

## The Sum of the “Harmonic Series” Is Not Enough

In *Problem 06-007*, OVIDIU FURDUI conjectured that for each integer  $m \geq 2$ , there are corresponding rational numbers  $a_m$  and  $b_m$  such that

$$S_m := \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n} = a_m \pi^2 + b_m \log^2 m.$$

(Here  $\log x$  is the natural logarithm of  $x$ .) The reader was asked to prove or disprove this conjecture.

*Editorial note.* The problem as stated is apparently quite hard. The “solutions” below determine  $S_m$  for all  $m \geq 2$ , but that does not settle the issue. For example, one finds that

$$S_5 = \frac{7\pi^2}{30} - \frac{5 \log^2 5}{8} - \frac{1}{2} \log^2 \phi, \quad \text{where} \quad \phi = \frac{1 + \sqrt{5}}{2},$$

which suggests that the conjecture is false. To be complete, however, a counterexample claim needs to include a proof that there do not exist rational numbers  $a$  and  $b$  such that

$$\log^2 \phi = a\pi^2 + b \log^2 5.$$

Neither the solvers nor the editors have such a proof.

*Solution 1 by OMRAN KOUBA*<sup>1</sup> (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

We will prove that for each  $m \geq 2$ ,

$$(1) \quad \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n} = \frac{(m-1)(m+2)}{24m} \pi^2 - \frac{1}{2} \log^2 m - \frac{1}{2} \sum_{j=1}^{m-1} \log^2 \left( 2 \sin \frac{j\pi}{m} \right).$$

In particular,

$$S_2 = \frac{1}{12} \pi^2 - \log^2 2, \quad S_3 = \frac{5}{36} \pi^2 - \frac{3}{4} \log^2 3, \quad S_4 = \frac{3}{16} \pi^2 - \frac{11}{16} \log^2 4,$$

$$S_5 = \frac{7}{30} \pi^2 - \frac{5}{8} \log^2 5 - \frac{1}{2} \log^2 \left( \frac{1 + \sqrt{5}}{2} \right), \quad S_6 = \frac{5}{18} \pi^2 - \frac{1}{2} \log^2 6 - \frac{1}{4} \log^2 3 - \frac{1}{2} \log^2 2.$$

This answers completely the question of evaluating the sums  $(S_m)_{m \geq 2}$ .

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For  $m \geq 2$ , let us define  $Q_m(t) = 1 + t + \cdots + t^{m-1}$ . Since  $(1-t)Q_m(t) = 1 - t^m$  we have  $(1-t)Q'_m(t) = Q_m(t) - mt^{m-1}$ , and consequently, for  $n \geq 1$  and  $t \in (0, 1)$ ,

$$\frac{Q'_m(t)}{Q_m(t)}(1 - t^{nm}) = (1-t)Q'_m(t) \frac{1 - t^{nm}}{1 - t^m} = \frac{1 - t^{nm}}{1 - t^m}(Q_m(t) - mt^{m-1}),$$

that is,

$$\begin{aligned} \frac{Q'_m(t)}{Q_m(t)}(1 - t^{nm}) &= (1 + t^m + t^{2m} + \cdots + t^{(n-1)m}) (1 + t + t^2 + \cdots + t^{m-1} - mt^{m-1}) \\ &= \sum_{j=1}^{nm} t^{j-1} - m \sum_{\ell=1}^n t^{m\ell-1}. \end{aligned}$$

We conclude that

$$\begin{aligned} \int_0^1 \frac{Q'_m(t)}{Q_m(t)} t^{nm} dt &= \int_0^1 \frac{Q'_m(t)}{Q_m(t)} dt - \sum_{j=1}^{nm} \int_0^1 t^{j-1} dt + m \sum_{\ell=1}^n \int_0^1 t^{m\ell-1} dt \\ &= \log m - H_{mn} + H_n. \end{aligned}$$

But the functions  $\left(t \mapsto \frac{Q'_m(t)t^{mn}}{nQ_m(t)}\right)_{n \geq 1}$  are positive and continuous on  $[0, 1]$ , so

$$\int_0^1 \frac{Q'_m(t)}{Q_m(t)} \left(\sum_{n=1}^{\infty} \frac{t^{mn}}{n}\right) dt = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \frac{Q'_m(t)}{Q_m(t)} t^{mn} dt,$$

that is,

$$- \int_0^1 \frac{Q'_m(t)}{Q_m(t)} \log(1 - t^m) dt = \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n} = S_m.$$

Now  $\log(1 - t^m) = \log(1 - t) + \log Q_m(t)$ , so

$$\begin{aligned} S_m &= - \int_0^1 \frac{Q'_m(t)}{Q_m(t)} \log(1 - t) dt - \int_0^1 \frac{Q'_m(t)}{Q_m(t)} \log(Q_m(t)) dt \\ &= - \int_0^1 \frac{Q'_m(t)}{Q_m(t)} \log(1 - t) dt - \left[ \frac{1}{2} \log^2 Q_m(t) \right]_0^1. \end{aligned}$$

Finally,

$$(2) \quad S_m = -\frac{1}{2} \log^2 m + J_m \quad \text{with} \quad J_m = - \int_0^1 \frac{Q'_m(t)}{Q_k(t)} \log(1 - t) dt,$$

and the problem is reduced to evaluating the integral  $J_m$ . Let  $\omega$  denote the  $m^{\text{th}}$  root of unity:  $e^{2\pi i/m}$ . Then  $Q_m(t) = \prod_{j=1}^{m-1} (t - \omega^j)$  and

$$\frac{Q'_m(t)}{Q_m(t)} = \sum_{j=1}^{m-1} \frac{1}{t - \omega^j}.$$

It follows that

$$J_m = \sum_{j=1}^{m-1} \int_0^1 \frac{\log(1-t)}{\omega^j - t} dt = \sum_{j=1}^{m-1} \int_0^1 \frac{\log(1-t)}{\omega^{m-j} - t} dt,$$

hence

$$(3) \quad 2J_m = \sum_{j=1}^{m-1} \left( \int_0^1 \frac{\log(1-t)}{\omega^j - t} dt + \int_0^1 \frac{\log(1-t)}{\omega^{-j} - t} dt \right).$$

We will use the following lemma, the proof of which is postponed to the end.

LEMMA. Let  $\Omega = \mathbb{C} \setminus [0, +\infty[$ , that is, the set of complex numbers with a cut along the set of nonnegative real numbers. For  $z$  in  $\Omega$  we define  $F(z)$  by

$$F(z) = \int_0^1 \frac{\log(1-t)}{z-t} dt.$$

Then  $F$  satisfies the functional equation

$$\forall z \in \Omega, \quad F(z) + F\left(\frac{1}{z}\right) = \frac{\pi^2}{6} - \text{Log}(1-z) \text{Log}\left(1 - \frac{1}{z}\right),$$

where  $\text{Log}$  is the principal branch of the logarithm.

With the notation of the lemma we can write (3) as follows:

$$\begin{aligned} 2J_m &= \sum_{j=1}^{m-1} \left( F(\omega^j) + F\left(\frac{1}{\omega^j}\right) \right) \\ &= \frac{(m-1)\pi^2}{6} - \sum_{j=1}^{m-1} \text{Log}(1 - \omega^j) \text{Log}(1 - \bar{\omega}^j) \\ &= \frac{(m-1)\pi^2}{6} - \sum_{j=1}^{m-1} |\text{Log}(1 - \omega^j)|^2. \end{aligned}$$

For  $1 \leq j < m$  we have  $1 - \omega^j = 2 \sin(j\pi/m)e^{i\pi/m - \pi/2}$ , and consequently  $\text{Log}(1 - \omega^j) = \log(2 \sin(j\pi/m)) + i\pi \left(\frac{j}{m} - \frac{1}{2}\right)$ . Therefore,

$$2J_m = \frac{(m-1)\pi^2}{6} - \sum_{j=1}^{m-1} \log^2 \left( 2 \sin \frac{j\pi}{m} \right) - \pi^2 \sum_{j=1}^{m-1} \left( \frac{j}{m} - \frac{1}{2} \right)^2,$$

and

$$\begin{aligned} \sum_{j=1}^{m-1} \left( \frac{j}{m} - \frac{1}{2} \right)^2 &= \frac{1}{m^2} \cdot \frac{(m-1)m(2m-1)}{6} - \frac{1}{m} \cdot \frac{(m-1)m}{2} + \frac{m-1}{4} \\ &= \frac{(m-1)(m-2)}{12m}, \end{aligned}$$

hence

$$(4) \quad J_m = \frac{(m-1)(m+2)\pi^2}{24m} - \frac{1}{2} \sum_{j=1}^{m-1} \log^2(2 \sin(j\pi/m)).$$

Clearly (1) follows from (2) and (4). □

*Proof of the lemma.* Note first that both  $F$  and  $z \mapsto \text{Log}(1 - z)$  are holomorphic in the connected region  $\Omega$ , and since  $\Omega$  is invariant under the holomorphic mapping  $z \mapsto 1/z$ , we conclude that

$$z \mapsto G(z) = F(z) + F\left(\frac{1}{z}\right) + \text{Log}(1 - z) \text{Log}\left(1 - \frac{1}{z}\right)$$

is holomorphic in  $\Omega$ , so to prove the lemma, we have to prove only that  $G(x) = \pi^2/6$  for each negative real  $x$ . Now for  $x \in (-\infty, 0)$  we have (integrating by parts)

$$\begin{aligned} F'(x) &= - \int_0^1 \frac{\log(1-t)}{(t-x)^2} dt = \left[ \left( \frac{1}{t-x} - \frac{1}{1-x} \right) \log(1-t) \right]_0^1 + \frac{1}{1-x} \int_0^1 \frac{dt}{t-x} \\ &= \frac{1}{1-x} \log\left(1 - \frac{1}{x}\right), \end{aligned}$$

and consequently  $G'(x) = 0$  for every negative real  $x$ . This proves that, for some constant  $c$ , we have  $G(x) = c$  for all  $x$  in the interval  $(-\infty, 0)$ . Now, letting  $x$  tend to  $0^-$ , and noting that  $\lim_{x \rightarrow -\infty} F(x) = 0$ , we conclude that

$$c = F(0) = \int_0^1 \frac{-\log(1-t)}{t} dt = \int_0^1 \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} dt = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 t^{n-1} dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This concludes the proof of the lemma. □

*Solution 2 by THE EDITOR*<sup>2</sup> (University of Memphis, Memphis, TN). In view of the asymptotic relation  $H_n = \log n + \gamma + O(1/n)$ , it is evident that the series defining  $S_m$  does indeed converge. Let

$$F(z) := \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n} z^n.$$

Then  $F$  is analytic in the unit disc and  $F(1) = S_m$ . By Abel's Limit Theorem [1, p. 42],  $S_m = \lim_{x \uparrow 1} F(x)$ . It remains to find a suitable representation of  $F$  and calculate the required limit. Start with the well-known series<sup>3</sup>

$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} z^{n+1} = \frac{1}{2} \text{Log}^2(1-z), \quad |z| < 1,$$

where  $\text{Log } z = \log |z| + i \text{Arg } z$  denotes the principal branch of the logarithm. Then for  $|z| < 1$ ,

$$G(z) := \sum_{n=1}^{\infty} \frac{H_n}{n} z^n = z + \sum_{n=1}^{\infty} \frac{H_n + 1/(n+1)}{n+1} z^{n+1} = \frac{1}{2} \text{Log}^2(1-z) + \text{Li}_2(z),$$

where  $\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$  is the dilogarithm.

By the Multisection Formula [2, pp. 97–98], if  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , then

$$\mathcal{M}_m[f(z)] := \sum_{n=0}^{\infty} c_{mn} z^{mn} = \frac{1}{m} \sum_{k=0}^{m-1} f(\omega^k z), \quad \omega = e^{2\pi i/m}.$$

In view of the relation

$$m \sum_{n=1}^{\infty} \frac{H_{mn}}{mn} z^n = m \mathcal{M}_m[G(\sqrt[m]{z})],$$

it follows that

$$F(z) = -\log m \text{Log}(1-z) - m \mathcal{M}_m[G(\sqrt[m]{z})] + G(z), \quad |z| < 1.$$

Split  $F(z)$  into “continuous” and “singular” terms according to their behavior near  $z = 1$ . The contribution to  $S_m$  of a continuous term is simply its value at  $z = 1$ . For example, the dilogarithm terms in  $G(z)$  and  $-m \mathcal{M}_m[G(\sqrt[m]{z})]$  together contribute

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - m \sum_{n=1}^{\infty} \frac{1}{(mn)^2} = \left(\frac{m-1}{m}\right) \text{Li}_2(1) = \left(\frac{m-1}{m}\right) \frac{\pi^2}{6}.$$

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<sup>3</sup>Differentiate  $\frac{1}{2} \text{Log}^2(1-z)$  to get  $\sum_{n=1}^{\infty} H_n z^n$  and then integrate.

The Multisection Formula gives

$$m\mathcal{M}_m \left[ \frac{1}{2} \text{Log}^2(1 - \sqrt[m]{z}) \right] = \frac{1}{2} \sum_{k=0}^{m-1} \text{Log}^2(1 - \omega^k \sqrt[m]{z}).$$

The  $k = 0$  term, namely  $\frac{1}{2} \text{Log}^2(1 - \sqrt[m]{z})$ , is singular and the remaining  $m - 1$  terms are continuous. The contribution to  $S_m$  from the latter is  $-\frac{1}{2} \sum_{k=1}^{m-1} \text{Log}^2(1 - \omega^k)$ .

It is necessary to calculate the limit of the sum of all singular terms as  $x \uparrow 1$ :

$$L = \lim_{x \uparrow 1} \left\{ -\log m \log(1 - x) + \frac{1}{2} \log^2(1 - x) - \frac{1}{2} \log^2(1 - \sqrt[m]{x}) \right\}.$$

To this end, set  $x = 1 - e^{-u}$ . Then  $x \uparrow 1$  as  $u \rightarrow \infty$  and

$$\begin{aligned} L &= \lim_{u \rightarrow \infty} \left\{ u \log m + \frac{1}{2} u^2 - \frac{1}{2} \log^2(1 - (1 - e^{-u})^{1/m}) \right\} \\ &= \lim_{u \rightarrow \infty} \left\{ u \log m + \frac{1}{2} u^2 - \frac{1}{2} \log^2(e^{-u}/m + O(e^{-2u})) \right\} \\ &= \lim_{u \rightarrow \infty} \left\{ u \log m + \frac{1}{2} u^2 - \frac{1}{2} (u + \log m)^2 + O(ue^{-u}) \right\} \\ &= \lim_{u \rightarrow \infty} \left\{ -\frac{1}{2} \log^2 m + O(ue^{-u}) \right\} = -\frac{1}{2} \log^2 m. \end{aligned}$$

Assembling all contributions, we have

$$(1) \quad S_m = -\frac{\log^2 m}{2} + \left( \frac{m-1}{m} \right) \frac{\pi^2}{6} - \frac{1}{2} \sum_{k=1}^{m-1} \text{Log}^2(1 - \omega^k).$$

To simplify the formula for  $S_m$ , note that the principal logarithm of  $1 - \omega^k$  is

$$\text{Log}(1 - \omega^k) = \log |1 - \omega^k| + i \text{Arg}(1 - \omega^k) = \log \left( 2 \sin \frac{k\pi}{m} \right) + i\pi \left( \frac{k}{m} - \frac{1}{2} \right).$$

Hence

$$\begin{aligned} \sum_{k=1}^{m-1} \text{Log}^2(1 - \omega^k) &= \text{Re} \sum_{k=1}^{m-1} \text{Log}^2(1 - \omega^k) \\ &= \sum_{k=1}^{m-1} \log^2 \left( 2 \sin \frac{k\pi}{m} \right) - \pi^2 \sum_{k=1}^{m-1} \left( \frac{k}{m} - \frac{1}{2} \right)^2 \\ &= \sum_{k=1}^{m-1} \log^2 \left( 2 \sin \frac{k\pi}{m} \right) - \pi^2 \frac{(m-1)(m-2)}{12m}. \end{aligned}$$

Substitution of this result into (1) yields

$$S_m = \frac{(m-1)(m+2)}{24m} \pi^2 - \frac{1}{2} \log^2 m - \frac{1}{2} \sum_{k=1}^{m-1} \log^2 \left( 2 \sin \frac{k\pi}{m} \right).$$

## REFERENCES

- [1] L. V. AHLFORS, *Complex Analysis*, McGraw-Hill, New York, 1966.
- [2] Z. A. MELZAK, *Companion to Concrete Mathematics*, John Wiley & Sons, New York, 1973.

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*Editorial note.* The approach of THOMAS A. DICKENS was similar to that in *Solution 1*. In addition to formal analysis, DICKENS used an integer-relation detection algorithm to study numerically the possibility of representing  $S_m$  by a linear combination of  $\pi^2$  and  $\log^2 m$  with rational coefficients. Using an integral representation of  $S_m$ , he applied the PSLQ algorithm to look for integers  $a, b, c$  such that  $aS_m + b\pi^2 + c\log^2 m = 0$ . Using 500-digit arithmetic, this method gave the known results for  $m = 2, 3$ , and  $4$  and found no solutions for  $m \geq 5$ . NGUYEN VAN VINH and NGO PHUOC NGUYEN NGOC first show that

$$S_m = \left( \frac{m-1}{m} \right) \frac{\pi^2}{6} - \frac{1}{m} \int_0^1 \left( \sum_{k=1}^{m-1} t^{k/m} - (m-1) \right) \frac{\log(1-t)}{t(1-t)} dt$$

and then use computer algebra to evaluate the integral for specific cases.