IP10 The Mysterious Infinity Laplacian

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The ∞-Laplacian

$(L^\infty P)$ $\min_u \|Du\|_{L^\infty(U)} \ni u = \varphi$ on ∂U

? Euler Equation ?

Approx by L^p

$(L^p P)$ $\min_u \int_U |Du|^p dx \ni u = \varphi$ on ∂U

$$-\Delta_p u = -\mathrm{div}(|Du|^{p-2} Du)$$
$$= -|Du|^{p-2}\Delta u - (p-2)|Du|^{p-4} \underbrace{u_{x_j} u_{x_k} u_{x_j x_k}}_{\Delta_\infty u}$$
$$= 0$$

$$-\frac{1}{p-2}\frac{1}{|Du|^{p-4}}\Delta_p u = -\frac{|Du|^2 \Delta u}{p-2} - \Delta_\infty u = 0$$

$p \to \infty$

$$-\Delta_\infty u = -\sum_{j,k} u_{x_j} u_{x_k} u_{x_j, x_k} = 0$$

Justified by "viscosity sol'n" theory

$-\Delta_p u_p = 0$ & $u_p \to u \Rightarrow -\Delta_\infty u = 0.$

Functional in $(L^\infty P)$ <u>not</u> strictly convex, sol'ns not unique

$u = \varphi$ on ∂U

$|\varphi(x) - \varphi(y)| \le L|x-y|, \; x, y \in \partial U$

$|u(x) - u(y)| \le L|x-y|, \; x, y \in \bar{U}$

$\Leftrightarrow \|Du\|_{L^\infty(U)} \le L.$

U convex (Simple case)

$$-L|x-y| \le u(x) - \varphi(y) \le L|x-y|, \; y \in \partial U$$

$$\bar{u}(x) = \min_{y \in \partial U} \varphi(y) + L|x-y| \quad \text{Largest}$$

$$\underline{u}(x) = \max_{y \in \partial U} \varphi(y) - L|y-x| \quad \text{Smallest}$$

E. J. McShane 1934

"Absolutely Minimizing Lipschitz"
(AML)

$\forall\, V \subset\subset U$ & $v = u$ on ∂V

$$\|Du\|_{L^\infty(V)} \leq \|Dv\|_{L^\infty(V)}$$

G. Aronsson 1967, 1968, 1986

R. JENSEN 1993

$$-\Delta_\infty u = 0 \iff u \text{ (AML)}.$$

Bhattacharya, DiBenedetto, Manfredi 1989
($p \to \infty$ in $\Delta_p u_p = f$)

Barron, Jensen, Wang preprints

Barron review 1999

Juutinen, Wu

Juutinen, Lindquist, Manfredi

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Barles & Busca, in preparation

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Even with all this work

$? \; -\Delta_\infty u = 0 \Rightarrow u \in C^1 \; ?$

Unknown.

Properties of u if smooth $+ -\Delta_\infty u = 0$

$\dot{X} = Du(X) \Rightarrow$

$$\frac{d}{dt} |Du(X)|^2 = \Delta_\infty u(X) = 0$$

$|Du(X)|$ is Constant.

$n=2$ then if u is not Constant Du Can never vanish + Global $\Rightarrow$ linear

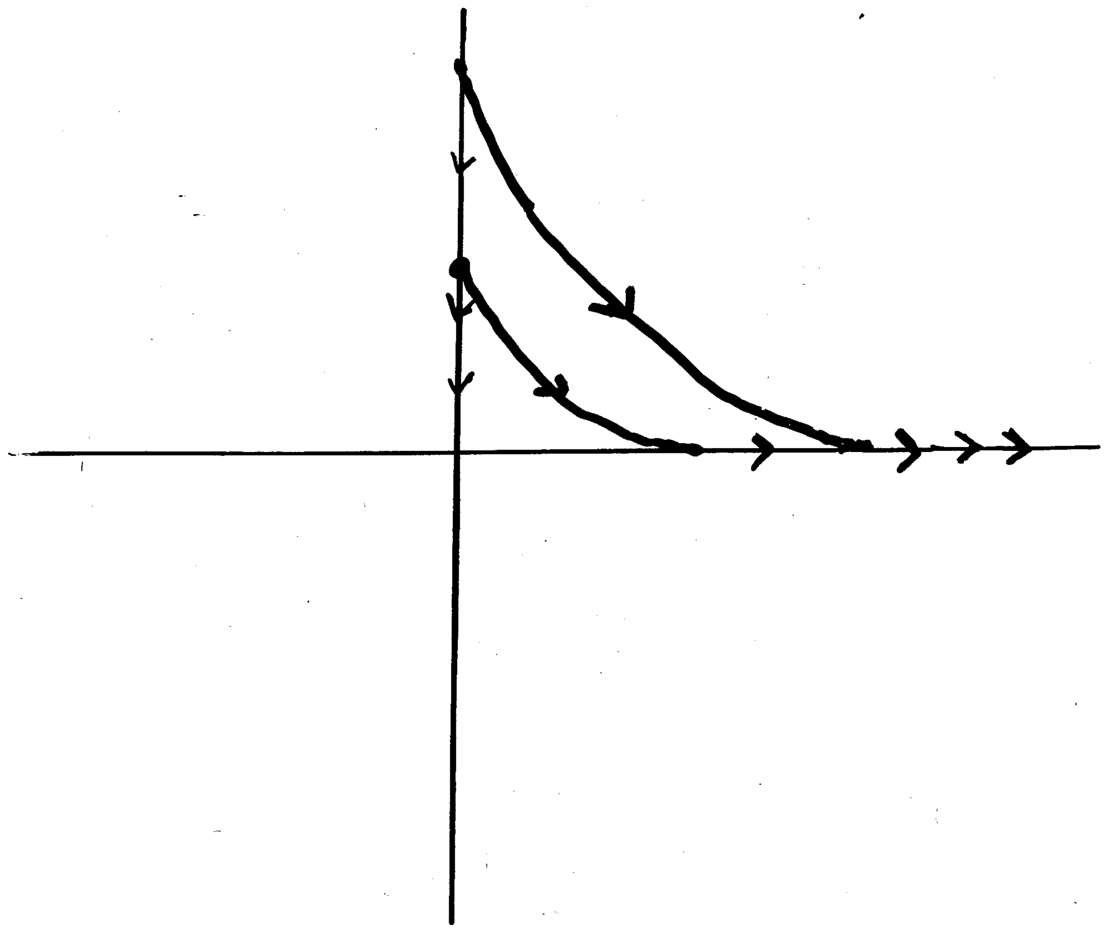
Aronsson

Example: $u = x_1^{4/3} - x_2^{4/3}$ solves $-\Delta_\infty u = 0$ in $\mathbb{R}^2$

All properties above fail for $x_1^{4/3} - x_2^{4/3}$.

3.5

$\dot{x} = \frac{4}{3} x^{1/3}, \dot{y} = -\frac{4}{3} y^{1/3}$



Example: "Cone functions"

$C(x) = a + b|x - x_0|$

$|DC| = |b|$ for $x \neq x_0$

$\Rightarrow -\Delta_\infty C = 0$ for $x \neq x_0$.

"Comparison with cones:"

$u \leq C$ on $\partial(V \setminus \{x_0\})$

$\Rightarrow u \leq C$ in V

- this is "from above".
- Similarly define from below
- "u enjoys comparison with cones" if comparison holds from above + from below

<u>C, Evans, Gariepy</u>

$-\Delta_\infty u = 0 \Leftrightarrow u$ enjoys comparison with cones. (Jensen)

$$u(x)\begin{cases} \geq u(y) + \Big(\overbrace{\min_{|z-y|=r} \frac{u(z)-u(y)}{r}}^{=:\, L_r^-(y)}\Big)|x-y| \\[2ex] \leq u(y) + \Big(\underbrace{\max_{|z-y|=r} \frac{u(z)-u(y)}{r}}_{=:\, L_r^+(y)}\Big)|x-y| \end{cases}$$

$\underbrace{L_r^+(y)}_{\geq 0} \nearrow$ as $r \nearrow$, $\underbrace{L_r^-(y)}_{\leq 0} \searrow$ as $r \nearrow$

$L^+(y) = \lim_{r \downarrow 0} L_r^+(y)$, $L^-(y) = \lim_{r \downarrow 0} L_r^-(y)$

$-L^-(y) = L^+(y)$ proxy for $|Du(y)|$

$$\|Du\|_{L^\infty(V)} = \max_{y \in V} L^+(y)$$

- u enjoys comparison with cones $\Rightarrow$ u AML

- u enjoys comparison with cones from above in $\mathbb{R}^n$
  $u \le$ linear $\Rightarrow$ $u \equiv$ linear
  $\Rightarrow$ if $D^+u(y)$ nonempty, $Du(y)$ exists.

(CEG)

(CE) $r_j \downarrow 0$

$$V(x) = \lim_{j \to \infty} \frac{u(r_j x + x_0) - u(x_0)}{r_j}$$

$\Rightarrow$ V linear.

Consequence

$L^+(x_0) = 0 \Rightarrow Du(x_0) = 0$

$L^+(x_0) > 0 \Rightarrow Du(x_0)$ exists

iff

$|z_r - x_0| = r$ + $u(z_r) = \max_{|z-x_0|=r} u(z)$

$\Rightarrow \lim_{r \downarrow 0} \frac{z_r - x_0}{r}$ exists.

<u>Preiss</u> $n = 1$ every

$v(x) = \lim_{j\to\infty} \frac{u(r_j x) - u(0)}{r_j}$

is linear if $u(x) = x \sin(\log(|\log|x||))$

but u not diff'ble @ 0.

$x_0 = 0$

- v enjoys comparison with cones

- $\pm L^{\pm}_{r,v}(y) \le L^{+}_{v}(0) = L^{+}_{u}(0)$

$$=: L_0$$

least Lip const of v

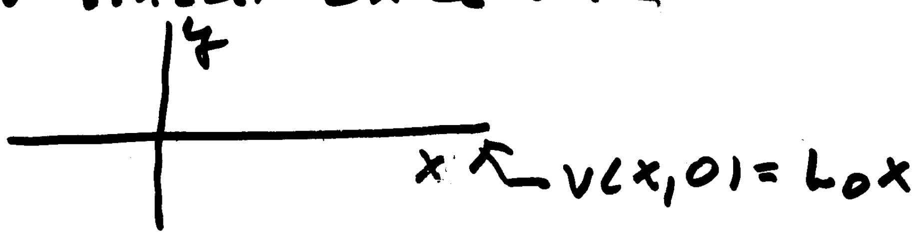
- $|z_r^{\pm}| = r, \quad v(z_r^{\pm}) = L^{\pm}_{r,v}(0)$

$$\Rightarrow \quad v(z_r^{+}) - v(z_r^{-}) = L_0 \, 2r$$

$$\Rightarrow \quad z_r^{+} = -z_r^{-} \quad \&$$

v linear on $[z_r^{-}, z_r^{+}]$

- v linear on a line



$$|v(x,y) - v(s,0)|^2$$

$$= |v(x,y) - L_0 x + L_0 x - L_0 s|^2$$

$$= |v(x,y) - L_0 x|^2 + 2L_0(x-s)(v(x,y) - L_0 x) + \underline{L_0 |x-s|^2}$$

$$\leq L_0 |y|^2 + \underline{L_0 |x-s|^2}$$

$$2L_0(x-s)(v(x,y) - L_0 x) \leq L_0 |y|^2$$

s free $\Rightarrow v(x,y) \equiv L_0 x.$