Five Numerical Problems in the Style and Spirit of the SIAM 100-Digit Challenge

Problem 04-003, *by* FOLKMAR BORNEMANN (Technische Universität München, Munich, Germany), DIRK LAURIE (University of Stellenbosch, Stellenbosch, South Africa), STAN WAGON (Macalester College, St. Paul, MN), AND JÖRG WALDVOGEL (ETH Zurich, Zurich, Switzerland).

The next five problems are from the just-published book:

> Folkmar Bornemann, Dirk Laurie, Stan Wagon, Jörg Waldvogel: *The SIAM* 100-*Digit Challenge: A Study in High-Accuracy Numerical Computing. With a Foreword by David H. Bailey.* SIAM, Philadelphia, 2004.

That book, available at `http://www.ec-securehost.com/SIAM/ot86.html`, includes an appendix with 22 problems in the same style and spirit as those of the original challenge, a contest posed by Lloyd N. Trefethen of Oxford University in the January/February 2002 issue of *SIAM News*.

Each of the problems is answered by a single real number. The task is to calculate at least 10 significant digits of that number and to give some rationale of their correctness.

(a) If N point charges are distributed on the unit sphere, the potential energy is

$$E = \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} |x_j - x_k|^{-1},$$

where $|x_j - x_k|$ is the Euclidean distance between x_j and x_k. Let E_N denote the minimal value of E over all possible configurations of N charges. What is E_{100}?

(contributed by Lloyd N. Trefethen)

(b) Riemann's prime counting function (introduced in the seminal 1859 memoir in which he stated his famous hypothesis) is defined as

$$R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \operatorname{li}(x^{1/k}),$$

where $\mu(k)$ is the Möbius function, which is $(-1)^\rho$ when k is a product of ρ different primes and zero otherwise, and $\operatorname{li}(x) = \int_0^x dt/\log t$ is the logarithmic integral, taken as a principal value in Cauchy's sense. What is the largest positive zero of R?

(contributed by Jörg Waldvogel)

Remark. The answer to this problem is truly shocking.

(c) What is the value of

$$\int_0^\infty x\, J_0(x\sqrt{2}) J_0(x\sqrt{3}) J_0(x\sqrt{5}) J_0(x\sqrt{7}) J_0(x\sqrt{11})\, dx,$$

where J_0 denotes the Bessel function of the first kind of order zero?

(contributed by Folkmar Bornemann)

Remark. Integrals over products of Bessel functions appear frequently in physics and electrical engineering. There is a rich body of literature, also quite recent work, on their evaluation in terms of special functions. Such expressions are known for the corresponding integrals with three factors [2], e.g.,

$$\int_0^\infty x\, J_0(x\sqrt{2}) J_0(x\sqrt{3}) J_0(x\sqrt{5})\, dx = \frac{1}{\pi\sqrt{6}},$$

and with four factors [1], e.g.,

$$\int_0^\infty x\, J_0(x\sqrt{2}) J_0(x\sqrt{3}) J_0(x\sqrt{5}) J_0(x\sqrt{7})\, dx = \frac{1}{\pi^2 \cdot 210^{1/4}} K\left(\tfrac{1}{4}\sqrt{8 + 23\sqrt{\tfrac{5}{42}}}\right).$$

Here $K(k)$ denotes the complete elliptic integral of the first kind of modulus k. Remarkably, the positivity of the integrals with three Bessel factors was used by G. Szegő in his 1933 proof of a conjecture of Friedrichs and Lewy that arose from a finite difference approximation of the wave equation [3].

(d) A particle's movement in the x-y plane is governed by the kinetic energy $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ and the potential energy

$$U = y + \frac{\epsilon^{-2}}{2}(1 + \alpha x^2)(x^2 + y^2 - 1)^2.$$

The particle starts at the position $(0, 1)$ with the velocity $(1, 1)$. For which parameter α does the particle hit $y = 0$ first at time 10 in the limit $\epsilon \to 0$?

(contributed by Folkmar Bornemann)

(e) At what time t_∞ does the solution of the equation $u_t = \Delta u + e^u$ on a 3×3 square with zero boundary and initial data blow up to ∞?

(contributed by Lloyd N. Trefethen)

REFERENCES

[1] J. W. Nicholson, *Generalisation of a theorem due to Sonine*, Quart. J. Math., 48 (1920), pp. 321–329.

[2] N. J. Sonine, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, Math. Ann., 16 (1880), pp. 1–80.

[3] G. Szegő, *Über gewisse Potenzreihen mit lauter positiven Koeffizienten*, Math. Z., 37 (1933), pp. 674–688.

Status. The proposers have solutions. Additional solutions are invited.