Permanents of Circulants: a Transfer Matrix Approach*
(Extended Abstract)

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Abstract

Calculating the permanent of a $(0,1)$ matrix is a $\#P$-complete problem but there are some classes of *structured* matrices for which the permanent is calculable in polynomial time. The most well-known example is the *fixed-jump* $(0,1)$ circulant matrix which, using algebraic techniques, was shown by Minc to satisfy a constant-coefficient fixed-order recurrence relation.

In this note we show how, by interpreting the problem as calculating the number of cycle-covers in a directed circulant *graph,* it is straightforward to reprove Minc's result using combinatorial methods. This is a two step process: the first step is to show that the cycle-covers of directed circulant graphs can be evaluated using a *transfer matrix* argument. The second is to show that the associated transfer matrices, while very large, actually have much smaller characteristic polynomials than would a-priori be expected.

An important consequence of this new viewpoint is that, in combination with a new recursive decomposition of circulant-graphs, it permits extending Minc's result to calculating the permanent of the much larger class of circulant matrices with *non-fixed* (but linear) jumps.

1 Introduction

Definition 1.1. *Let $A = (a_{i,j})$ be an $n \times n$ matrix. Let S_n be the set of permutations of the integers $[1, \ldots, n]$. The* permanent *of A is*

$$\mathrm{Perm}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)} \quad \textit{where} \quad \pi = [\pi(1), \ldots, \pi(n)].$$

If A is a $(0,1)$ matrix, then A can be interpreted as the adjacency matrix of some directed graph G and $\mathrm{Perm}(A)$ is the number of *directed cycle-covers* in G, where a directed cycle-cover is a collection of disjoint cycles that cover all of the vertices in the graph. Alternatively, A can be interpreted as the adjacency matrix of a bipartite graph $\bar{G}$, in which case $\mathrm{Perm}(A)$ is the number of *perfect-matchings* in $\bar{G}$. The permanent is a classic well-studied combinatorial object (see the book and later survey by Minc[13, 16]).

Calculating the permanent of a $(0,1)$ matrix is a $\#P$-Complete problem [19] even when A is restricted to have only 3 non-zero entries per row [8]. The best known algorithm for calculating a general permanent is a straightforward inclusion-exclusion technique due to Ryser [13] running in $\Theta(n2^n)$ time and polynomial space. By allowing super-polynomial space, Bax and Franklin [1] developed a slightly faster (although still exponential) algorithm for the $(0,1)$ case. We point out, in another direction, that just recently, Jerrum, Sinclair and Vigoda [11] developed a fully polynomial approximation scheme for *approximating* the permanent of nonnegative matrices.

On the other hand, for certain special structured classes of matrices one can *exactly* calculate the permanent in "polynomial time". The most studied example of such a class is probably the *circulant matrices,* which, as discussed in [7], can be thought of as the borderline between the easy and hard cases.

An $n \times n$ circulant matrix $A = (a_{i,j})$ is defined by specifying its first row; the $(i+1)^{\mathrm{st}}$ row is a cyclic shift i units to the right of the first row, i.e., $a_{i,j} = a_{1,1+(n+j-i) \bmod n}$. Let P_n denote the $(0,1)$ $n \times n$ matrix with **1**s in positions $(i, i+1)$, $i = 1, \ldots, n-1$, and $(n, 1)$ and **0**s everywhere else. Many of the early papers on this topic express circulant matrices in the form

$$A_n = a_1 P_n^{s_1} + a_2 P_n^{s_2} + \cdots + a_k P_n^{s_k} \tag{1.1}$$

where $0 \le s_1 < s_2 < \cdots < s_k < n$ and $a_i = a_{1,s_i+1}$.

The first major result on permanents of $(0,1)$ circulants is due to Metropolis, Stein and Stein [12]. Let $k > 0$ be fixed and $A_{n,k} = \sum_{i=0}^{k-1} P_n^i$, be the $n \times n$ circulant matrix whose first row is composed of **1**s in its first k columns and **0**s everywhere else. Then [12] showed that, as a function of n, $\mathrm{Perm}(A_{n,k})$ satisfies a fixed order constant-coefficient recurrence relation in n

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and therefore, could be calculated in polynomial time in n (after a superpolynomial "start-up cost" in k for deriving the recurrence relation).

This result was greatly improved by Minc who showed that it was only a very special case of a general rule. Let $0 \le s_1 < s_2 < \cdots < s_k < n$ be *any* fixed sequence and set $A_n = A_n(s_1, \ldots, s_k) = P_n^{s_1} + P_n^{s_2} + \cdots + P_n^{s_k}$. In [14, 15] Minc proved that $\mathrm{Perm}(A_n)$ always satisfies a constant-coefficient recurrence relation in n of order $2^{s_k} - 1$. Minc's theorem was proven by manipulating algebraic properties of A_n. Note, that as mentioned by Minc, this result is difficult to apply for large s_k since, in order to derive the coefficients of the recurrence relation it is first necessary to evaluate $\mathrm{Perm}(A_n)$ for $n \le 2(2^{s_k} - 1)$ and, using, Ryser's algorithm, this requires $\Omega\left(2^{2^{s_k}}\right)$ time.

Later Codenotti, Resta and various coauthors improved these results in various ways; e.g. in [2] showing how to evaluate *sparse* circulant matrices of size ≤ 200; in [4, 5] showing that the permanents of circulants with only three $\mathbf{1}$s per row can be evaluated in polynomial time; in [6] showing how the permanents of some special sparse circulants can be expressed in terms of determinants and are therefore solvable in polynomial time; in [2] showing that the permanents of *dense* circulants are hard to calculate and in [7] that even approximating the permanent of an arbitrary circulant modulo a prime p is "hard" unless $\mathbf{P}^{\#\mathbf{P}} = \mathbf{BPP}$.

In this paper we return to the original problem of Minc. Our first main result will be to show that if circulant *matrix* $A_n(s_1, \ldots, s_k)$ is interpreted as the adjacency matrix of a directed circulant *graph* C_n, then counting the number of cycle-covers of C_n using a *transfer matrix* approach immediately reproves Minc's result. As well as reproving Minc's original result this new technique will then permit us extend the result to a much larger set of circulant graphs as well as address other related problems. To explain, we first need to introduce some notation.

Definition 1.2. *See Figure 1. Let $C_n^{s_1,s_2,\cdots,s_k}$. be the n-node* directed circulant graph *with jumps $S = \{s_1, s_2, \ldots s_k\}$. (Note that in this definition we allow negative s_i.) Formally,*

$$C_n^{s_1,s_2,\ldots,s_k} = (V(n), E_C(n))$$

where

$$V(n) = \{0, 1, \ldots, n-1\}$$

and

$$E_C(n) = \Big\{(i,j) : (j-i) \bmod n \in S\Big\}.$$

Note: we will assume that S contains at least one non-negative s_i since if all the s_i were negative we could multiply them by -1 and get an isomorphic graph. Also, we will often write C_n as shorthand for $C_n^{s_1,s_2,\cdots,s_k}$.

Let $G = (V, E)$ be a graph, $T \subseteq E$ and $v \in V$. Define $\mathtt{ID}_T(v)$ to be the *indegree* of v in graph (V, T) and $\mathtt{OD}_T(v)$ to be the *outdegree* of v in (V, T). $T \subseteq E$ is a *cycle-cover* of G if

$$\forall v \in V, \quad \mathtt{ID}_T(v) = \mathtt{OD}_T(v) = 1.$$

Definition 1.3. *Let $S = \{s_1, s_2, \ldots s_k\}$ be given. Set*

$$\mathcal{CC}(n) = \{T \subseteq C_n \ : \ T \text{ is a cycle-cover of } C_n\}$$

and

$$T(n) = |\mathcal{CC}(n)| = \text{No. of cycle-covers of } C_n.$$

Note that, by the standard correspondence mentioned before, $A_n(s_1, \ldots, s_k)$ is the adjacency matrix of $C_n^{s_1,s_2,\cdots,s_k}$ and $T(n) = \mathrm{Perm}(A_n(s_1, \ldots, s_k))$. So, calculating $T(n)$ is equivalent to calculating permanents of $A_n(s_1, \ldots, s_k)$.

In [9, 10] the authors of this paper were interested in counting spanning trees and other structures in *undirected circulant* graphs. The main tool introduced there was a recursive decomposition of such graphs. In Section 2 we describe a related recursive decomposition of *directed circulant* graphs. Our technique will be to use this decomposition to show that for some constant m there is a $m \times 1$ (column) vector function $\bar{T}(n)$ such that

$$\forall n \ge 2\bar{s}, \quad T(n) = \beta\, \bar{T}(n) \quad \text{and} \quad \bar{T}(n+1) = A\, \bar{T}(n) \tag{1.2}$$

where $\bar{s}$ is a constant to be defined later (but reduces to $\bar{s} = s_k$ for the Minc formulation described previously), β is a $1 \times m$ constant row-vector and A is a constant $m \times m$ matrix. Such an A is known as a *transfer-matrix* see, e.g., [18].

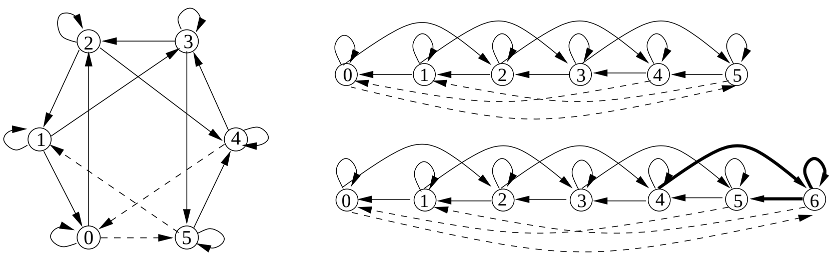
Let $P(x) = \sum_{i=0}^{t} p_i x^i$ be any polynomial that annihilates A, i.e., $P(A) = 0$. Then it is easy to see that $\forall n \ge 2\bar{s}$,

$$\begin{aligned}
\sum_{i=0}^{t} p_i T(n+i) &= \beta \left(\sum_{i=0}^{t} p_i A^{n+i-2\bar{s}} \right) \bar{T}(2\bar{s}) \\
&= \beta\, A^{n-2\bar{s}} \left(\sum_{i=0}^{t} p_i A^i \right) \bar{T}(2\bar{s}) \\
&= \beta\, A^{n-2\bar{s}}\, \mathbf{0}\, \bar{T}(2\bar{s}) \\
&= 0
\end{aligned}$$

where $\mathbf{0}$ denotes the $m \times m$ zero matrix and 0 a scalar; $T(n)$ thus satisfies the degree-t constant coefficient recurrence relation $T(n+t) = \sum_{i=0}^{t-1} -\frac{p_i}{p_t} T(n+i)$ in n. By

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) $P_n^{-1} + I + P_n^2$ $(n = 6)$



(b) $C_n^{-1,0,2}$ $(n = 6)$ (c) $C_n^{-1,0,2}$ $(n = 6, 7)$

Figure 1: $C_n^{-1,0,2}$: Circulant matrix (a) is the adjacency matrix of circulant graph $C_6^{-1,0,2}$ in (b). In (b), the solid edges are L_n and the dashed edges are $\texttt{Hook}(n)$. (c) illustrates $C_n^{-1,0,2}$ $(n = 6, 7)$ drawn in lattice graph representation. The bold edges in $C_7^{-1,0,2}$ are $\texttt{New}(n)$. Note that the $\texttt{Hook}(n)$ edges for $n = 6, 7$ are "independent" of n.

the Cayley-Hamilton theorem, the characteristic polynomial of A, which has degree $\leq m$, must annihilate A, so such a polynomial exists and $T(n)$ satisfies a recurrence relation of at most degree m. In our notation, Minc's theorem is that $T(n)$ satisfies a recurrence relation of degree $2^{\bar{s}} - 1$. Unfortunately, in our construction, $m = 2^{2\bar{s}}$ so the characteristic polynomial does not suffice for our purposes. Our next step will involve showing that even though A is of size $2^{2\bar{s}} \times 2^{2\bar{s}}$, there is a much smaller P, of degree $2^{\bar{s}} - 1$, that annihilates A, thus reproving Minc's theorem. We point out that this degree reduction of the transfer matrix (to the square-root of the original size) is, a-priori, quite unexpected, and does not occur in the undirected-circulant counting problems analyzed in [9, 10].

One interesting consequence of this new derivation is that, unlike in Minc's proof, to derive the recurrence relation it is no longer necessary to start by spending $\Omega\left(2^{2^{\bar{s}}}\right)$ time calculating the first $2^{\bar{s}}$ values of $T(n)$ using Ryser's method. Instead one only has to calculate A, β, the polynomial P and the first $2^{\bar{s}}$ values of $\bar{T}(n)$ which, as we will see later, can all be done in $O(\bar{s}2^{5\bar{s}})$ time, reducing the start-up complexity from doubly-exponential in $\bar{s}$ to singularly exponential.

Another, albeit minor, consequence of this new derivation is that it can also handle non $(0, 1)$ circulants. That is, given *any* matrix A_n of the form (1), even when the a_i are not restricted to be in $\{0, 1\}$ the technique shows that $\mathrm{Perm}(A_n)$ satisfies a recurrence relation of degree $2^{\bar{s}} - 1$. This is only a minor consequence, though, since working through the details of Minc's original proof it is possible to modify it to get the same result.

A much more important new consequence, and the major motivation for this paper, is the fact that the proof can be extended to evaluate the permanents of *non-constant* jump circulant matrices, something which has not been addressed before. To explain this, we generalize Definition 1.2 to

Definition 1.4. *See Figure 2(a). Let $p, s, p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be fixed integral constants with such that $\forall i$, $0 \leq p_i < p$. Set $S = \{p_1 n + s_1, p_2 n + s_2, \cdots, p_k n + s_k\}$. Denote the $(pn + s)$-node* directed circulant graph *with jumps S by*

$$C_n = C_{pn+s}^{p_1 n + s_1, p_2 n + s_2, \cdots, p_k n + s_k} = (V(n), E_C(n))$$

where

$$V(n) = \{0, 1, \ldots, pn + s - 1\}$$

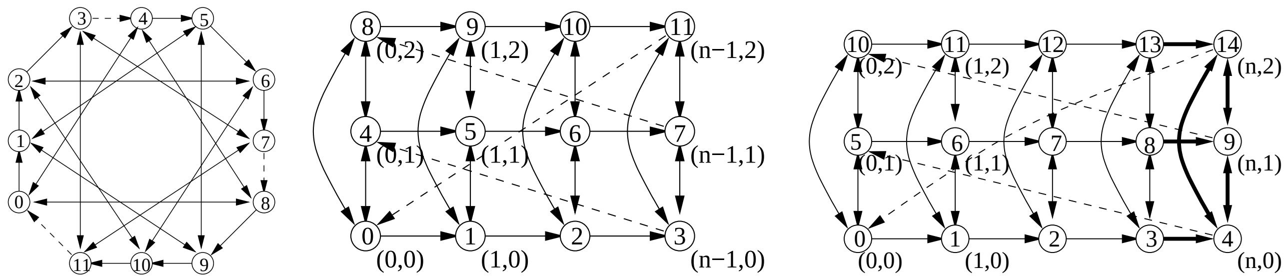
and

$$E_C(n) = \Big\{(i, j) : (j - i) \bmod (pn + s) \in S\Big\}.$$

Note that $A_{pn+s}(p_1 n + s_1, p_2 n + s_2, \cdots, p_k n + s_k)$ is the adjacency matrix of C_n so, counting the cycle-covers in C_n is equivalent to evaluating $\mathrm{Perm}(A_{pn+s}(p_1 n + s_1, p_2 n + s_2, \cdots, p_k n + s_k))$. Our method of counting the cycle covers in C_n will be to derive a new recursive decomposition of C_n (which might be of independent interest) and use it to show that an analogue of (1.2) holds in the non-constant jump case as well; thus $T(n)$ still satisfies a constant-coefficient recurrence relation in n. For example, in Table 1, we show the recurrence relation for the number of cycle covers in $C_{3n}^{1,n+1,2n}$ and $C_{3n}^{0,n,2n-1}$.

In the next section we describe the new recursive decompositions of C_n, for both constant and non-constant jumps, upon which our technique is based. In Section 3 we show how this permits easily reproving Minc's result for non-constant circulants. We then describe the minor modifications that are needed to extend the proof to non-constant circulants.

Note: Due to space limitations in this extended abstract only the proof skeleton is given, with many



(a) $C_{3n}^{1,n,2n}$ $(n = 4)$ (b) $C_{3n}^{1,n,2n}$ $(n = 4)$ (c) $C_{3n}^{1,n,2n}$ $(n = 5)$

Figure 2: $C_{3n}^{1,n,2n}$, a non-constant jump circulant: Dashed edges are $\texttt{Hook}(n)$. Solid edges are L_n. (a) and (b) are two representations of the graph when $n = 4$. Note the lattice representation in (b). (c) is is the case $n = 5$. The bold solid edges on the right are $\texttt{New}(n)$. The 3 vertices on the right are $VN(n)$. Note that the dashed $\texttt{Hook}(n)$ edges for both $n = 4, 5$ are "independent" of n.

$C_n^{-1,0,1}$	$T(n) = 2T(n-1) - T(n-3)$	$T(n) \sim \phi^n$
$C_n^{0,1,2}$	initial values $9, 13, 12$ for $n = 4, 5, 6$	$\phi = (1 + \sqrt{5})/2$
$C_{3n}^{1,n+1,2n}$	$T(n) = 5T(n-1) - 5T(n-2) - 5T(n-3) + 6T(n-4)$	
$C_{3n}^{0,n,2n-1}$	initial values $17, 45, 113, 309$ for $n = 2, 3, 4, 5$	$T(n) \sim 3^n$

Table 1: The number of cycle-covers $T(n)$ in directed circulant graphs with constant jumps $C_n^{-1,0,1}$ and $C_n^{0,1,2}$, and with non-constant jumps $C_{3n}^{1,n+1,2n}$ and $C_{3n}^{0,n,2n-1}$, as derived by the technique in this paper. Note that inside each pair of graphs, the number of cycle covers is the same. This is to be expected, since their adjacency matrices are just linear circular shifts of each other so the permanents of their adjacency matrices are the same.

of the details omitted. Also, when reproving Minc's result, we only prove that $\mathrm{Perm}(A_n)$ *satisfies a degree* $2^{\bar{s}}$ *recurrence relation and not a* $2^{\bar{s}} - 1$ *one.*

2 A Recursive Decomposition of Directed Circulant Graphs

The main conceptual difficulty with deriving a recurrence relation for $T(C_n)$ is that larger circulant graphs can not be built recursively out of smaller ones. The crucial observation, though, is that, there is *another graph*, L_n, the *lattice graph*, that *can* be built recursively, and C_n can then be constructed from L_n through the addition of a constant number of edges[1]. In [9, 10] the authors of this paper developed such a recursive decomposition for *undirected* circulant graphs as a tool for counting the number of spanning trees in such graphs. In what follows we develop a corresponding decomposition for *directed circulants* that will permit counting cycle-covers.

We first show this for the restricted case in which S, the set of jumps, is constant (independent of n), where it is easy to visualize. After deriving the relevant properties we extend the decomposition to the more complicated case in which the set of jumps can depend upon n.

Definition 2.1. *See Figure 1. Let* $S = \{s_1, s_2, \ldots s_k\}$*, where the* s_i *are fixed integers. Define the* n*-node* lattice graph *with jumps* S

$$L_n^{s_1,s_2,\ldots,s_k} = (V(n), E_L(n))$$

where

$$E_L(n) = \Big\{(i,j) : j - i \in S\Big\}.$$

Now set

$$\texttt{Hook}(n) = E_C(n) - E_L(n)$$

and

$$\texttt{New}(n) = E_L(n+1) - E_L(n).$$

Note that this implies

(2.3) $L_{n+1} = L_n \cup \texttt{New}(n)$ *and* $C_n = L_n \cup \texttt{Hook}(n)$.

The simple but important observation is that, *when n is viewed as a label rather than as a number*, $\texttt{Hook}(n)$ and $\texttt{New}(n)$ are *independent* of the actual *value* of n.

[1] To put this into context, this is very similar to the definition of *Recursive families* for undirected graphs [3, 17].

LEMMA 2.1. *Set* $S^+ = \{s \in S \,:\, s \geq 0\}$ *and* $S^- = \{s \in S \,:\, s < 0\}$. *Then*

$$\begin{aligned}\mathtt{Hook}(n) &= \left(\bigcup_{s\in S^+} \{\,(n-j,\, s-j) \,:\, 1 \leq j \leq s\}\right)\\ &\quad \cup \left(\bigcup_{s\in S^-} \{\,(j,\, n+s+j) \,:\, 0 \leq j < |s|\}\right),\\ \mathtt{New}(n) &= \left(\bigcup_{s\in S^+} \{(n-s,n)\}\right) \cup \left(\bigcup_{s\in S^-} \{(n,n+s)\}\right).\end{aligned}$$

Further set $s^+ = \max_{s\in S^+} s$,
and $s^- = \max_{s\in S^-} |s|$ *(if* $S^- = \emptyset$ *set* $s^- = 0$*).*
For later use we define $\bar{s} = s^+ + s^-$. *Now define*

$$\begin{aligned}L^+(n) &= \{0,\ldots s^+ - 1\},\\ R^+(n) &= \{n - s^+,\ldots,n-1\},\\ L^-(n) &= \{0,\ldots s^- - 1\},\\ R^-(n) &= \{n - s^-,\ldots,n-1\}.\end{aligned}$$

Then

$$\begin{aligned}\mathtt{Hook}(n) &\subseteq \big(R^+(n)\times L^+(n)\big) \cup \big(L^-(n)\times R^-(n)\big)\\ \mathtt{New}(n) &\subseteq \big(R^+(n)\times\{n\}\big) \cup \big(\{n\}\times R^-(n)\big)\\ &\quad \cup \{(n,n)\}\end{aligned} \tag{2.4}$$

Important Note: In this section and the next we will always assume that $n \geq 2\bar{s}$ *since this will guarantee that* $(L^+(n)\cup L^-(n)) \cap (R^+(n)\cup R^-(n)) = \emptyset$. *Without this assumption some of our proofs would fail. Also note that the* $\{(n,n)\}$ *term in* $\mathtt{New}(n)$ *is only needed when* $0 \in S$.

We now extend the above definitions and lemmas to the case of non-constant circulants. This will require a change in the way that we visualize the nodes of C_n; until, now, as in Figure 1(c), we visualized them as points on a line with the edges in $\mathtt{Hook}(n)$ connecting the left and right endpoints of the line. In the non-constant jump case it will be convenient to visualize them as points on a bounded-height *lattice*, where $\mathtt{Hook}(n)$ connects the left and right boundaries of the lattice. We start by introducing a new graph:

DEFINITION 2.2. *See Figure 2. Let* $p, s,\ p_1, p_2, \ldots, p_k$ *and* $s_1, s_2, \ldots, s_k$ *be given integral constants such that* $\forall i,\ 0 \leq p_i < p$. *Set* $S = \{p_1n+s_1, p_2n+s_2, \cdots, p_kn+s_k\}$. *For* u, v *and integer* n, *set* $f(n;u,v) = un+v$. *Define*

$$\widehat{C}_n = \left(\widehat{V}_C(n),\ \widehat{E}_C(n)\right)$$

where

$$\begin{aligned}\widehat{V}(n) &= \{\,(u,v) \,:\, 0 \leq u \leq p-1,\ 0 \leq v \leq n-1\}\\ &\quad \cup \{\,(p-1,v) \,:\, n \leq v \leq n+s-1\}\end{aligned}$$

and $\widehat{E}_C(n)$ *is the set of the union of all edges*

$$((u_1,v_1),(u_2,v_2)),$$

where the union is taken over all

$$(u_1,v_1),\ (u_2,v_2) \in \widehat{V}(n)$$

such that

$$f(n;u_2,v_2) - f(n;u_1,v_1) \bmod (pn+s) \in S.$$

Directly from the definition we see $\widehat{C}_n$ is *isomorphic* to $C_n = C^{p_1n+s_1,p_2n+s_2,\cdots,p_kn+s_k}_{pn+s}$. In particular, cycle-covers of $\widehat{C}_n$ are in 1-1 correspondence with cycle covers of C_n so we can restrict ourselves to counting cycle covers of $\widehat{C}_n$. We now introduce the generalization of Definition 2.1.

DEFINITION 2.3. *Let* $p, s,\ \ p_1, p_2, \ldots, p_k$ *and* $s_1, s_2, \ldots, s_k$ *and* S, f *be as in Definition 2.2. Define the* $pn+s$*-node* lattice graph *with jumps* S

$$L_n = (\widehat{V}(n), \widehat{E}_L(n))$$

where $\widehat{E}_L(n)$ *is the set of the union of all edges*

$$((u_1,v_1),(u_2,v_2))$$

where the union is taken over all

$$(u_1,v_1),\ (u_2,v_2) \in \widehat{V}(n),$$

such that

$$f(n;u_2,v_2) - f(n;u_1,v_1) = p_in + s_i \bmod (pn+s)$$

and

$$u_2 - u_1 = p_i \bmod p.$$

Now set

$$\mathtt{Hook}(n) = \widehat{E}_C(n) - \widehat{E}_L(n)$$

and

$$\mathtt{New}(n) = \widehat{E}_L(n+1) - \widehat{E}_L(n).$$

Note that this implies

$$\begin{aligned}L_{n+1} &= L_n \cup \mathtt{New}(n)\\ &\quad\text{and}\\ \widehat{C}_n &= L_n \cup \mathtt{Hook}(n).\end{aligned} \tag{2.5}$$

It is now straightforward to derive an analogue of Lemma 2.1 showing that $\mathtt{Hook}(n)$ and $\mathtt{New}(n)$ are *independent* of the actual *value* of n. Let $NV(n) = V_L(n+1) - V_L(n)$. $NV(n)$ will be the *new* vertices in $V_L(n+1)$. Note that we did not define this for *fixed-jump* circulant graphs since in the fixed-jump case there is only the one new vertex $V_L(n+1) - V_L(n) = \{n\}$ and $NV(n)$ would be constant.

LEMMA 2.2. *Set* $S^+ = \{s_i \in S : s_i \geq 0\}$, $S^- = \{s_i \in S : s_i < 0\}$, *and* $s^+ = \max_{s_i \in S^+} s_i$, $s^- = \max_{s_i \in S^-} |s_i|$ *(if* $S^- = \emptyset$ *set* $s^- = 0$*).*
Let $t_1 = s^+ - 1$, $t_2 = s^- - 1$ *and* $r = \min\{-s, 0\}$.
Now let $0 \leq u \leq p-1$ *and define*

$$\begin{aligned} L^+(n) &= \{(u,v) : 0 \leq v \leq \max\{t_1, t_1+s\}\}, \\ R^+(n) &= \{(u, n-1-v) : r \leq v \leq \max\{t_1, t_1-s\}\}, \\ L^-(n) &= \{(u,v) : 0 \leq v \leq \max\{t_2+s, t_2+2s\}\}, \\ R^-(n) &= \{(u, n-1-v) : r \leq v \leq \max\{t_2, t_2+s\}\}. \end{aligned}$$

Then

$$\begin{aligned} \mathtt{Hook}(n) &\subseteq \left(R^+(n) \times L^+(n)\right) \\ &\quad \cup \left(L^-(n) \times R^-(n)\right), \\ \mathtt{New}(n) &\subseteq \left(R^+(n) \times NV(n)\right) \\ &\quad \cup \left(NV(n) \times R^-(n)\right) \\ &\quad \cup \left(NV(n) \times NV(n)\right). \end{aligned} \tag{2.6}$$

3 A New Proof of Minc's result

Let CC be a cycle-cover of C_n. Then, in $T = CC - \mathtt{Hook}(n)$, almost all vertices v except possibly those that have an edge of $\mathtt{Hook}(n)$ hanging off of them, have $\mathtt{ID}_T(v) = \mathtt{OD}_T(v) = 1$. Referring to (2.4) this motivates

DEFINITION 3.1. $T \subseteq E_L(n)$ *is a* legal cover of L_n *if*

- $\forall v \in V, \quad \mathtt{ID}_T(v) \leq 1$ *and* $\mathtt{OD}_T(v) \leq 1$.
- $\forall v \in V - (L^+(n) \cup R^-(n)), \quad \mathtt{ID}_T(v) = 1$.
- $\forall v \in V - (L^-(n) \cup R^+(n)), \quad \mathtt{OD}_T(v) = 1$.

Then, from (2.4) we have

LEMMA 3.1.
(a) If $T \subseteq E_C(n)$ *is a cycle-cover of* C_n*, then* $T - \mathtt{Hook}(n)$ *is a legal-cover of* L_n.

(b) If $T \subseteq E_L(n+1)$ *is a legal-cover of* L_{n+1}*, then* $T - \mathtt{New}(n)$ *is a legal-cover of* L_n.

From the definition of legal covers we can classify and partition legal covers by the appropriate in/out degrees of their vertices in $L^+(n), L^-(n), R^+(n), R^-(n)$.

DEFINITION 3.2. *A is a* binary r-tuple *if* $A = (A(0), A(1), \ldots, A(r-1))$ *where* $\forall i, A(i) \in \{0,1\}$.

Let $\mathcal{P}$ *be the set of* $2^{2\bar{s}}$ *tuples* (L_+, L_-, R_+, R_-) *where* L_+, L_-, R_+, R_- *are, respectively, binary* s^+, s^-, s^+, s^- *tuples.*

Let T be a legal-cover *of* L_n. *The* classification of T *will be* $C(T) = (L^T_+, L^T_-, R^T_+, R^T_-) \in \mathcal{P}$ *where*

$$\begin{aligned} \forall 0 \leq i < s^+, \quad L^T_+(i) &= \mathtt{ID}_T(i) \\ R^T_+(i) &= \mathtt{OD}_T(n-1-i), \\ \forall 0 \leq i < s^-, \quad R^T_-(i) &= \mathtt{ID}_T(n-1-i), \\ L^T_-(i) &= \mathtt{OD}_T(i). \end{aligned}$$

If T is not *a legal-cover then we will use the convention that* $C(T) = \emptyset$. *Finally, set*

$$\begin{aligned} \mathcal{L}(n) &= \{T \subseteq E_L(n) : T \text{ is a legal cover of } L_n\} \\ \mathcal{L}_X(n) &= \{T \in \mathcal{L}(n) : C(T) = X\} \\ T_X(n) &= |\mathcal{L}_X(n)| \end{aligned}$$

so $T_X(n)$ *is the number of legal-covers of* L_n *with classification* X.

The main reason for introducing these definitions is that checking whether a legal cover T of L_n can be completed to a cycle-cover of C_n doesn't depend upon all of T but only on its classification $C(T)$. Furthermore, how a legal-cover in L_n expands to a legal cover in L_{n+1} will also only depend upon $C(T)$.

LEMMA 3.2. *See Figures 3 and 4.*
(a) Let $X = (L^X_+, L^X_-, R^X_+, R^X_-) \in \mathcal{P}$ *and* $S \subseteq \mathtt{Hook}(n)$. *Let T be a legal cover in* L_n *with* $C(T) = X$.

Then whether $T \cup S$ *is a cycle cover of* C_n *depends only upon X and S (and* not at all on n*). In particular, if T is a legal-cover of* L_n *and* T' *is a legal cover of* $L_{n'}$ *with* $C(T) = C(T')$ *then*

$T \cup S$ is a cycle-cover of C_n

iff

$T' \cup S$ is a cycle-cover of $C_{n'}$

Note: We will write $X \cup S$ *is a cycle cover* to denote that $T \cup S$, with $C(T) = X$, is a cycle cover.

(b)Let T' *be a legal cover in* L_n *with* $C(T') = X' \in \mathcal{P}$. *and* $S \subseteq \mathtt{New}(n)$.

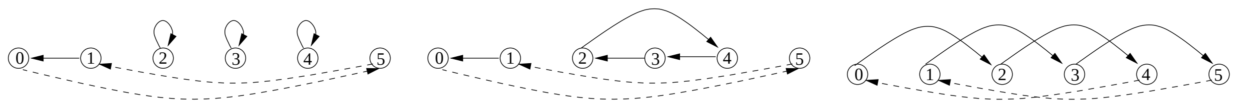
Then whether $C(T' \cup S) = X$ *depends only upon* X' *and S (and* not at all on n*). In particular, if* T' *is a legal-cover of* L_n *and* T'' *is a legal cover of* $L_{n'}$ *with* $C(T') = C(T'')$ *then*

$$C(T' \cup S) = C(T'' \cup S)$$

Note: We will write $(X' \cup S) = X$ to denote that, when $C(T') = X'$, $C(T' \cup S) = X$.

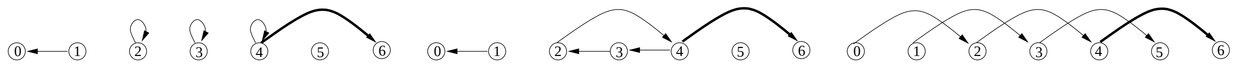
Proof. To prove (a) recall that $T \cup S$ is a legal-cover of L_n if and only if,
$\forall v \in V$, $\mathtt{ID}_{T \cup S}(v) = \mathtt{OD}_{T \cup S}(v) = 1$ or

$$\forall v \in V, \mathtt{ID}_S(v) = 1 - \mathtt{ID}_T(v) \quad \text{and} \quad \mathtt{OD}_S(v) = 1 - \mathtt{OD}_T(v) \tag{3.7}$$



(a) CC_1 and T_1 (b) CC_2 and T_2 (c) CC_3 and T_3

Figure 3: All of the figures are in $C_6^{-1,0,2}$. Dashed edges are $\texttt{Hook}(n)$. The solid plus dashed edges are three different cycle covers CC_i, $i = 1, 2, 3$ in C_6. Removing the dashed $\texttt{Hook}(n)$ edges leaves three *legal* covers T_i, $i = 1, 2, 3$, in L_6. Note that $s^+ = 2$ and $s^- = 1$ so classifications are of the form $(L_+^T, L_-^T, R_+^T, R_-^T)$ where L_+^T and R_+^T are pairs and L_-^T and R_-^T are singletons. Calculation gives $C(T_1) = C(T_2) = X_1' = (\,(1,0),\ (0),\ (0,1),\ (0)\,)$ and $C(T_3) = X_3' = (\,(0,0),\ (1),\ (0,0),\ (1)\,)$.



(a) $T_1 \cup \{(n-2, n)\}$ (b) $T_2 \cup \{(n-2, n)\}$ (c) $T_3 \cup \{(n-2, n)\}$

Figure 4: n was increased from 6 to 7 and $S = \{(4, 6)\} \subseteq \text{New}(6)$ was added to the T_i of the previous figure. Note that, in L_7, $C(T_1 \cup S) = C(T_2 \cup S) = \emptyset$ since they are no longer legal covers. Also, $C(T_3 \cup S) = X_3 (= X_3') = (\,(0,0),\ (1),\ (0,0),\ (1)\,)$. Thus, $C(X_1' \cup S) = \emptyset$ and $C(X_3' \cup S) = X_3'$.

From Lemma 2.1 and the definition of a legal cover we have that this is true if and only if

$$\begin{array}{lrcl} \forall i \le s^+, & \texttt{ID}_S(i) & = & 1 - L_+^X(i), \\ & \texttt{OD}_S(i) & = & 1 - L_-^X(i), \\ \forall i \le s^-, & \texttt{ID}_S(n-1-i) & = & 1 - L_+^X(i,) \\ & \texttt{OD}_S(n-1-i) & = & 1 - L_-^X(i). \end{array}$$

and this is only dependent upon X and S and not upon n or any other properties of T.

The proof of (b) is similar and omitted here.

□

Definition 3.3. *For $X, X' \in \mathcal{P}$, $S \subseteq \texttt{Hook}(n)$ and $S' \subseteq \texttt{New}(n)$ set*

$$\beta_{X,S} = \begin{cases} 1 & \text{if } X \cup S \text{ is a cycle cover} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha_{X,X',S'} = \begin{cases} 1 & \text{if } C(X' \cup S') = X \\ 0 & \text{otherwise} \end{cases}.$$

Now set

$$\beta_X = \sum_{S \subseteq \texttt{Hook}(n)} \beta_{X,S}$$

and

$$\alpha_{X,X'} = \sum_{S' \subseteq \texttt{New}(n)} \alpha_{X,X',S'}.$$

Note that β_X and $\alpha_{X,X'}$ are constants that can be mechanically calculated. Then Lemmas 3.1 and 3.2 immediately imply our main technical result, which is equivalent to (1.2).

Lemma 3.3.

$$T(n) = \sum_{X \in \mathcal{P}} \beta_X T_X(n)$$

and

$$T_X(n+1) = \sum_{X' \in \mathcal{P}} \alpha_{X,X'} T_X(n).$$

Let $m = |\mathcal{P}| = 2^{2\bar{s}}$. Take any arbitrary ordering of $\mathcal{P}$ and define the $1 \times m$ constant vector $\beta = (\beta_X)_{X \in \mathcal{P}}$ and $m \times m$ constant matrix $A = (\alpha_{X,X'})_{X,X' \in \mathcal{P}}$. Finally, set $\bar{T}(n) = \text{col}(T_X(n))_{X \in \mathcal{P}}$ to be a $m \times 1$ column vector. Then, Lemma 3.3 is exactly equation (1.2) which immediately implies that $T(n)$ satisfies a fixed-degree constant coefficient recurrence relation where the degree of the recurrence is at most the degree of any polynomial $P(x)$ such that $P(A) = 0$. By the Cayley-Hamilton theorem, $Q(A) = 0$, $Q(x)$ is the degree $m = 2^{2\bar{s}}$ characteristic polynomial $Q(x) = \det(IX - A)$.

We will now see that it is possible to reduce this degree from $2^{2\bar{s}}$ down to below $2^{\bar{s}}$.

Lemma 3.4. *Let $A = (\alpha_{X,X'})$. Then there is a degree $2^{\bar{s}} - 1$ polynomial $P(x)$ such that $P(A) = 0$.*

Proof. Recall that $\bar{s} = s^+ + s^-$. Suppose $X = (L_+^X, L_-^X, R_+^X, R_-^X)$ and $X' = (L_+^{X'}, L_-^{X'}, R_+^{X'}, R_-^{X'})$.

Recall that $\alpha_{X,X'} = \sum_{S \subseteq \texttt{New}(n)} \alpha_{X,X',S}$ where $\alpha_{X,X',S} = 1$ if and only if $C(X' \cup S) = X$, and is otherwise 0.

Now let L_+, L_- be any 2^{s^+} and 2^{s^-} binary tuples and partition $\mathcal{P}$ up into $2^{\bar{s}}$ sets of size $2^{\bar{s}}$, $\mathcal{P}_{L_+,L_-} = \{X \in \mathcal{P} : L_+^X = L_+,\ L_-^X = L_-\}$.

Note that, if $S \subseteq \texttt{New}(n)$, none of S's edges have endpoints in $L^+(n)$ or $L^-(n)$. Intuitively, this is because edges in $\texttt{New}(n)$ only connect vertices near the *right* side of the lattice and do not touch any vertices on the *left* side of the lattice.

Thus, if $\alpha_{X,X',S} = 1$, then $L_+^X = L_+^{X'}$ and $L_-^X = L_-^{X'}$. In particular this means that if $\alpha_{X,X',S} = 1$ then X, X' are both in the same partition set $\mathcal{P}_{L_+,L_-}$.

Now suppose that $\alpha_{X,X',S} = 1$. Let $\bar{L}_+$, $\bar{L}_-$ be *any* other 2^{s^+} and 2^{s^-} binary tuples and set

$$\bar{X} = (\bar{L}_+,\, \bar{L}_-,\, R_+^X,\, R_-^X) \quad \text{and} \quad \bar{X}' = (\bar{L}_+,\, \bar{L}_-,\, R_+^{X'},\, R_-^{X'}). \tag{3.8}$$

Then, again using the fact that none of the endpoints of S are in $L^+(n)$ or $L^-(n)$ we have that $C(X' \cup S) = X$ if and only if $C(\bar{X}' \cup S) = \bar{X}$ so $\alpha_{X,X'} = \alpha_{\bar{X},\bar{X}'}$.

When constructing matrix $A = (\alpha_{X,X'})_{X,X' \in \mathcal{P}}$ we previously allowed any arbitrary ordering of $\mathcal{P}$. Now order the $X \in \mathcal{P}$ lexicographically; this groups all of the X in a particular $\mathcal{P}_{L_+,L_-}$ consecutively. The observations above imply that A is partitioned into $2^{\bar{s}} \times 2^{\bar{s}}$ blocks where each block is of size $2^{\bar{s}} \times 2^{\bar{s}}$. The non-diagonal blocks correspond to $\alpha_{X,X'}$ where X, X' are in different partitions so all of the non-diagonal blocks are 0. On the other hand, the fact that $\alpha_{X,X'} = \alpha_{\bar{X},\bar{X}'}$ for the $\bar{X}, \bar{X}'$ defined in (3.8), tells us that all the diagonal blocks are copies of each other.

Let $\bar{A}$ be one of the $2^{\bar{s}} \times 2^{\bar{s}}$ diagonal blocks in A. A can then be denoted as $A = \text{diag}(\bar{A}, \bar{A}, \ldots, \bar{A})$ where A contains $2^{\bar{s}}$ copies of $\bar{A}$ on its diagonal. Thus, $\forall i$, $A^i = \text{diag}(\bar{A}^i, \bar{A}^i, \ldots, \bar{A}^i)$. In particular, this means that any polynomial $P(x)$ that annihilates $\bar{A}$ also annihilates A. Since $\bar{A}$ is a $2^{\bar{s}} \times 2^{\bar{s}}$ matrix, the Cayley-Hamilton theorem says that the characteristic polynomial $\bar{P}(x)$ of $\bar{A}$, which is of degree $2^{\bar{s}}$, annihilates $\bar{A}$.

By a more careful analysis of the structure of $\bar{A}$ it is possible to show that $\bar{P}(x)$ actually has degree $2^{\bar{s}}-1$ but, as mentioned in the introduction, that further analysis will be omitted here. $\square$

Lemma 3.3 tells us that (1.2) holds while Lemma 3.4 tells us that matrix A is annihilated by polynomial $P(x)$ of degree $2^{\bar{s}} - 1$. Combining them gives that $T(n)$ satisfies a degree-$(2^{\bar{s}} - 1)$ constant coefficient recurrence relation. In order to actually *derive* the recurrence relation, though, it is necessary to calculate the $\alpha_{X,X'}$, β_X, $\bar{T}$ and P as well as the first $2^{\bar{s}} - 1$ values of $T(n) = \beta\bar{T}(n)$. It is relatively straightforward (but omitted in this extended abstract) to see how to evaluate all of these in $O(\bar{s}2^{5\bar{s}})$ time by evaluating $O(2^{2\bar{s}})$ permanents of size $2\bar{s}$ and $O(2^{4\bar{s}})$ of size $\bar{s}$.

We just saw how to calculate the number of cycle-covers in constant-jump circulant graphs. Reviewing the proof, everything followed directly as a consequence from the recursive decomposition of circulant graphs in (2.3) combined with the structural properties of the decomposition given in Lemma 2.1. But, as also derived in Section 2, non-constant jump circulants have exactly the *same* structural properties, given in (2.5) and Lemma 2.2. Therefore, the entire proof developed in Section 3 can be rewritten for non-constant jump circulants. The only difference is in the degree of the recurrence relation for the number of cycle-covers. Reviewing the proof for the constant-jump case we can see that the order of the recurrence relation is really $2^{|R^+(n)|+|R^-(n)|}$ which worked out to $2^{\bar{s}} - 1$. In the non-constant case, from Lemma 2.2, we can calculate that $|R^+(n)| + |R^-(n)| = p(|s| + s^+ + s^-) + 2s$ so the order of the recurrence relation will then be $2^{p(|s|+s^++s^-)+2s}$. Note that in the constant jump case we had $p = s = 0$ so this collapses down to $2^{s^++s^-} = 2^{\bar{s}}$ which is what we had previously derived. For an example of such a recurrence relation, see the second set of graphs in Table 1.

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Appendices

A Worked example for $C_n^{0,1,2}$

As discussed in the paper we have that $T(n)$, the number of cycle covers in $C_n^{0,1,2}$, satisfies

$$\forall n \geq 2\bar{s}, \quad T(n) = \beta\,\bar{T}(n) \quad \text{and} \quad \bar{T}(n+1) = A\,\bar{T}(n)$$

where $\beta = (\beta_X)_{X\in\mathcal{P}}$ and $A = (\alpha_{X,X'})_{X,X'\in\mathcal{P}}$.

For $C_n^{0,1,2}$ we have $s^+ = 2$, $s^- = 0$ so $\bar{s} = s^+ + s^- = 2$. Definition 3.2 then says that every $X \in \mathcal{P}$ is in the form $X = (L_+^X, L_-^X, R_+^X, R_-^X)$ where $L_+^X, R_+^X \in \{0,1\}^2$ and L_-^X, R_-^X are empty. We can therefore represent every X by a four-bit binary vector in which the first two bits represent L_+^X and the last two R_+^X; there are 16 such $X \in \mathcal{P}$. Ordering the X lexicographically we calculate that β is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\bar{T}(4)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^t$$

(where the t denotes taking the transpose), and *Transfer matrix* $A(X)$ is

$$\left(\begin{array}{cccc|cccc|cccc|cccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

As predicted by Lemma 3.4, A is partitioned into 16 4×4 blocks where all but the diagonal blocks are 0 and all of the diagonal blocks are equal to some 4×4 matrix $\bar{A}$ which in this case is

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This means that

$$Q(x) = det(Ix - \bar{A}) = (x^2 - x - 1)(x-1)^2,$$

the characteristic polynomial of $\bar{A}$, annihilates A.

Working through the details we can then solve to find that $C_n^{0,1,2} = 2T(n-1) - T(n-3)$ with initial values $T(4) = 9$, $T(5) = 13$, and $T(6) = 12$.

B Other Applications

In this appendix we quickly mention two other applications of the technique introduced in this paper.

The first is in analyzing the number of cycles in certain classes of random restricted permutations. Using the standard cycle-decomposition of a permutation there is 1-1 correspondence between permutations

$C_n^{-1,0,1}$	$T_1(n) = 3T(n-1) - T_1(n-2) - 3T_1(n-3)$ $+T_1(n-4) + T_1(n-5)$ initial values $22, 42, 80, 149, 274$ for $n = 4, 5, 6, 7, 8$	$T_1(n) \sim \frac{\phi^4}{\phi^2+\phi^4} n\phi^n$ $\sim .7236n\phi^n$	$\frac{T_1(n)}{T_0(n)} \sim .7236n$
$C_n^{0,1,2}$	$T_1(n) = 3T(n-1) - 6T_1(n-3) + 2T_1(n-4)$ $+4T_1(n-5) - T_1(n-6) - T_1(n-7)$ initial values $21, 32, 56, 93, 161, 275, 475$ for $n = 4, 5, \ldots, 10$	$T_1(n) \sim \frac{\phi^2}{\phi^2+\phi^4} n\phi^n$ $\sim .2764n\phi^n$	$\frac{T_1(n)}{T_0(n)} \sim .2764n$

Table 2: $T_1(n)$ is the number of cycles in the given graph with n vertices.

$\pi \in S_n$ and cycle-covers of the complete directed graph on n-vertices. For given parameters p, s, p_i, s_i and S as in Definition 1.4 define

$$S_{pn+s}(S) = \{\pi \in S_{pn+s} \,:\, \pi[i] - i \bmod (pn+s) \in S\} \tag{2.9}$$

to be the set of permutations in which $\pi[i]$ is restricted by (2.9). Now suppose that we pick a permutation π uniformly at random from $S_{pn+s}(S)$ and set $X =$ # of cycles in π. What can be said about the distribution of X?

By the 1-1 correspondence between permutations and cycle-covers, $\pi \in S_{pn+s}(S)$ if and only if the corresponding cycle-cover is in C_n. Thus, the number of such permutations satisfies $|S_{pn+s}(S)| = T(n)$ where $T(n)$ is the number of cycle-covers in C_n. Suppose now that for cycle cover $T \in \mathcal{CC}(n)$ we define $\#_C(T)$ to be the *number* of cycles composing cover T and set

$$TC_i(n) = \sum_{T \in \mathcal{CC}(n)} (\#_C(T))^i.$$

That is, $TC_0(n) = T(n)$ while $TC_1(n)$ is the total number of cycles summed over all cycle-covers in C_n. Then, again by the correspondence, we have that the moments of X are given by

$$\forall i \geq 0, \quad E(X^i) = \frac{TC_i(n)}{TC_0(n)}.$$

The interesting point is that the transfer matrix approach introduced in this paper can mechanically be extended to counting the total number of cycles in the cycle-covers, to show that for every i, $TC_i(n)$ satisfies a fixed-order constant coefficient recurrence relation. For given, $p, s, p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ this permits, for example, calculating $E(X)$ and $Var(X)$.

As an illustration recall the results from Table 1 counting the number of cycle covers in $C_n^{-1,0,1}$ and $C_n^{0,1,2}$. Even though these two graphs are *not isomorphic* they had the same number of cycle-covers because the adjacency matrix of the second is just the adjacency matrix of the first with every row (cyclicly) shifted over one step. Since permanents are invariant under cyclic shifts both matrices have the same permanent which is $\sim \phi^n$ where $\phi = (1+\sqrt{5})/2$.

Using our technique we calculated $TC_1(n)$ for both cases with the results given in Table 2.

In both cases we have that $TC_1(n) \sim cn\phi^n$. This means that if a permutation on n items is chosen at random from the corresponding distribution then, on average, it will have $\frac{T_1(n)}{T_0(n)} \sim cn$ cycles. It is interesting to note that that c is different for the two cases.

The second application of the technique we note is that a minor modification permits using it to show that the number of *Hamiltonian Cycles* in a directed circulant graph C_n also satisfies a constant-coefficient recurrence relation in n. This fact was previously known for *undirected* circulant graphs [9, 10] but doesn't seem to have been known for directed circulants, with the exception of the special case of in(out)-degree 2 circulants [20].