Abstract

Our input instance is a bipartite graph $G = (A \cup B, E)$ where A is a set of applicants, B is a set of jobs, and each vertex $u \in A \cup B$ has a preference list ranking its neighbors in a strict order of preference. For any two matchings M and T in G, let $\phi(M, T)$ be the number of vertices that prefer M to T. A matching M is *popular* if $\phi(M, T) \geq \phi(T, M)$ for all matchings T in G. There is a utility function $w : E \to \mathbb{Q}$ and we consider the problem of matching applicants to jobs in a popular and utility-optimal manner. A popular *mixed matching* could have a much higher utility than all popular matchings, where a mixed matching is a probability distribution over matchings, i.e., a mixed matching $\Pi = \{(M_0, p_0), \ldots, (M_k, p_k)\}$ for some matchings $M_0, \ldots, M_k$ and $\sum_{i=0}^{k} p_i = 1$, $p_i \geq 0$ for all i. The function $\phi(\cdot, \cdot)$ easily extends to mixed matchings; a mixed matching Π is popular if $\phi(\Pi, \Lambda) \geq \phi(\Lambda, \Pi)$ for all mixed matchings Λ in G. Motivated by the fact that a popular mixed matching could have a much higher utility than all popular matchings, we study the popular fractional matching polytope $\mathcal{P}_G$. Our main result is that this polytope is half-integral and in the special case where a stable matching in G is a perfect matching, this polytope is integral. This implies that there is always a max-utility popular mixed matching Π such that $\Pi = \{(M_0, \frac{1}{2}), (M_1, \frac{1}{2})\}$ where M_0 and M_1 are matchings in G. As Π can be computed in polynomial time, an immediate consequence of our result is that in order to implement a max-utility popular mixed matching in G, we need just *a single* random bit. We analyze $\mathcal{P}_G$ whose description may have exponentially many constraints via an extended formulation with a linear number of constraints. The linear program that gives rise to this formulation has an unusual property: *self-duality*. In other words, this linear program is identical to its dual program. This is a rare case where an LP of a natural problem has such a property. The self-duality of this LP plays a crucial role in our proof of half-integrality of $\mathcal{P}_G$. We also show that our result carries over to the *roommates* problem, where the graph G need not be bipartite. The polytope of popular fractional matchings is still half-integral here and so we can compute a max-utility popular half-integral matching in G in polynomial time. To complement this result, we also show that the problem of computing a max-utility popular (integral) matching in a roommates instance is NP-hard.