A Multiple Fractional Integral

Problem 07-002, *by* OVIDIU FURDUI[1] (Western Michigan University).

Let $n \geq 1$ be a natural number and let I_n be the integral defined by

$$I_n = \int_0^1 \cdots \int_0^1 \left\{ \frac{1}{x_1 x_2 \cdots x_n} \right\} dx_1 dx_2 \cdots dx_n,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the *fractional part* of a.

(a) Prove that

$$I_2 = \int_0^1 \int_0^1 \left\{ \frac{1}{xy} \right\} dx dy = 1 - \gamma - \gamma_1,$$

where $\gamma_1 = \lim_{n\to\infty}(\sum_{k=1}^n \frac{\ln k}{k} - \frac{\ln^2 n}{2})$ and $\gamma = \lim_{n\to\infty}(H_n - \ln n)$.

(b) For $n \geq 3$ evaluate I_n. By elementary calculations, it can be shown that

$$I_1 = \int_0^1 \left\{ \frac{1}{x} \right\} dx = 1 - \gamma.$$

With this result and the one found in part (a) in mind, it is natural to conjecture that I_n, $n \geq 3$, can be expressed in terms of the *Stieltjes constants*, γ_m, given by

$$\gamma_m = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{\ln^m k}{k} - \frac{\ln^{m+1} n}{m+1} \right).$$

Note that $\gamma_0 = \gamma$ is the Euler–Mascheroni constant. Prove or disprove this conjecture.

Status. The proposer has a solution of part (a). Additional solutions are welcome. Part (b) is open.

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