Abstract

We investigate the impact the negative curvature has on the traffic congestion in large-scale networks. We prove that every Gromov hyperbolic network G admits a core, thus answering in the positive a conjecture by Jonckheere, Lou, Bonahon, and Baryshnikov, Internet Mathematics, 7 (2011) which is based on the experimental observation by Narayan and Saniee, Physical Review E, 84 (2011) that real-world networks with small hyperbolicity have a core congestion. Namely, we prove that for every subset X of n vertices of a graph with δ-thin geodesic triangles (in particular, of a δ-hyperbolic graph) G there exists a vertex m of G such that the ball $B(m, 4\delta)$ of radius 4δ centered at m intercepts at least one half of the total flow between all pairs of vertices of X, where the flow between two vertices $x, y \in X$ is carried by geodesic (or quasi-geodesic) (x, y)-paths. Moreover, we prove a primal-dual result showing that, for any commodity graph R on X and any $r \geq 8\delta$, the size $\sigma_r(R)$ of the least r-multi-core (i.e., the number of balls of radius r) intercepting all pairs of R is upper bounded by the maximum number of pairwise $(2r - 5\delta)$-apart pairs of R and that an r-multi-core of size $\sigma_{r-5\delta}(R)$ can be computed in polynomial time for every finite set X. Our result about total r-multi-cores is based on a Helly-type theorem for quasiconvex sets in δ-hyperbolic graphs (this is our second main result). Namely, we show that for any finite collection $\mathcal{Q}$ of pairwise intersecting ϵ-quasiconvex sets of a δ-hyperbolic graph G there exists a single ball $B(c, 2\epsilon + 5\delta)$ intersecting all sets of $\mathcal{Q}$. More generally, we prove that if $\mathcal{Q}$ is a collection of $2r$-close (i.e., any two sets of $\mathcal{Q}$ are at distance $\leq 2r$) ϵ-quasiconvex sets of a δ-hyperbolic graph G, then there exists a ball $B(c, r^*)$ of radius $r^* := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$ intersecting all sets of $\mathcal{Q}$. These kind of Helly-type results are also useful in geometric group theory. Using the Helly theorem for quasiconvex sets and a primal-dual approach, we show algorithmically that the minimum number of balls of radius $2\epsilon + 5\delta$ intersecting all sets of a family $\mathcal{Q}$ of ϵ-quasiconvex sets does not exceed the packing number of $\mathcal{Q}$ (maximum number of pairwise disjoint sets of $\mathcal{Q}$). We extend the covering and packing result to set-families ${}^{\kappa}\mathcal{Q}$ in which each set is a union of at most κ ϵ-quasiconvex sets of a δ-hyperbolic graph G. Namely, we show that if $r \geq \epsilon + 2\delta$ and $\pi_r({}^{\kappa}\mathcal{Q})$ is the maximum number of mutually $2r$-apart members of ${}^{\kappa}\mathcal{Q}$, then the minimum number of balls of radius $r + 2\epsilon + 6\delta$ intersecting all members of ${}^{\kappa}\mathcal{Q}$ is at most $2\kappa^2\pi_r({}^{\kappa}\mathcal{Q})$ and such a hitting set and a packing can be constructed in polynomial time for every finite ${}^{\kappa}\mathcal{Q}$ (this is our third main result). For set-families consisting of unions of κ balls in δ-hyperbolic graphs a similar result was obtained by Chepoi and Estellon (2007). In case of $\delta = 0$ (trees) and $\epsilon = r = 0$, (subtrees of a tree) we recover the result of Alon (2002) about the transversal and packing numbers of a set-family in which each set is a union of at most κ subtrees of a tree.