Irreducible Powerful Ray Patterns*

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1 Notations and Preliminaries

To explore the qualitative or combinatorial properties of nonnegative matrices, many authors made use of Boolean matrix. And as a natural generalization of Boolean matrix, many authors considered sign pattern. Sign pattern is a matrix each of whose entries is 0, −1 or 1 with its own algebra (See [2]). Sign pattern can be considered as abstraction of real matrix. So it is natural to consider abstraction of complex matrix. The authors of the recent paper (See [3]) studied this topic. Ray pattern is a matrix each of whose entries is either 0 or a ray $e^{i\theta}$ where $\theta$ is real number. Table 1 shows the addition and the multiplication of 0 and rays.

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Table 1. Addition and multiplication of 0 and rays

We denote by # any sum of rays where at least two of the rays are distinct. And we call # ambiguous entry. The product of $m \times p$ ray pattern $A = [a_{ij}]$ and $p \times n$ ray pattern $B = [b_{ij}]$ is defined as usual; the $(s, t)$ entry of $AB$ is $\sum_{k=1}^{p} a_{sk} b_{kt}$. Note that the product of two ray patterns does not always yield ray pattern, since some entries of the product can be #.

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Let $A = (a_{rs})$ be an $n \times n$ ray pattern. The *digraph of $A, denoted $D(A),$ is the digraph with vertex set $\{1, 2, \ldots, n\}$ such that there is an arc from $r$ to $s$ iff $a_{rs} \neq 0$. By a *walk* of length $k$ in $A$ we mean a formal product of some nonzero entries of $A$ of the form $W = a_{i_0 j_1}a_{i_1 j_2} \cdots a_{i_{k-1} j_k}$ such a walk $W$ is a called *path* if the indices $i_0, i_1, i_2, \ldots, i_k$ are distinct, except possibly $i_0 = i_k$. Note that a walk $W$ may be identified with the corresponding walk in the digraph $D(A)$. A *cycle* of length $k$ in $A$ is a nonzero product of the form $\gamma = a_{i_k j_1}a_{i_1 j_2} \cdots a_{i_{k-1} j_k}$ where the indices $i_1, i_2, \ldots, i_k$ are distinct. For a walk of $W$ in $A$, we define the *actual product* of $W$, denoted by $ap(W)$, to be the product of the entries in $W$.

We say that an $n \times n$ ray pattern $A$ is *powerful* if for each positive integer $k$, the matrix $A^k$ has no #. For a powerful ray pattern $A$, consider the sequence $A = A^1, A^2, A^3, \ldots$. If this sequence has repetitions, we say the ray pattern $A$ is *periodic*. Let $A^l$ be the first one that is repeated. Write $A^l = A^{l+p}$ with the minimal $p > 0$. Then $l$ is called the base of $A$, and $p$ the period of $A$. Denote the base of $A$ by $l(A)$, and the period of $A$ by $p(A)$.

For a ray pattern $A = [a_{ij}]$, we define $|A| = [a'_{ij}]$ where $a'_{ij} = 1$ if $a_{ij} \neq 0$ and $a'_{ij} = 0$ if $a_{ij} = 0$. Note that $|A|$ is a Boolean matrix. If $|D|$ is an identity matrix, we say $D$ is a *diagonal ray pattern*. For ray patterns $A = [a_{ij}]$ and $B = [b_{ij}]$, we say $B$ is *ray diagonally similar to $A$* if there exists a diagonal ray pattern $D$ such that $A = DBD^*$. And if $a_{ij} = \delta_{ij}b_{ij}$ where $\delta_{ij}$ is 1 or 0 for all $i, j$, then we say $A$ is a *subpattern* of $B$.

In this paper we consider the power of square ray patterns. Note that each powerful sign pattern $A$ is periodic (See [2]). But for the ray pattern

$$A = e^t \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$A$ is powerful but not periodic. In short, powerfulness does not guarantee that the ray pattern is periodic. The ray $\omega$ is periodic if there exists some positive integer $m$ such that $\omega^m = 1$. And if $\omega$ is periodic the smallest $m$ satisfies the equation $\omega^m = 1$ is called the period of $\omega$ and denote it by $p(\omega)$. In this example, we can say that $A$ is not periodic since the ray $e^t$ is not periodic.

Let $A$ be an irreducible ray pattern with *index of imprimitivity* denoted by $k(A) = k$, where $k$ is equal to the greatest common divisor of the lengths of the cycles in $A$. By adapting arguments on irreducible nonegative matrices, we see that $A$ is permutationally similar to a ray pattern in block cyclic form, see [1]. For simplicity of notation, we may assume that $A$ is already in block cyclic form:

$$A = \begin{bmatrix} O & A_{1,2} & \cdots & O \\ O & A_{2,3} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ A_{k-1,1} & O & \cdots & O \end{bmatrix}$$

(1)

where the zero diagonal blocks are square, and the nonzero blocks have no zero row or zero column.
Note that if $A$ is periodic, then $A^k$ is also periodic. So each diagonal block of $A^k$, $A_{i,i+1}A_{i+1,i+2}\cdots A_{i-1,i}$ where the indices are modulo $k$, is periodic for all $i$. Now we represent some previous results which will be used in this paper.

**Proposition 1.** (See Lemma 1.2 in [3]) The set of powerful ray patterns is closed under the following operations:

(i) multiplication by any ray;
(ii) transposition;
(iii) conjugate transposition (denoted by $^*$);
(iv) diagonal similarity;
(v) permutational similarity;
(vi) direct sum;
(vii) taking subpatterns.

**Proposition 2.** (See Theorem 2.1 in [3]) Let $A$ be an $n \times n$ entrywise nonzero ray pattern. Then $A$ is powerful iff $A$ is ray diagonally similar to $e^{\theta}J$ for some $\theta \in \mathbb{R}$.

**Proposition 3.** (See Theorem 3.5 in [3]) Every irreducible powerful ray pattern is a subpattern of an entrywise nonzero powerful ray pattern.

By combining the above two propositions, we obtain the following theorem.

**Theorem 4.** Suppose a ray pattern $A$ is irreducible. Then $A$ is powerful iff $A$ is ray diagonally similar to $\omega|A|$ where $\omega$ is a ray.

**Proof.** ($\Leftarrow$) It is trivial since $\omega|A|$ is powerful.
($\Rightarrow$) Since $A$ is an irreducible powerful ray pattern, there exists an entrywise nonzero powerful ray pattern $\hat{A}$ such that $A$ is subpattern of $\hat{A}$ by Proposition 3. And there exists a diagonal ray pattern $D$ such that $D\hat{A}D^* = \omega J$ for some ray $\omega$ by Proposition 2. Since $D\hat{A}D^*$ is a subpattern of $D\hat{A}D^*$, $D\hat{A}D^* = \omega K$ where $K$ is a Boolean matrix. Let $A = (a_{ij})$ and $D = \text{diag}(d_1, \ldots, d_n)$. Then the $(i,j)$-th entry of $D\hat{A}D^*$ is $d_i a_{ij} d_j$. So we have $|D\hat{A}D^*| = |A| = |K|$. Therefore $A$ is ray diagonally similar to $\omega|A|$. \[\square\]

In this paper, we consider the relationship between the ray pattern $A$ and $|A|$ in section 2. And we consider the base and the period of ray patterns in section 3.

## 2 Ray pattern $A$ and $|A|$

Suppose an irreducible ray pattern $A$ is powerful. By Theorem 4, we have a new block form for the irreducible ray pattern $A$. For simplicity of notation, we may
assume that $A$ is already of block cyclic form

$$
A = \omega \begin{bmatrix}
0 & |A_{1,2}| & 0 \\
0 & 0 & |A_{2,3}|
\vdots & \vdots & \ddots \\
|A_{k-1,k}| & 0 & 0
\end{bmatrix}.
$$

(2)

This means that there is $\omega$ such that $A$ is ray diagonally similar to $\omega|A|$. Then how can we find such $\omega$? First, consider the case when $A$ is a cycle.

**Lemma 5.** Suppose ray pattern $A$ is in cyclic form

$$
A = \begin{bmatrix}
0 & \alpha_1 & 0 & \cdots & 0 \\
0 & 0 & \alpha_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_k & 0 & 0 & \cdots & 0
\end{bmatrix},
$$

where $\alpha_i$ is a ray for each $i$, $\alpha_1 \alpha_2 \cdots \alpha_k = \alpha \neq 0$ and each of off-diagonal entries is 0. Then $A$ is ray diagonally similar to

$$
\omega = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}
$$

for each $\omega$ such that $\omega^k = \alpha$.

**Proof.** Suppose $\alpha = 1$. Let $\text{arg}(\alpha_i) = \theta_i$ for each $i$ and $\theta \in R$. Consider $d_i$ where $\text{arg}(d_1) = \theta$ and $\text{arg}(d_i) = \theta + \sum_{j=1}^{i-1} \theta_j$ for $i \geq 2$, and take $D = \text{diag}\{d_1, d_2, \cdots, d_k\}$. Then the arguments of $(i, i + 1)$ entry of $DAD^*$ is

$$
\text{arg}(d_i \alpha_i d_{i+1}^*) = (\theta + \sum_{j=1}^{i-1} \theta_j) + \theta_i - (\theta + \sum_{j=1}^{i} \theta_j) = 0
$$

where $2 \leq i \leq k - 1$ and the argument is modulo $2\pi$. And we can have $\text{arg}(d_1 \alpha_1 d_2^*) = 0$ and $\text{arg}(d_k \alpha_k d_k^*) = 0$ since $\text{arg}(\alpha) = \sum_{j=1}^{k} \theta_j = 0$ modulo $2\pi$. Therefore if $\alpha = 1$, $A$ is ray diagonally similar to

$$
DAD^* = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
$$
In general case, let $B_\omega = \frac{1}{\omega} A$ for each $\omega$ such that $\omega^k = \alpha$. Since $B_\omega = \frac{1}{\omega} A$ is ray
diagonally similar to
\[
\begin{bmatrix}
0 & 1 & \cdots & 1 \\
0 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0
\end{bmatrix},
\]
we have $A$ is ray diagonally similar to $\omega |A|$ for each $\omega$ such that $\omega^k = \alpha$.

Consider the following matrix
\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]
Following the proof of the Lemma 5, we may find two diagonal ray patterns
\[
D_1 = \begin{bmatrix}
1 & 0 \\
0 & i
\end{bmatrix}, \quad D_2 = \begin{bmatrix}
1 & 0 \\
0 & -i
\end{bmatrix}
\]
such that $D_1 AD_1^* = -i |A|$ and $D_2 AD_2^* = i |A|$ hold. Does there exist a third ray $\omega$
such that $\omega \neq i, -i$ and $DAD^* = \omega |A|$ for some diagonal ray pattern $D$?
Formally speaking, for an irreducible powerful ray pattern $A$, let
\[
\Omega(A) = \{ \omega |A| \text{ is ray diagonally similar to } \omega |A| \}.
\]
What is the cardinality of $\Omega(A)$? In the following, we answer this question.
If a matrix $A$ is of the form
\[
\begin{bmatrix}
A_1 & & & \\
& A_2 & & \\
& & \ddots & \\
& & & A_s
\end{bmatrix},
\]
where each $A_i$ is square for $1 \leq i \leq s$ and each of off-diagonal block submatrices is
a zero matrix, then we denote $A$ by $\bigoplus_{i=1}^s A_i$.

**Lemma 6.** If an irreducible ray pattern $A$ is of block cyclic form (2), then $A$ is ray
diagonally similar to $\alpha |A|$ for each $\alpha$ such that $\alpha^k = \omega^k$.

**Proof.** By Lemma 5, for each $\alpha$ such that $\alpha^k = \omega^k$, there exists a diagonal ray
pattern $D = \text{diag}\{d_1, \cdots, d_k\}$ such that
\[
D = \begin{bmatrix}
0 & \omega & 0 & \omega & \cdots & 0 & \omega \\
\omega & 0 & \omega & 0 & \cdots & 0 & \omega \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\omega & 0 & \omega & 0 & \cdots & 0 & \omega
\end{bmatrix}, \quad D^* = \alpha
\begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & 0
\end{bmatrix}.
Let $E = \bigoplus_{i=1}^{k} d_i I_i$ where $I_i$ is identity matrix for each $i$. Then the $(i, i + 1)$ block of $E A E^t$ is 
\[ d_i I_i(\omega | A_{i,i+1}) d_{i+1} I_{i+1} = \alpha |A_{i,i+1}|. \]
Therefore we have $A$ is ray diagonally similar to $\omega |A|$ for each $\alpha$ such that $\alpha^k = \omega^k$. 

Note that Lemma 6 says that $|\Omega(A)| \geq k(A)$ for irreducible powerful ray pattern $A$.

**Lemma 7.** Suppose irreducible ray pattern $A$ is powerful. If $A$ is ray diagonally similar to $\omega |A|$ and $\omega' |A|$, then $\omega^k(A) = (\omega')^k(A)$.

**Proof.** Let $k(A) = k$ and $L(A) = \{l_1, l_2, \cdots, l_m\}$ be the set of lengths of the cycles in $A$. First we show that if $\gamma$ is a cycle in $D(A)$ whose length is $s$ then $\omega^s = ap(\gamma)$. We can take a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$ such that $D A D^t = \omega |A|$. And we have $D A^s D^t = \omega^s |A|^s$. Let $i$ be a vertex on $\gamma$, then the $(i, i)$ entry of $D A^s D^t$ is $d_i ap(\gamma) d_i$. Since $d_i d_i = 1$, we have $\omega^s = ap(\gamma)$.

Since $k$ is the greatest common divisor of $L(A)$, we can take integers $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that $\sum_{i=1}^{m} \alpha_i l_i = k$. Assume that $\omega$, $\omega' \in \Omega(A)$. Then we have 
\[ \omega^k = (\omega^{l_1})^{\alpha_1} (\omega^{l_2})^{\alpha_2} \cdots (\omega^{l_m})^{\alpha_m}. \]
And for each $i$, $(\omega^{l_i})^{\alpha_i} = (ap(\gamma_i))^{\alpha_i}$ where $\gamma_i$ is a cycle of length $l_i$. So we have 
\[ \omega^k = (ap(\gamma_1))^{\alpha_1} (ap(\gamma_2))^{\alpha_2} \cdots (ap(\gamma_m))^{\alpha_m}, \]
where $\gamma_i$ is a cycle of length $l_i$. Same reasoning shows that 
\[ (\omega')^k = (ap(\gamma_1))^{\alpha_1} (ap(\gamma_2))^{\alpha_2} \cdots (ap(\gamma_m))^{\alpha_m}. \]
So $\omega^k = (\omega')^k$. 

Note that Lemma 7 means that $|\Omega(A)| \leq k(A)$.

By combining Lemma 6 and Lemma 7, we can obtain the following theorem.

**Theorem 8.** Suppose irreducible ray pattern $A$ is powerful. Then $|\Omega(A)| = k(A)$.

In [2], the authors introduce the concept of cyclically nonnegative sign pattern. We generalize this concept to ray patterns. A ray pattern $A$ is cyclically nonnegative if the actual product of each cycle in $A$ is 1.

**Corollary 9.** Suppose an irreducible powerful ray pattern $A$ is ray diagonally similar to $\omega |A|$ for some ray $\omega$. $A$ is cyclically nonnegative iff $\omega^k(A) = 1$.

**Proof.** Let $k(A) = k$, and let $L(A)$ and $\alpha_1, \alpha_2, \cdots, \alpha_m$ be the same as in the proof of Theorem 7. Suppose $\omega^k = 1$. Since $k|l_i$ for each $i$, $\omega^{l_i} = 1$. Next, suppose $A$ is cyclically nonnegative. Same reasoning as in the proof of Theorem 7 shows that 
\[ \omega^k = (ap(\gamma_1))^{\alpha_1} (ap(\gamma_2))^{\alpha_2} \cdots (ap(\gamma_m))^{\alpha_m}. \]
By assumption, \( ap(\gamma_i) = 1 \) for each \( i \). Thus \( \omega^k = 1 \). This completes the proof \( \square \)

In the proof of Lemma 7, we actually prove the following proposition.

**Proposition 10.** Suppose irreducible ray pattern \( A \) is powerful. If \( A \) is ray diagonally similar to \( \omega|A| \), then \( \omega^s = ap(\gamma) \) for each cycle \( \gamma \) in \( D(A) \) whose length is \( s \).

Now we can answer the question which was given in the middle of this section. Let \( A \) be a ray pattern

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Note that \( |\Omega(A)| = k(A) = 2 \) by Theorem 8. Furthermore, since \( D(A) \) has only one cycle whose length is \( 2 \) with actual product \( -1 \), thus we can find \( \Omega(A) = \{i, -i\} \) without considering the diagonal ray pattern \( D \) by Theorem 8 and Proposition 10.

### 3 The bases and the periods of ray patterns

For brevity, when we say a ray pattern \( A \) is periodic, we assume that \( A \) is powerful. In this section, we suggest another characterizations of periodic ray patterns.

**Proposition 11.** (See Lemma 1.2 in [2]) Suppose powerful ray pattern \( A \) is periodic. Then for positive integers \( m, k \), \( A^m = A^{m+k} \) iff \( m \geq l(A) \) and \( p(A)|k \).

Note that Proposition 11 in [2] considered the case that \( A \) is a powerful sign pattern. But we can obtain the Proposition 11 by a slight modification of the statement in origin. Now, the following result is easily obtained from Theorem 4.

**Theorem 12.** If a periodic irreducible ray pattern \( A \) is ray diagonally similar to \( \omega|A| \), then we have \( l(A) = l(|A|) \) and \( p(A) = \text{lcm}\{p(\omega), p(|A|)\} \). In particular, if \( k(A) = k \) then we have \( p(A) = p(\omega^k)k \).

**Proof.** We may assume \( A = \omega|A| \) since the base and the period are invariant under ray diagonal similarity. Let \( \text{lcm}\{p(\omega), p(|A|)\} = p \). Then

\[
A^{l(A)+p(A)} = A^{l(A)},
\]

\[
\omega^{l(A)+p(A)}|A|^{l(A)+p(A)} = \omega^{l(A)}|A|^{l(A)},
\]

\[
\omega^{p(A)}|A|^{l(A)+p(A)} = |A|^{l(A)}.
\]

Since each nonzero entry of \( |A| \) is 1, \( \omega^{p(A)} \) must be 1. Hence \( p(\omega)|p(A) \). So from the last equality, we have \( |A|^{l(A)+p(A)} = |A|^{l(A)} \). Thus \( l(A) \geq l(|A|) \) and \( p(|A|)|p(A) \) by Proposition 11. Therefore \( l(A) \geq l(|A|) \) and \( p|p(A) \). Also we can obtain the following equations.
\[ |A|^{l(|A|)+p(|A|)} = |A|^{l(|A|)} ,
|A|^{l(|A|)+p} = |A|^{l(|A|)} ,
\omega^{l(|A|)+p|A|^{l(|A|)+p}} = \omega^{l(|A|)+p} |A|^{l(|A|)},
\omega^{l(|A|)+p|A|^{l(|A|)+p}} = \omega^{l(|A|)} |A|^{l(|A|)},
A^{l(|A|)+p} = A^{l(|A|)} .
\]

So we have \( l(|A|) \geq l(A) \) and \( p(A) \). Therefore we have \( l(A) = l(|A|) \) and \( p(A) = p = \text{lcm}\{p(\omega), p(|A|)\} \).

Let \( p = \text{lcm}\{p(\omega), k\} \) and \( \alpha = p(\omega^k) \) and remind \( p(|A|) = k\) (See [1]). We have \((\omega^k)^n = \omega^{nk} = 1\). Since \( p(\omega)|\alpha k \), we have \( p|\alpha k \). Conversely, we can have \( (\omega^p)^k = 1\). Since \( \alpha|p \) and \( k|p \), we have \( \alpha k|p \). So \( \alpha k = p \). This completes the proof. \( \square \)

Let \( A \) be an irreducible powerful sign pattern. Then any actual product of a cycle in \( A \) is 1 or \(-1\). Suppose \( A \) is ray diagonally similar to \( \omega|A| \). Then by Corollary 9, \( \omega^{k(A)} = 1 \) iff \( A \) is cyclically nonnegative. Otherwise \( \omega^{k(A)} = -1 \). So from Theorem 12, we have the following result.

\[
p(A) = \begin{cases} 
  k & \text{if } A \text{ is cyclically nonnegative} \\
  2k & \text{if } A \text{ has a negative cycle}
\end{cases}
\]

and

\[ l(A) = l(|A|) .\]

So we can consider Theorem 12 as a generalization of Theorine 4.3 in [2].

Now we characterize a periodic irreducible ray pattern whose period is \( p \).

**Theorem 13.** Suppose \( A \) is an irreducible ray pattern with \( k(A) = k \). \( A \) is periodic with \( p(A) = p \) iff \( k \) divides \( p \), and \( A \) is ray diagonally similar to \( \omega|A| \) where \( p(\omega^k) = p/k \).

**Proof.** (\( \Rightarrow \)) Note that \( A \) is ray diagonally similar to \( \omega|A| \) for some \( \omega \) by Theorem 4 and \( p(A) = p(\omega^k)k \) by Theorem 12. Therefore \( A \) is ray diagonally similar to \( \omega|A| \) where \( p = p(\omega^k)k \).

(\( \Leftarrow \)) Note that \( A \) is periodic since \( \omega \) is periodic, and \( p(A) = p(\omega|A|) = p(\omega^k)k = p \) since \( k(|A|) = k \). \( \square \)

By the above Theorem 13, we can obtain the following corollary about pattern \( p \)-potents, which was already presented in [4].

**Corollary 14.** Suppose \( A \) is an irreducible ray pattern in block cyclic form (1) with \( k(A) = k \). \( A \) is pattern \( p \)-potent for some positive integer \( p \) iff \( k \) divides \( p \), and \( A \)
is ray diagonally similar to

\[
\omega \begin{pmatrix}
0 & J_1 & 0 \\
0 & J_2 & 0 \\
& \ddots & \ddots \\
J_k & 0 & J_{k-1} \\
0 & 0 & 0
\end{pmatrix},
\]

where \( p(\omega^k) = p/k \) and \( J_i \) is an all ones matrix that is the same size as the corresponding block \( A_{i,i+1} \).

**Proof.** Note that \( A \) is ray diagonally similar to \( \omega|A| \) where \( p(\omega^k) = p/k \) by Theorem 13. Since \( l(A) = l(|A|) = 1 \), we have

\[
|A| = \begin{pmatrix}
0 & J_1 & 0 \\
0 & J_2 & 0 \\
& \ddots & \ddots \\
J_k & 0 & J_{k-1} \\
0 & 0 & 0
\end{pmatrix},
\]

where \( J_i \) is an all ones matrix that is the same size as the corresponding block \( A_{i,i+1} \). This completes the proof. \( \square \)

## 4 Closing remark

In this paper, we study irreducible powerful ray patterns using Theorem 4. One of key observation of our paper is that if a powerful ray pattern \( A \) is irreducible, then \( A \) is ray diagonally similar to \( \omega|A| \) for some ray \( \omega \). Now consider the following powerful ray pattern set

\[
S = \{ A | A \text{ is ray diagonally similar to } \omega|A| \text{ for some ray } \omega \}.
\]

Note every powerful irreducible ray patterns is an element of \( S \). Then how can we characterize the elements in \( S \), and what are the bases and periods of the elements in \( S \)?

For examples, consider the following reducible powerful ray patterns

\[
A = \begin{pmatrix}
0 & i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & -i & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

Then \( A \) is ray diagonally similar to \( i|A| \) and \( -i|A| \). Thus \( A \) is in \( S \), but \( B \) is not. How can we determine whether a powerful ray pattern is in \( S \) or not?
Bibliography


