Constraint Satisfaction and Graph Theory

Pavol Hell

SIAM DM, June 2008
NP versus Colouring

Problems in NP

NP-complete

\[ \vdots \]

P

k–colouring problems

NP-complete

\[ k > 2 \]

P

k=1,2

Constraint Satisfaction and Graph Theory
CSP
Assign values to variables so that all constraints are satisfied
Constraint Satisfaction Problems

CSP
Assign values to variables so that all constraints are satisfied

Examples
- SAT
Constraint Satisfaction Problems

CSP
Assign values to variables so that all constraints are satisfied

Examples
- SAT
- 3-COL
Constraint Satisfaction Problems

CSP
Assign values to variables so that all constraints are satisfied

Examples
- SAT
- 3-COL
- 
  \((x, y) \in \{(1, 1), (2, 3)\}\) and
  \((x, z, w) \in \{(2, 2, 1), (1, 3, 2), (2, 2, 2)\}\)
NP versus CSP

Problems in NP

NP-complete

\[ \vdots \]

P

Problems in CSP

NP-complete

? 

P
Generalizing Colouring

Constraint Satisfaction and Graph Theory
Generalizing Colouring

Which Problems / Complexities

What is this problem?

Constraint Satisfaction and Graph Theory
Which Problems / Complexities

1

2

1

3
Which Problems / Complexities

What is this problem?
Generalizing Colouring

\[ \begin{array}{c}
1 \\
\downarrow \\
1
\end{array} \quad \begin{array}{c}
2 \\
\downarrow \\
3
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3
\end{array} = \text{HOM}(H) \]

Constraint Satisfaction and Graph Theory
Homomorphism problem $\text{HOM}(H)$

A colouring of a graph $G$ without the above pattern is exactly a homomorphism to $H$
Each constraint satisfaction problem is polynomially equivalent to $\text{HOM}(H)$ for some digraph $H$.
NP versus CSP

Problems in NP
- NP-complete
- ...
- P

Problems in CSP
- NP-complete
- ?
- HOM(H)
- P
Which Problem

Clique Cutset Problem

Matrix partition problems (Feder-Hell-Motwani-Klein)
Which Problem

Clique Cutset Problem
Generalizing Colouring

Which Problem

Clique Cutset Problem

Matrix partition problems (Feder-Hell-Motwani-Klein)
NP versus MPP

Problems in NP

NP-complete

\dots

P

Matrix Partition Problems

NP-complete

?
Given a 3-edge-coloured $K_n$, colour the vertices without a monochromatic edge.
Cameron-Eschen-Hoang-Sritharan
Stubborn problem

Feder-Hell
Given a 3-edge-coloured $K_n$, colour the vertices without a monochromatic edge
Cameron-Eschen-Hoang-Sritharan

Stubborn problem

Feder-Hell

Given a 3-edge-coloured $K_n$, colour the vertices without a monochromatic edge

Complexity ? (Feder-Hell-Kral-Sgall)
Which Problem

Is

$H$

partitionable into a triangle-free graph and a cograph?

Partition problems, generalized colouring problems, subcolouring problems, etc., Alekseev, Farrugia, Lozin, Broersma, Fomin, Nesetril, Woeginger, Ekim, de Werra, Stacho, MacGillivray, Yu, Hoang, Le, etc.

Pavol Hell

Constraint Satisfaction and Graph Theory
Which Problem

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What Problem is This?
What Problem is This?
Generalizing Colouring

What Problem is This?

¿
Which Problems / Complexities

How general are these "pattern-forbidding colouring problems"?
Which Problems / Complexities

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Fagin + Feder-Vardi + Kun-Nesetril + Nesetril-Tardif
NP versus CSP

Which Problems / Complexities
How general are these "pattern-forbidding colouring problems"?

Fagin + Feder-Vardi + Kun-Nesetril + Nesetril-Tardif

- Every problem in NP is polynomially equivalent to a pattern-forbidding colouring problem.
NP versus CSP

Which Problems / Complexities

How general are these "pattern-forbidding colouring problems"?

Fagin + Feder-Vardi + Kun-Nesetril + Nesetril-Tardif

- Every problem in NP is polynomially equivalent to a pattern-forbidding colouring problem
- Every problem in CSP is polynomially equivalent to a pattern-forbidding colouring problem with patterns on single edges
Which Problems / Complexities

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- Every problem in CSP is polynomially equivalent to a pattern-forbidding colouring problem with patterns on single edges (or... trees)
NP versus CSP

Which Problems / Complexities

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- Every problem in NP is polynomially equivalent to a pattern-forbidding colouring problem.
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Some finite set of patterns corresponds to isomorphism complete problems. What does it look like?
NP versus CSP

Problems in NP

NP-complete

\[ \vdots \]

P

Problems in CSP

NP-complete

? 

P
NP versus CSP

Pattern–forbidding Problems

NP–complete

...?
P

Pattern–forbidding Problems
with Single–edge Patterns

NP–complete

P
NP versus MPP

Pattern–forbidding Problems

NP–complete

Pattern–forbidding Problems
with Two–vertex Patterns

NP–complete

P

?
NP versus MPP

Problems in NP

NP–complete

P

Matrix Partition Problems

NP–complete

?
NP versus CSP

Problems in NP

NP–complete

P

Problems in CSP

NP–complete

P
Polymorphisms

**POL(H) for a digraph H**

\[ f : V(H)^k \rightarrow V(H) \text{ such that } a_i b_i \in E(H) \ \forall i \implies f(a_1, a_2, \ldots, a_k)f(b_1, b_2, \ldots, b_k) \in E(H). \]
Polymorphisms

POL($H$) for a digraph $H$

$f : V(H)^k \rightarrow V(H)$ such that

$a_ib_i \in E(H) \ \forall i \implies f(a_1, a_2, \ldots, a_k)f(b_1, b_2, \ldots, b_k) \in E(H)$.

Jeavons

If $POL(H) \subseteq POL(H')$, then $HOM(H')$ reduces to $HOM(H)$
Polymorphisms

**POL(H) for a digraph H**

\[ f : V(H)^k \rightarrow V(H) \text{ such that } \]
\[ a_i b_i \in E(H) \quad \forall i \quad \implies \quad f(a_1, a_2, \ldots, a_k) f(b_1, b_2, \ldots, b_k) \in E(H). \]

**Jeavons**

If \( \text{POL}(H) \subseteq \text{POL}(H') \), then \( \text{HOM}(H') \) reduces to \( \text{HOM}(H) \).

The more polymorphisms \( H \) has, the more likely is \( \text{HOM}(H) \) to be polynomial.
How small can $\text{POL}(H)$ be?

**Projective $H$**

$\text{POL}(H)$ consists only of all projections, composed with automorphisms of $H$
How small can $\text{POL}(H)$ be?

**Projective $H$**

$\text{POL}(H)$ consists only of all projections, composed with automorphisms of $H$

**$K_n$**

The graph $K_n$, with $n \geq 3$, is projective
How small can $\text{POL}(H)$ be?

**Projective $H$**

$\text{POL}(H)$ consists only of all projections, composed with automorphisms of $H$.

**$K_n$**

The graph $K_n$, with $n \geq 3$, is projective.

**Therefore**

If $H$ is projective, then $\text{HOM}(H)$ is NP-complete.
Example Polymorphisms

Majority polymorphism:
\[ f(u, u, v) = f(u, v, u) = f(v, u, u) = u \]

Near unanimity polymorphism:
\[ f(v, u, \ldots, u) = \cdots = f(u, u, \ldots, v) = u \]

Weak unanimity polymorphism:
\[ f(u, u, \ldots, u) = u, f(v, u, \ldots, u) = \cdots = f(u, u, \ldots, v) \]

Maltsev polymorphism:
\[ f(u, u, v) = f(v, u, u) = v \]
Example Polymorphisms

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Examples

Example Polymorphisms

- Majority polymorphism:
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- Near unanimity polymorphism:
  \[ f(v, u, \ldots, u) = \cdots = f(u, u, \ldots, v) = u \]

- Weak unanimity polymorphism:
  \[ f(u, u, \ldots, u) = u, \quad f(v, u, \ldots, u) = \cdots = f(u, u, \ldots, v) \]

- Maltsev polymorphism:
  \[ f(u, u, v) = f(v, u, u) = v \]
If $H$ has a near unanimity polymorphism, or a Maltsev polymorphism, then the problem $\text{HOM}(H)$ is in $P$. 
All known polynomial cases are attributable to some nice polymorphism.
HOM(H) is NP-complete if

HOM(H) is K3-partitionable

HOM(H) is block-projective

some algebra in VAR((V(H), POL(H))) is projective,

and all these conditions are equivalent.

Conjecture

In all other cases HOM(H) can be solved in polynomial time.
HOM(\(H\)) is NP-complete if

- \(H\) has no weak near unanimity polymorphism
HOM(H) is NP-complete if

- H has no weak near unanimity polymorphism
- H is $K_3$-partitionable
HOM(H) is NP-complete if

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Conjecture
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Pavol Hell
Constraint Satisfaction and Graph Theory
HOM($H$) is NP-complete if

- $H$ has no weak near unanimity polymorphism
- $H$ is $K_3$-partitionable
- $H$ is block-projective
- some algebra in $VAR((V(H), POL(H))$ is projective,
HOM(H) is NP-complete if

- H has no weak near unanimity polymorphism
- H is $K_3$-partitionable
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- some algebra in $VAR((V(H), POL(H))$ is projective,
- and all these conditions are equivalent
HOM($H$) is NP-complete if

- $H$ has no weak near unanimity polymorphism
- $H$ is $K_3$-partitionable
- $H$ is block-projective
- some algebra in $VAR((V(H),POL(H)))$ is projective,
- and all these conditions are equivalent

Conjecture

In all other cases HOM($H$) can be solved in polynomial time
The Dichotomy Conjecture

The problem HOM(H) is
The Dichotomy Conjecture

The problem $\text{HOM}(H)$ is

- In P is $H$ admits a weak near unanimity polymorphism
The Dichotomy Conjecture

The problem \( \text{HOM}(H) \) is

- In P is \( H \) admits a weak near unanimity polymorphism
- NP-complete if \( H \) does not admit a weak near unanimity polymorphism
Example application:
Example application:

**Barto-Kozik-Niven 2008**

If $H$ has neither sources nor sinks, then

- $\text{HOM}(H)$ is in P if $H$ retracts to a cycle
- $\text{HOM}(H)$ is NP-complete otherwise
Example application:

**Barto-Kozik-Niven 2008**

If $H$ has neither sources nor sinks, then

- $\text{HOM}(H)$ is in P if $H$ retracts to a cycle
- $\text{HOM}(H)$ is NP-complete otherwise
  (there is no weak near unanimity polymorphism)
Reflexive Graphs

The following are equivalent

- $G$ has a majority polymorphism
Reflexive Graphs

The following are equivalent

- $G$ has a majority polymorphism
- $G$ is a retract of a product of paths
Reflexive Graphs

The following are equivalent

- $G$ has a majority polymorphism
- $G$ is a retract of a product of paths
- $G$ is cop-win and clique-Helly
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Hell-Rival, Nowakowski-Rival, Pesch-Poguntke
Reflexive Graphs

The following are equivalent

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Hell-Rival, Nowakowski-Rival, Pesch-Poguntke
Feder-Vardi - decidable in polynomial time
Reflexive Graphs

- Any chordal graph $G$ admits a near-unanimity polymorphism
Reflexive Graphs

- Any chordal graph $G$ admits a near-unanimity polymorphism
- A chordless cycle of length $>3$ does not admit a near-unanimity polymorphism
Reflexive Graphs

The following statements are equivalent

- $G$ admits a near-unanimity polymorphism

$G$ dismantles to the diagonal

Decidable in polynomial time
Reflexive Graphs

The following statements are equivalent

- $G$ admits a near-unanimity polymorphism
- $G^2$ dismantles to the diagonal
Reflexive Graphs

The following statements are equivalent

- $G$ admits a near-unanimity polymorphism
- $G^2$ dismantles to the diagonal

Decidable in polynomial time
Reflexive Graphs

The following statements are equivalent

- $G$ has a conservative near-unanimity polymorphism
Reflexive Graphs

The following statements are equivalent

- $G$ has a conservative near-unanimity polymorphism
  
  $\left( f(a, b, \ldots) \in \{a, b, \ldots\} \right)$
Reflexive Graphs

The following statements are equivalent

- $G$ has a conservative near-unanimity polymorphism
  \[ f(a, b, \ldots) \in \{a, b, \ldots\} \]
- $G$ is an interval graph
For a connected $H$ the following are equivalent

- $H$ admits a near-unanimity operation
For a connected $H$ the following are equivalent:

- $H$ admits a near-unanimity operation
- $H$ has bounded treewidth duality
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- $\text{HOM}(H)$ can be expressed in Datalog
For a connected $H$ the following are equivalent

- $H$ admits a near-unanimity operation
- $H$ has bounded treewidth duality
- $\text{HOM}(H)$ can be expressed in Datalog
- $\text{RET}(H)$ can be expressed by a first order formula
Rossman, Atserias, Larose-Loten-Zadori

- $H$ has finite duality $\iff$

$\text{HOM}(H)$ can be expressed by a first order formula

some retract $G$ of $H$ has the property that $G \times G$ dismantles to the diagonal

and then CSP($H$) is in P

Nesetril-Tardif

If $H$ has finite duality then $H$ has tree duality
Rossman, Atserias, Larose-Loten-Zadori

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Rossman, Atserias, Larose-Loten-Zadori

- $H$ has finite duality $\iff$
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Rossman, Atserias, Larose-Loten-Zadori

- $H$ has finite duality $\iff$
- $\text{HOM}(H)$ can be expressed by a first order formula
- some retract $G$ of $H$ has the property that $G \times G$ dismantles to the diagonal
- and then $\text{CSP}(H)$ is in $P$
Rossman, Atserias, Larose-Loten-Zadori

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If $H$ has finite duality then $H$ has tree duality