K \) are the mass, damping, and stiffness matrices. Since this is a second-order system, its complexity is twice as high (45,384 states). For details on eigenvalue problems for second-order systems, see [325]. This problem was provided by Karl Meerbergen of Free Field Technologies, Leuven, Belgium. See also [235].

### 2.8 CVD reactors

An important problem in semiconductor manufacturing is the control of chemical reactors, for instance, CVD reactors. This problem is addressed in the literature using POD methods. For details, see the work of Banks and coworkers [199], [200], [49]. The dimension of the resulting linear systems is on the order of a few thousand state variables.

Another issue concerning CVD reactors is the determination of the stability of steady states. To address this problem, the transient behavior is linearized around a steady state. This leads to a generalized eigenvalue problem. The eigenvalues with largest real part are calculated using the Arnoldi iteration. A model problem of three-dimensional incompressible flow and heat transfer in a rotating disk CVD reactor is used to analyze the effect of parameter change on the performance of the eigenvalue algorithm. The description of this system requires a full three-dimensional Navier–Stokes model. The calculation of leading eigenvalues for matrix systems of order 4 million to 16 million is required. These calculations lead to the critical Grashof, Rayleigh, and Reynolds numbers for a Hopf bifurcation. For details, see [226].

### 2.9 Microelectromechanical devices

Microelectromechanical systems (MEMS) are integrated systems combining electrical and mechanical components. They are usually fabricated using IC techniques and can range in size from micrometers to millimeters. In their most general form, MEMS consist of mechanical microstructures, microsensors, microactuators, and electronics, all integrated onto the same chip.

Finite element model (FEM) simulation is often used in MEMs and results in complex systems. System-level models with reduced order and accuracy have to be generated as a basis of system simulation [291].

sugar is a simulation tool for MEMS devices based on nodal analysis techniques of integrated circuit simulation. Beams, electrostatic gaps, circuit elements, etc., are modeled by small, coupled systems of differential equations. For a description, see [89] and http://www-bsac.eecs.berkeley.edu/cadtools/sugar/sugar/.

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2.9.1 Micromirrors

Two such MEMS devices are the micromirror and micromirror arrays. They are used for precision light manipulation, e.g., as an optical switch in fiber optics. A mirror tilts to reflect light from one fiber to another. Large arrays, up to 1000 by 1000 mirrors, will be needed for telecommunication applications. As the fiber-to-mirror distance increases, strict precision of mirror tilt control is necessary. For a distance of 500 microns, at least 0.10 degree of precision is required. A second application of micromirrors is maskless lithography (virtual masks). Feedback control is needed to achieve precision positioning of a mirror of 2 degrees of freedom. Advantages of feedback control as an optical switch are that it is faster, smaller, and cheaper than electrical switches.

2.9.2 Elk sensor

A few years ago, production of the Mercedes-Benz A Class cars had to be stopped just after their launch because they failed the elk test. This test consists of forcing the car to take a sharp turn to avoid an obstacle that suddenly appears on the road. To remedy this situation, the company incorporated a rollover (elk) sensor that could detect turning movement and apply the brakes to slow the rotational movement. The first rollover sensors were mechanical. Subsequently, Bosch AG developed a microelectromechanical sensor at reduced cost and reduced size. Now, the elk sensor is standard equipment in many cars. A similar angular velocity sensor is shown in Figure 2.8.

Once such a device has been designed, the next issue consists in testing its performance by simulation. One method is a physically oriented modeling (see, e.g., Schwarz [290] and Teegarden, Lorenz, and Neul [322]), using appropriate simulation packages, as described in [291]. The more detailed the modeling, the higher the complexity (i.e., the number of differential equations) of the resulting model. As the available simulation packages are built to handle low complexities, there is a need for simplification of the model through model reduction.

Figure 2.8. MEMS angular velocity sensor used in car navigation systems and for rollover detection in passenger restraint systems. Picture courtesy of Robert Bosch Co.
2.9. Microelectromechanical devices

Figure 2.9. Optimal cooling: discretization grid.

Figure 2.10. Progression of the cooling of a steel profile (from left to right and from top to bottom); the bars are cooled from $1000^\circ C$ to $500^\circ C$.

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2.10 Optimal cooling of steel profile

Many problems in control are both structured and computationally intensive. An application arising in tractor steering can be found in [61]. For an overview, see Fassbender [112]. Here we will describe an application worked out by Benner [60]. In rolling mills, steel bars have to be heated and cooled quickly and uniformly to achieve a fast production rate. Cooling takes place by spraying the bars with cooling fluids. The problem consists of devising and applying an optimal cooling strategy.

The heat equation is used to model the cooling process. The steel bar is assumed to have infinite length, thus reducing the problem to two dimensions. The domain is obtained by cutting the steel bar vertically; this domain can be further halved due to symmetry. It is assumed that there are eight nozzles spraying cooling liquid uniformly on the boundary of the bar. Thus a two-dimensional boundary control problem results.

The heat-diffusion equation with Neumann boundary conditions is discretized in the spatial variables using the FEM described in the package albert. The initial mesh leads to a system of order $n = 106$. Subsequently, the mesh is refined, leading to systems of orders $n = 371$, $1357$, and $5177$ (see Figure 2.9). The resulting mass and stiffness matrices are sparse (with approximately $3 \cdot 10^5$ and $3 \cdot 10^4$ nonzero elements, respectively). The control law is obtained by means of linear quadratic regulator (LQR) design.

Finally, the model uses water at $20^\circ$C and has the spraying intensity as control parameter. Figure 2.10 shows the progression of the cooling.