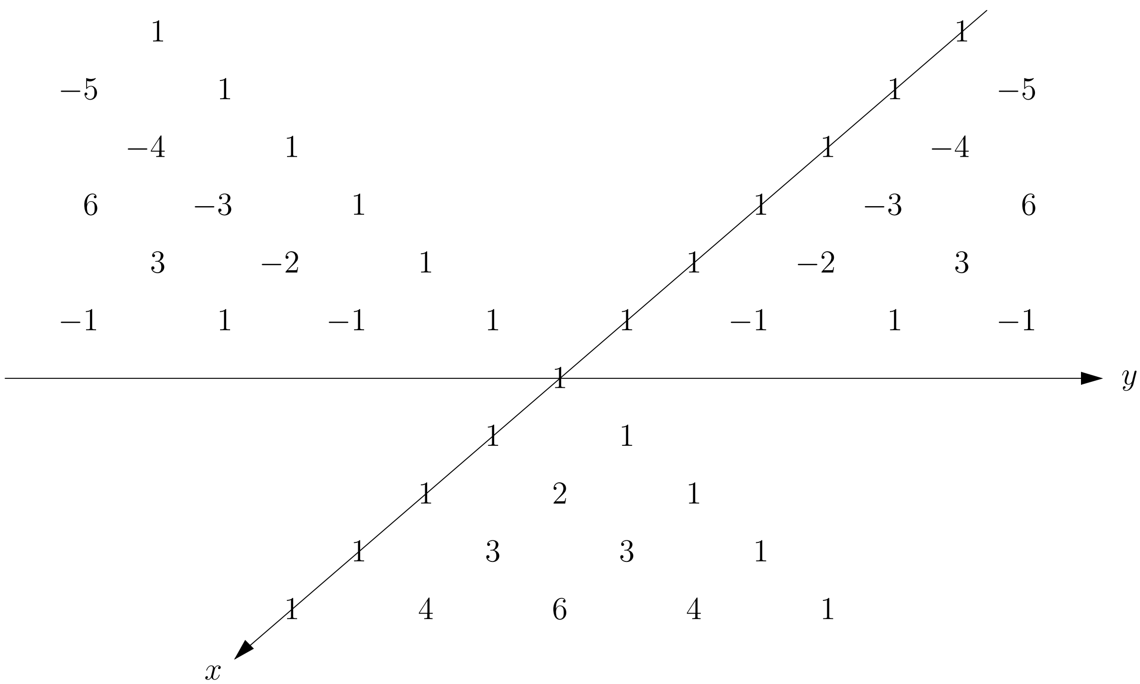
Bilateral Binomial Theorem

Problem 03-001, *by* Martin Erik Horn (University of Potsdam, Germany).

Let

$$\binom{x}{y} = \lim_{h\to 0} \frac{\Gamma(x+1+h)}{\Gamma(y+1+h)\Gamma(x-y+1+h)}.$$

If integers of distance 1 in the Pascal Plane



are added and the definition of the *bilateral hypergeometric function* of [1], namely

$$(1) \qquad {}_1H_1(a;b;z) = \cdots + \frac{(b-1)(b-2)}{(a-1)(a-2)}z^{-2} + \frac{b-1}{a-1}z^{-1} + 1 + \frac{a}{b}\,z + \frac{a(a+1)}{b(b+1)}z^2 + \cdots$$

is used, the following bilateral hypergeometric identity is obtained [2]:

$$(2) \qquad 2^x = \frac{x!}{y!(x-y)!}\,{}_1H_1[(y-x);(y+1);-1], \qquad x,y\in\mathbb{R}.$$

This is a special case of the bilateral binomial theorem

$$(3) \qquad (1+z)^x = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}\,{}_1H_1[(y-x);(y+1);-z], \qquad x,y\in\mathbb{R},\ z\in\mathbb{C},$$

with $|z| = 1$. Give a proof of (3) provided the bilateral series converges.

Hint by Jonathan M. Borwein. Formula (2) may be reformulated in a rather pretty symmetric form and can then be proved easily:

$$\begin{aligned}
\frac{1}{x}\,{}_2F_1&\left[\begin{matrix}(1-y),\ 1\\ (1+x)\end{matrix};-1\right]+\frac{1}{y}\,{}_2F_1\left[\begin{matrix}(1-x),\ 1\\ (1+y)\end{matrix};-1\right]\\
&=\int_0^1 (1-t)^{x-1}(1+t)^{y-1}\,dt+\int_0^1 (1-t)^{y-1}(1+t)^{x-1}\,dt\\
&=\int_{-1}^1 (1+t)^{x-1}(1-t)^{y-1}\,dt\\
&=2\int_0^1 (2s)^{y-1}(2-2s)^{x-1}\,ds\\
&=2^{x+y-1}\,\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\end{aligned}$$

REFERENCES

[1] William N. Bailey, *Series of hypergeometric type which are infinite in both directions*, Quart. J. Math, 7 (1936), pp. 105–115.

[2] Martin E. Horn, *Lantacalan*, unpublished.

[3] Arthur Erdélyi, ed., *Higher Transcendental Functions*, Vol. I, compiled by the staff of the Bateman Manuscript Project, reprinted edition, Robert E. Krieger Publishing Co., Malabar, FL, 1981.

Status. Formula (2) is published in concealed form in [3] as formula (48) of chapter 2.8 on page 104 without proof. The proof given above was suggested by J. Borwein. There are suggestions that (3) is already published in the literature, but the proposer couldn't find it. One proof of this formula was found by the proposer, with considerable help from J. Borwein, by modifying the symmetric formula into

$$\begin{aligned}
\frac{\sqrt{z}}{y}\,&{}_1H_1[(1-x);(1+y);-z]\\
&=\frac{1}{x\sqrt{z}}\,{}_2F_1\left[\begin{matrix}(1-y),\ 1\\ (1+x)\end{matrix};-\frac{1}{z}\right]+\frac{\sqrt{z}}{y}\,{}_2F_1\left[\begin{matrix}(1-x),\ 1\\ (1+y)\end{matrix};-z\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{z}} \int_0^1 (1-t)^{x-1} \left(1+\frac{t}{z}\right)^{y-1} dt + \sqrt{z} \int_0^1 (1-t)^{y-1}(1+tz)^{x-1}\, dt \\
&= -\int_0^{1/\sqrt{z}} (1+r\sqrt{z})^{x-1} \left(1-\frac{r}{\sqrt{z}}\right)^{y-1} dr + \int_0^{\sqrt{z}} \left(1-\frac{r}{\sqrt{z}}\right)^{y-1} (1+r\sqrt{z})^{x-1}\, dr \\
&= \int_{-1/\sqrt{z}}^{\sqrt{z}} (1+r\sqrt{z})^{x-1} \left(1-\frac{r}{\sqrt{z}}\right)^{y-1} dr \\
&= \int_0^1 (s(z+1))^{x-1} \left(1+\frac{1}{z} - s\left(1+\frac{1}{z}\right)\right)^{y-1} \left(\sqrt{z}+\frac{1}{\sqrt{z}}\right) ds \\
&= (z+1)^{x-1} \left(1+\frac{1}{z}\right)^{y-1} \left(\sqrt{z}+\frac{1}{\sqrt{z}}\right) \int_0^1 s^{x-1}(1-s)^{y-1}\, ds \\
&= (1+z)^{x+y-1} z^{\frac{1}{2}-y} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\end{aligned}
$$

Another proof was found by the proposer using *Kummer's series* (pp. 105–106 of [3]). Other solutions are invited, as are references in case that (3) is found in the literature.