Random partitions with parts in the range of a polynomial*

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Abstract

Let $\Omega(n, Q)$ be the set of partitions of n into summands that are elements of the set $\mathcal{A} = \{Q(k) : k \in Z^+\}$. Here $Q \in Z[x]$ is a fixed polynomial of degree $d > 1$ which is increasing on $\mathbf{R}^+$, and such that $Q(m)$ is a non–negative integer for every integer $m \geq 0$. For every $\lambda \in \Omega(n, Q)$, let $\mathbf{M}_n(\lambda)$ be the number of parts, with multiplicity, that λ has. Put a uniform probability distribution on $\Omega(n, Q)$, and regard $\mathbf{M}_n$ as a random variable. The limiting density of the random variable $\mathbf{M}_n$ (suitably normalized) is determined explicitly. For specific choices of Q, the limiting density has appeared before in rather different contexts such as Kingman's coalescent, and processes associated with the maxima of Brownian bridge and Brownian meander processes.

1 Introduction and statement of the result

In research on partitions, there have been great synergies between probabilistic, analytic, and combinatorial methods. The oldest literature on partition enumerations, dating back to Hardy and Ramanujan [15], has a purely analytic flavor. But Erdös and Lehner [11] introduced a probabilistic viewpoint that was quite fruitful. Random partitions were developed by Erdös, Szalay, Turan and others, [12, 27, 29, 30, 31, 32]. Some authors, e.g. [5, 16, 17, 22, 25], have studied random partitions with summands restricted to proper subsets of the set of positive integers. Increasingly sophisticated probabilistic ideas have been introduced [13, 2], and these ideas have led to remarkably strong theorems about the joint distribution of part sizes of random integer partitions [23].

In this abstract we concentrate on the limiting distribution of the number of parts in a random partition whose parts are restricted to the range of a polynomial. Specifically, let

$$Q(x) = a_d x^d + a_{d-1}x^{d-1} + ... + a_1 x + a_0 \tag{1.1}$$

be a fixed polynomial of degree $d \geq 2$ and we assume that $Q(x)$ is strictly increasing for $x > 0$ and that $Q(m)$ is a non-negative integer for an integer $m \geq 0$. Let $\Omega(n, Q)$ be the set of partitions of n into summands that are elements of the set $\mathcal{A} = \{Q(k) : k \in Z^+\}$. For every $\lambda \in \Omega(n, Q)$, let $\mathbf{M}_n(\lambda)$ be the number of parts, with multiplicity, that λ has. Put a uniform probability measure P_n on $\Omega(n, Q)$, and regard $\mathbf{M}_n$ as a random variable. Note that $\mathbf{M}_n(\lambda) = \sum_a M_a(\lambda)$, where $M_a(\lambda)$ is the multiplicity of the part size a in the P_n-random partition λ. These random variables M_a are clearly not independent since they must satisfy the condition $\sum_{a\in\mathcal{A}} aM_a = n$. Fristedt [13] used a conditioning device that enables one to cope with this dependence. It quickly proved to be a powerful tool and has been used by several authors in the past decade, see e.g. [1, 2, 5, 8, 23, 24, 26]. Given a parameter $q \in (0, 1)$, let $\{G_a\}_{a\in\mathcal{A}}$ be mutually independent geometric random variables with respective parameters $1 - q^a$, i.e. for all $a \in \mathcal{A}$, and for all non-negative integers k, we have $\mathsf{P}(G_a = k) = (1 - q^a)q^{ak}$. As was observed by Fristedt [13] the joint distribution of the random variables $\{M_a\}_{a\in\mathcal{A}}$ (with respect to P_n) is exactly *equal* to the *conditional* distribution of the $\{G_a\}_{a\in\mathcal{A}}$, where the event conditioned on is that $\sum_{a\in\mathcal{A}} aG_a = n$. This is true for any choice of the parameter q. Hence the parameter $q = q_n$ can be chosen in such a way that asymptotic estimates as $n \to \infty$ are tractable. Analogous methods have been used (with Poisson distributions in place of geometric distributions) in the context of random permutations [28]. As a matter of fact, it is quite common that the distribution of the components of random combinatorial structures are independent random variables conditioned on the sum of the sizes being fixed (see [1] for more information and references).

We write E_n for expected values computed using P_n. We likewise write P_q and E_q for computations with the independent geometric variables. We use Fristedt's device, as well as some additional probabilistic and analytic arguments to derive a limit theorem for $\mathbf{M}_n$.

For $n \geq 1$ we choose the parameter $q = q_n = \exp[-C_Q n^{-d/d+1}]$, with a specific value of the constant,

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namely C_Q is equal to

$$(1.2)\quad \frac{1}{a_d^{1/(d+1)}}\left(\frac{\zeta(1+d^{-1})\Gamma(1+d^{-1})}{d}\right)^{d/(d+1)}.$$

(A reason for that particular choice will become clear in Section 2.) We also set the normalizing constants $\mu_n = n^{\frac{d}{d+1}}/C_Q$. Finally, we let $r_1, r_2, \ldots, r_d$ be the (complex) roots of $Q(z)$ and for $j = 1, 2\ldots, \alpha_1(j), \ldots, \alpha_{d-1}(j)$ be those (complex) roots of $Q(z) - Q(j)$ that are not equal to j.

Our aim is to sketch a proof of the following:

THEOREM 1.1. *For any positive real number x, we have*

$$(1.3)\qquad \lim_{n\to\infty} P_n\left(\frac{\mathbf{M}_n}{\mu_n} \le x\right) = P(W_Q \le x),$$

where W_Q is a random variable whose characteristic function is

$$(1.4)\qquad \phi_Q(t) = \prod_{k\ge 1} \frac{1}{1 - it/Q(k)}$$

and whose density is

$$(1.5)\qquad f_Q(x) = \sum_{j=1}^{\infty} (-1)^{j-1} e^{-Q(j)x} \frac{Q'(j)}{(j-1)!} \frac{\prod_{m=1}^{d-1} \Gamma(1-\alpha_m(j))}{\prod_{m=1}^{d} \Gamma(1-r_m)},$$

for $x > 0$.

Our argument will be broken into several steps. We will first show that the distribution of $\mathbf{M}_n$ is close to that of a sum of independent geometric random variables with suitably chosen parameters. It then follows that the limiting distribution has a characteristics function given by (1.4). This means that W_Q is equidistributed with the infinite sum of independent exponential random variables with parameters $Q(k)$, $k = 1, 2\ldots$. We will carry out the Fourier inversion and, after some simplifications, will show that the density is given by (1.5).

Finally, we will point out that specific choices of Q lead to distributions that have already appeared in several, quite different, contexts. We will briefly mention a few such instances in the last section of the abstract.

2 Reduction to the case of independent summands

We consider a doubly infinite array $\{G_{n,a} :\ a \in \mathcal{A},\ n \ge 1\}$, where $G_{n,a}$ is a geometric random variable with parameter $1 - q_n^a$, and for each $n \ge 1$, $\{G_{n,a}\ a \in \mathcal{A}\}$ are independent.

As follows from an observation by Fristedt [13] (see also [5]), for any $x > 0$, and $n \ge 1$ we have

$$\begin{aligned} \mathsf{P}_n\left(\frac{\mathbf{M}_n}{\mu_n} \le x\right) &= \mathsf{P}_q\Big(\sum_{a\in\mathcal{A}} G_{n,a} \le \mu_n x \Big| \sum_{a\in\mathcal{A}} aG_{n,a} = n\Big) \\ (2.6)\qquad &= \frac{\mathsf{P}_q\big(\sum_{a\in\mathcal{A}} G_{n,a} \le \mu_n x,\ \sum_{a\in\mathcal{A}} aG_{n,a} = n\big)}{\mathsf{P}_q\big(\sum_{a\in\mathcal{A}} aG_{n,a} = n\big)}. \end{aligned}$$

The argument, whose details we omit here, is to show that the events in the numerator of (2.6) are asymptotically independent. This is done by arguing that for a suitably chosen sequence (k_n), each of the sums

$$\sum_{a\in\mathcal{A}} aG_{n,a} = \sum_{j=1}^{\infty} Q(j) G_{n,Q(j)},$$

and

$$\sum_{a\in\mathcal{A}} G_{n,a} = \sum_{j=1}^{\infty} G_{n,Q(j)},$$

can be split in two pieces ($j \le k_n$ and $j > k_n$) so that the dominant contribution to the value of $\sum_{j\ge1} G_{n,Q(j)}$ comes from indices $j \le k_n$ while the dominant contribution to $\sum_{j\ge1} Q(j)G_{n,Q(j)}$ comes from $j > k_n$.

This can be seen by extending the line of argument that was originally developed by Fristedt. First, in order to asymptotically maximize the denominator in (2.6) we choose q_n so that $\mathsf{E}_q\big(\sum_{a\in\mathcal{A}} aG_{n,a}\big) \sim n$. Since G's are geometric this means that we want

$$\sum_{a\in\mathcal{A}} a\frac{q^a}{1-q^a} = \sum_{j=1}^{\infty} Q(j)\frac{q^{Q(j)}}{1-q^{Q(j)}} \sim n.$$

After calculations and change of variables $y = Q(x)\ln(1/q)$ we get (the same computations were carried out in the case $Q(x) = \binom{x+d}{d}$ in [13] for $d = 1$ and in [5] for $d \ge 2$)

$$\begin{aligned} \mathsf{E}_q \sum_{a\in\mathcal{A}} aG_{n,a} &= \sum_{\ell=1}^{\infty} Q(\ell)\frac{q^{Q(\ell)}}{1-q^{Q(\ell)}} \\ &\sim \int_0^\infty Q(x)\frac{e^{-Q(x)\ln(1/q)}}{1-e^{-Q(x)\ln(1/q)}}dx \\ &= \frac{1}{\ln^2(1/q)}\int_0^\infty \frac{y}{Q'(Q^{-1}(y/\ln(1/q)))}\frac{e^{-y}}{1-e^{-y}}dy \\ &\sim \frac{1}{da_d^{1/d}\ln^{1+1/d}(1/q)}\int_0^\infty y^{1/d}\frac{e^{-y}}{1-e^{-y}}dy, \end{aligned}$$

which, using [20, formula 3.411-7] leads to $q_n = \exp(-C_Q/n^{d/(d+1)})$, where C_Q is given by (1.2).

By the same argument, if $k_n = o(n^{1/(d+1)})$, then

$$\mathsf{E}_q \sum_{j\le k_n} Q(j)G_{n,Q(j)} = \sum_{j=1}^{k_n} Q(j)\frac{e^{-Q(j)\ln(1/q)}}{1-e^{-Q(j)\ln(1/q)}}$$

$$\sim \frac{1}{\ln^2(q^{-1})}\int_0^{Q(k_n)\ln(q^{-1})} \frac{y}{Q'\left(Q^{-1}\left(\frac{y}{\ln(q^{-1})}\right)\right)}\frac{e^{-y}dy}{1-e^{-y}}$$

$$\sim n\int_0^{k_n^d/n^{d/(d+1)}} \frac{y^{1/d}e^{-y}}{1-e^{-y}}dy$$

$$\sim cn\frac{k_n}{n^{1/(d+1)}}, \tag{2.7}$$

which is of lower order than the expected value of the full sum $\sum_{j=1}^{\infty} Q(j)G_{n,Q(j)}$. (Here and throughout the rest of this abstract $c = c_Q$ is an unspecified constant which depends on Q only. Its value is unimportant and may change from one use to another.)

Similar reasoning applied to $\sum G_{n,Q(j)}$ gives,

$$\mathsf{E}_q \sum_{j>k_n} G_{n,Q(j)} \sim \int_{k_n}^{\infty} \frac{e^{-Q(x)\ln(1/q)}}{1-e^{-Q(x)\ln(1/q)}}dx$$

$$\sim \frac{1}{\ln(q^{-1})}\int_{Q(k_n)\ln(q^{-1})}^{\infty} \frac{e^{-y}}{(1-e^{-y})}\frac{dy}{Q'\left(Q^{-1}\left(\frac{y}{\ln(q^{-1})}\right)\right)}$$

$$\sim cn^{1/(d+1)}\int_{\frac{ck_n^d}{n^{d/(d+1)}}}^{\infty} \frac{y^{\frac{1}{d}-1}e^{-y}}{1-e^{-y}}dy$$

$$\sim cn^{1/(d+1)}\cdot\frac{n^{(d-1)/(d+1)}}{k_n^{d-1}} \sim c\frac{n^{d/(d+1)}}{k_n^{d-1}},$$

and

$$\mathsf{E}_q \sum_{j=1}^{\infty} G_{n,Q(j)} \sim cn^{d/(d+1)},$$

Hence, as long as $k_n \to \infty$, the expected value of the sum restricted to $j > k_n$ is of smaller order than that of the full sum. Thus one expects the contribution of $\sum_{j>k_n} G_{n,Q(j)}$ to be negligible. Similarly, (2.7) suggests that the contribution of the truncated sum $\sum_{j\le k_n} Q(j)G_{n,Q(j)}$ to the full sum is negligible.

Of course, the very fact that the two pieces have expectation of lower order than the respective sums over all of natural numbers, does not by itself suffice to argue that they may be dropped from the sums without affecting their magnitude. But both of these expressions, being sums of independent random variables are heavily concentrated about their expected value. This can be quantified by using methods based on exponential inequalities. When these estimates are carried out, we are left with two truncated sums over the disjoint sets of indices, plus error terms that are negligible even when divided by the denominator of (2.6). The gain is that, unlike the original sums, the truncated sums are independent and thus can be handled with relative ease. Since the estimates are very explicit, it is easy to trace down conditions that k_n's need to satisfy and it turns out that one may choose

$$k_n = \Theta(n^{\alpha}), \quad \text{where} \quad 0 < \alpha < \frac{1}{2(d+1)}. \tag{2.8}$$

The upshot of all this is that, for any $x > 0$,

$$\lim_{n\to\infty} \mathsf{P}_n(\mathbf{M}_n/\mu_n \le x) = \lim_{n\to\infty} \mathsf{P}_q(\sum_{j=1}^{k_n} G_{n,Q(j)}/\mu_n \le x).$$

Since the jth summand on the right–hand side above is geometric with parameters $1 - q^{Q(j)}$ its characteristic function is

$$\mathsf{E}_q e^{itG_{n,Q(j)}/\mu_n} = \frac{1-q^{Q(j)}}{1-e^{it/\mu_n}q^{Q(j)}}.$$

Since $j \le k_n = o(n^{1/(2(d+1))})$, $q = \exp(-C_Q/n^{d/(d+1)})$, and $\mu_n = n^{d/(d+1)}/C_Q$ using basic approximations we further have

$$\begin{aligned}
&\frac{1-q^{Q(j)}}{1-e^{it/\mu_n}q^{Q(j)}}\\
&= \frac{1-\exp(-Q(j)C_Q/n^{d/(d+1)})}{1-\exp(it/\mu_n - Q(j)C_Q/n^{d/(d+1)})}\\
&= \frac{C_Q\frac{Q(j)}{n^{d/(d+1)}} + O\left(\frac{Q^2(k)}{n^{2d/(d+1)}}\right)}{C_Q\frac{Q(j)}{n^{d/(d+1)}} - \frac{it}{\mu_n} + O\left(\frac{Q^2(k)}{n^{2d/(d+1)}}\right)}\\
&= \frac{1}{1-\frac{it}{Q(j)}}\left(1 + O\left(\frac{Q^2(k)}{n^{2d/(d+1)}}\right)\right).
\end{aligned}$$

Hence, by independence of the summands, for j's in our range, we get

$$\begin{aligned}
\phi_n(t) &:= \mathsf{E}e^{\frac{t}{\mu_n}\sum_{j=1}^{k_n} G_{n,Q(j)}}\\
&= \prod_{j=1}^{k_n}\left(\frac{1}{1-\frac{it}{Q(j)}}\left(1+O\left(\frac{Q^2(j)}{n^{2d/(d+1)}}\right)\right)\right)\\
&= \left(\prod_{j=1}^{k_n}\frac{1}{1-\frac{it}{Q(j)}}\right)\left(1+O\left(\frac{Q^2(k_n)}{n^{2d/(d+1)}}\right)\right)^{k_n}\\
&= \left(\prod_{j=1}^{k_n}\frac{1}{1-\frac{it}{Q(j)}}\right)\left(1+O\left(\frac{k_nQ^2(k_n)}{n^{2d/(d+1)}}\right)\right). \qquad (2.9)
\end{aligned}$$

Since $k_nQ^2(k_n) = O(k_n^{2d+1})$, for k_n satisfying (2.8), the "big Oh"term in (2.9) goes to zero. Thus we conclude that $\phi_n(t)$ converge pointwise to $\phi_Q(t)$ given by (1.4).

3 Fourier Inversion

In this section we derive an explicit representation for the density of the limit distribution. By inversion formula, this density is given for $x > 0$ by

$$f_Q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_Q(t)dt.$$

If we regard t as a complex variable, then $\phi_Q(t)$ is a meromorphic function with simple poles at $-iQ(j)$, $j \geq 1$. One may then apply residue theory to evaluate the integral and deduce that, for $x > 0$,

$$f_Q(x) = \sum_{j=1}^{\infty} e^{-Q(j)x}Q(j)\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(j)}. \tag{3.10}$$

Specifically, for a large natural number n, we let

$$N = \frac{Q(n)+Q(n+1)}{2}$$

and we let C_N to be a clockwise oriented rectangular contour in the complex plane with vertices at $\pm N$, $\pm N - iN$. We consider the contour integral

$$\frac{1}{2\pi}\oint_{C_N} e^{-itx}\phi_Q(t)dt, \quad x > 0,$$

and we show that the integrals along three non-real sides of C_N approach zero as $n \to \infty$. Since the residue of

$$e^{-itx}\prod_{\ell\geq 1}\frac{1}{1-\frac{it}{Q(\ell)}},$$

at $t = -iQ(j)$ is

$$iQ(j)e^{-xQ(j)}\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(j)},$$

using the residue theorem (taking into account the orientation of C_N) and passing to the limit with n we derive (3.10).

4 Simplification

The expression on the right–hand side of (3.10) may be further transformed by evaluating the product. Specifically, we will show that

$$\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(j)} = \frac{Q'(j)(-1)^{j+1}}{Q(j)(j-1)!}\frac{\prod_{t=1}^{d-1}\Gamma(1-\alpha_t(j))}{\prod_{t=1}^{d}\Gamma(1-r_t)}, \tag{4.11}$$

where $r_1, r_2, \ldots, r_d$ are the roots of $Q(z)$ and $\alpha_1(j), \ldots, \alpha_{d-1}(j)$ are those roots of $Q(z) - Q(j)$ that are not equal to j.

To this end we write

$$\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(j)} = \lim_{s\to j}\left(\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(s)}\right)$$
$$= \lim_{s\to j}\left(\frac{Q(j)-Q(s)}{Q(j)}\prod_{\ell\geq 1}\frac{Q(\ell)}{Q(\ell)-Q(s)}\right). \tag{4.12}$$

We factor both $Q(\ell)$ and $Q(\ell) - Q(s)$ as a product of linear terms

$$\begin{aligned} Q(\ell) &= a_d\prod_{m=1}^{d}(\ell - r_m) \\ Q(\ell)-Q(s) &= a_d\prod_{m=1}^{d}(\ell-\alpha_m(s)). \end{aligned}$$

We now use the following formula [33, Chaptex XII, Sec. 12.13]: if $a_1 + \ldots + a_r = b_1 + \ldots + b_r$ then

$$\prod_{n=1}^{\infty}\frac{(n-a_1)\cdot\ldots\cdot(n-a_r)}{(n-b_1)\cdot\ldots\cdot(n-b_r)} = \prod_{m=1}^{r}\frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}.$$

Applying this to the product in (4.12) we obtain

$$\prod_{\ell\neq j}\frac{Q(\ell)}{Q(\ell)-Q(j)}$$
$$= \lim_{s\to j}\left(\frac{Q(j)-Q(s)}{Q(j)}\prod_{m=1}^{d}\frac{\Gamma(1-\alpha_m(s))}{\Gamma(1-r_m)}\right).$$

We know that exactly one of $\alpha_m(s)$'s is equal to s and we assume without loss of generality that $\alpha_d(s) = s$. Since

$$\frac{1}{Q(j)\Gamma(1-r_d)}\prod_{m=1}^{d-1}\frac{\Gamma(1-\alpha_m(s))}{\Gamma(1-r_m)} \tag{4.13}$$

is continuous at $s = j$ we only need to be concerned with

$$\begin{aligned} &\lim_{s\to j}((Q(j)-Q(s))\Gamma(1-\alpha_d(s))) \\ &= \lim_{s\to j}\left(\frac{Q(j)-Q(s)}{j-s}(j-s)\Gamma(1-s)\right) \\ &= \lim_{s\to j}\left(\frac{Q(j)-Q(s)}{j-s}(j-s)(-s)\Gamma(-s)\right) \\ &= jQ'(j)\lim_{s\to j}((s-j)\Gamma(-s)). \end{aligned}$$

Since the residue of $\Gamma(z)$ at $-j$ is $(-1)^j/j!$ this last limit is $(-1)^{j-1}/j!$ which combined with (4.13) and (4.12) proves (4.11).

5 Further remarks

In this section we briefly discuss a few cases that are of special interest.

(i) One such case, $Q(z) = \frac{z(z+1)}{2}$ arises naturally in the context of iterated functions and the coalescent [14], [18]. There the characteristic function is

$$\begin{aligned}\phi(t) &= \prod_{m=2}^{\infty} \frac{\binom{m}{2}}{\binom{m}{2} - it} \\ &= \prod_{m=1}^{\infty} \frac{1}{1 - it/\binom{m+1}{2}}. \qquad (5.14)\end{aligned}$$

In (4.11) we have $r_1 = 0, r_2 = -1$, $\alpha_1(k) = -k-1$, and consequently

$$\begin{aligned}&\prod_{\ell \neq k} \frac{Q(\ell)}{Q(\ell) - Q(k)} \\ &= \frac{\frac{2k+1}{2}}{\frac{k(k+1)}{2}} \frac{\Gamma(1+1+k)}{\Gamma(1-0)\Gamma(1+1)} \frac{(-1)^{k-1}}{(k-1)!} \\ &= (-1)^{k-1}(2k+1).\end{aligned}$$

Hence inversion of (5.14) yields the probability density function

$$f(x) = \sum_{k=2}^{\infty} e^{-\binom{k}{2}x} \binom{k}{2} (-1)^k (2k-1), \; x > 0.$$

This latter density is well-known in certain circles, and is generally attributed to Kingman [18], [19]. See the unpublished manuscript [14] for a derivation that is related to the arguments in this paper.

(ii) Similarly, for the special case $Q(x) = x^3$, we consider the number of parts of random partitions of n into parts that are cubes. For this particular class of partitions, Richmond [25] provided asymptotic estimates for the moments. Carleman's conditions are satisfied, therefore the limit distribution is uniquely determined. However Richmond did not invert, and we are not aware of any previous work in which the limiting density is calculated. In fact, the density has an interesting form: for $x > 0$,

$$f(x) = 3 \sum_{k=1}^{\infty} e^{-k^3 x} \frac{(-1)^{k+1} k^3 c_k}{k!},$$

where

$$\begin{aligned}c_k &= \Gamma(1 - ke^{2\pi i/3})\Gamma(1 - ke^{-2\pi i/3}) \\ &= |\Gamma(1 - ke^{2\pi i/3})|^2.\end{aligned}$$

(iii) The next case corresponds to $Q(x) = \binom{x+d}{d}$, for some fixed positive integer d. (Since $d = 1$ does not impose any restrictions we will assume $d \geq 2$. Also, $d = 2$ was a special case discussed in (i).) Such partitions are in bijection with partitions with dth differences non-negative. Some of their properties (although the limiting distribution of the number of parts was not one of them) were studied in [5]. We have $r_m = -m$, $m = 1, \ldots, d$ and thus

$$\prod_{m=1}^{d} \Gamma(1 - r_m) = \prod_{m=1}^{d} m!.$$

Further,

$$Q'(x) = \frac{1}{d!} \sum_{j=1}^{d} \prod_{\substack{1 \leq \ell \leq d \\ \ell \neq j}} (x + \ell) = Q(x) \sum_{j=1}^{d} \frac{1}{x+j},$$

so that

$$Q'(k) = Q(k)\left(H_{k+d} - H_k\right),$$

where H_n is the nth harmonic number. Although there does not seem to be a simple way of handling the roots of $Q(x) - Q(k)$ in the general case, the case $d = 3$ can be managed (as can be any other polynomial of degree 3 since it leads to a quadratic equation after factoring $(x - k)$) and gives the density

$$\sum_{k=1}^{\infty} (-1)^{k-1} e^{-\binom{k+3}{3}x} \binom{k+3}{3} \frac{(H_{k+3} - H_k) f_k}{2! \cdot 3! \cdot (k-1)!},$$

where $f_k = \left|\Gamma\left(4 + \frac{k}{2} + \frac{i}{2}\sqrt{3k^2 + 12k + 8}\right)\right|^2$ and $x > 0$.

If $d = 4$ then $\binom{x+4}{4} - \binom{k+4}{4}$ has a real root $-k-5$ (in addition to k, of course) and the limiting density for $x > 0$ is given by

$$\sum_{k=1}^{\infty} (-1)^{k-1} e^{-\binom{k+4}{4}x} \binom{k+4}{4} \frac{(H_{k+4} - H_k) g_k (k+5)!}{2! \cdot 3! \cdot 4! \cdot (k-1)!},$$

where $g_k = \left|\Gamma\left(\frac{7}{2} + \frac{i}{2}\sqrt{4k^2 + 20k + 15}\right)\right|^2$.

(iv) Finally, we would like to conclude by observing that the choice $Q(x) = x^2$ corresponds to yet another interesting situation that arises in quite a different context. In view of (1.5) and (4.11) the probability density function corresponding to this choice is

$$f(x) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} k^2 e^{-k^2 x}, \quad x > 0.$$

Up to a scaling this is the density of the maximum of the Brownian bridge process or the Brownian meandering process (see [7, Section 3] and also [10, 9] for more details and information). Further interesting connections along with many more references to the literature are discussed in a relatively recent survey paper [3].

Distribution function corresponding to the last density is given by

$$\begin{aligned} F(x) &= 1 - 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k^2x} \\ &= \sum_{k=-\infty}^{\infty}(-1)^k e^{-k^2x}. \end{aligned}$$

Changing variables, $x \to 2x^2$ and differentiating gives a density

$$4\sum_{k=-\infty}^{\infty}(-1)^{k-1}k^2 x e^{-2k^2x^2},$$

which is the density of the Kolmogorov-Smirnov statistic used to measure the discrepancy between the true and empirical distribution functions. We refer the reader to [21] for the translation of the original work of Kolmogorov and to [4, Chapter 2, Sec. 13] for a detailed exposition.

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