Multilevel Methods: From Fourier to Gauss

Randolph E. Bank
Department of Mathematics
University of California, San Diego

- Introduction
- Classical Multigrid Methods
- Hierarchical Basis Methods
- Sparse Gaussian Elimination

Model Problem:

$$-\Delta u = f \quad \text{in } \Omega \subset \mathcal{R}^2,$$

$$u = 0 \quad \text{on } \partial\Omega$$

Weak Formulation: find $u \in \mathcal{H}_0^1(\Omega)$ such that

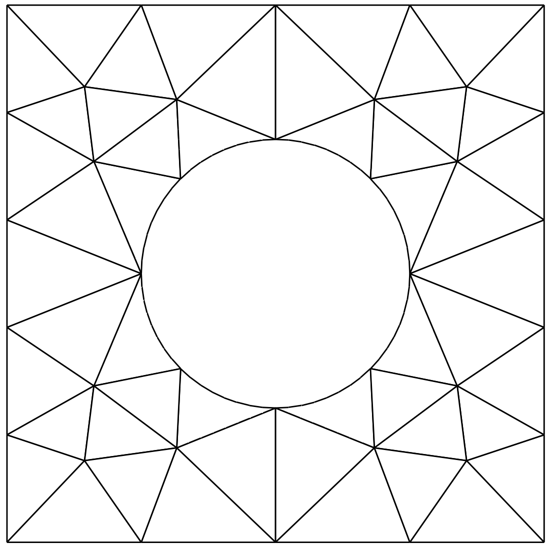
$$a(u, v) = (f, v)$$

for all $v \in \mathcal{H}_0^1(\Omega)$, where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad (f, v) = \int_\Omega f v \, dx$$

$$\|u\|^2 = a(u, u) \qquad ||u||^2 = (u, u)$$

Triangulation: (quasiuniform, shape regular)



Finite Element Subspace: $\mathcal{S} \subset \mathcal{H}_0^1(\Omega)$

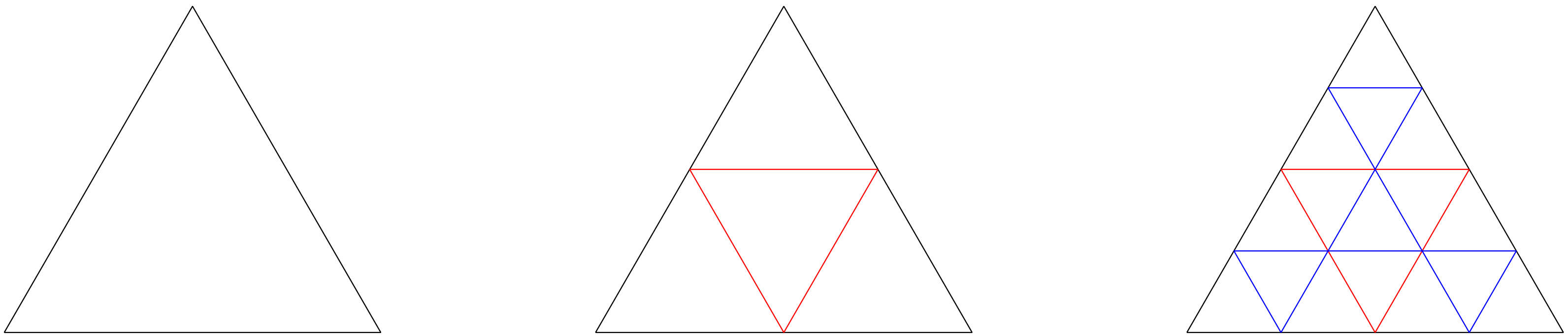
continuous piecewise linear polynomials

Finite Element Approximation: find $u_h \in \mathcal{S}$ such that

$$a(u_h, v) = (f, v)$$

for all $v \in \mathcal{S}$

Levels and Refinement:



$\ell = 1$ $\qquad$ $\ell = 2$ $\qquad$ $\ell = 3$

$$h_\ell = h_1 2^{1-\ell} = h_{\ell-1}/2$$

$$\mathcal{S}_{\ell-1} \subset \mathcal{S}_\ell$$

$$\boldsymbol{Dim}\ \ \mathcal{S}_\ell \approx 4 \times \boldsymbol{Dim}\ \ \mathcal{S}_{\ell-1}$$

Finite Element Problem on Level ℓ:

Find $u^{(\ell)} \in \mathcal{S}_\ell$ such that

$$a(u^{(\ell)}, v) = (f, v)$$

for all $v \in \mathcal{S}_\ell$

Classical A Priori Error Estimate:

$$|||u - u^{(\ell)}||| \leq C h_\ell \|u\|_{\mathcal{H}^2(\Omega)}$$

Remark: Adaptive Refinement can lead to nonuniform meshes

Classical Nodal Basis for $\mathcal{S}_\ell$:

$$\phi_i^{(\ell)}(v_j^{(\ell)}) = \delta_{ij}$$

Linear System of Equations:

$$A^{(\ell)}\, U^{(\ell)} = b^{(\ell)}$$

where

$$A_{ij}^{(\ell)} = a(\phi_j^{(\ell)}, \phi_i^{(\ell)}) \qquad b_i^{(\ell)} = (f, \phi_i^{(\ell)}) \qquad u^{(\ell)} = \sum U_i^{(\ell)}\, \phi_i^{(\ell)}$$

$A^{(\ell)}$ is large, sparse, symmetric, positive definite

Iterative Methods for $Ax = b$:

$$A = B - N$$

Preconditioner B is symmetric,positive definite, easy to solve.

Given x_0 (e.g., $x_0 = 0$), for $j = 0, 1, 2, \ldots$

$$r_j = b - Ax_j$$
$$B\delta_j = r_j$$
$$x_{j+1} = x_j + \delta_j$$

Remark: Should accelerate with Conjugate Gradients

Classical Convergence Analysis:

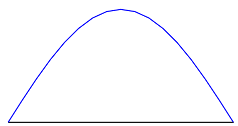
Consider Generalized Eigenvalue Problem

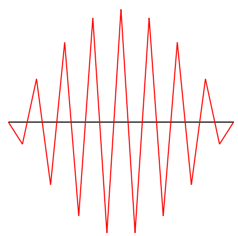
$$A\psi_i = \lambda_i B\psi_i$$

with $\|\psi_i\| = 1$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N \leq 1$ (scale B as needed)

$$e_k = x - x_k = (I - B^{-1}A)e_{k-1} = (I - B^{-1}A)^k e_0$$

$$e_0 = \sum c_i\psi_i \qquad \rightarrow \qquad e_k = \sum (1 - \lambda_i)^k c_i\psi_i$$



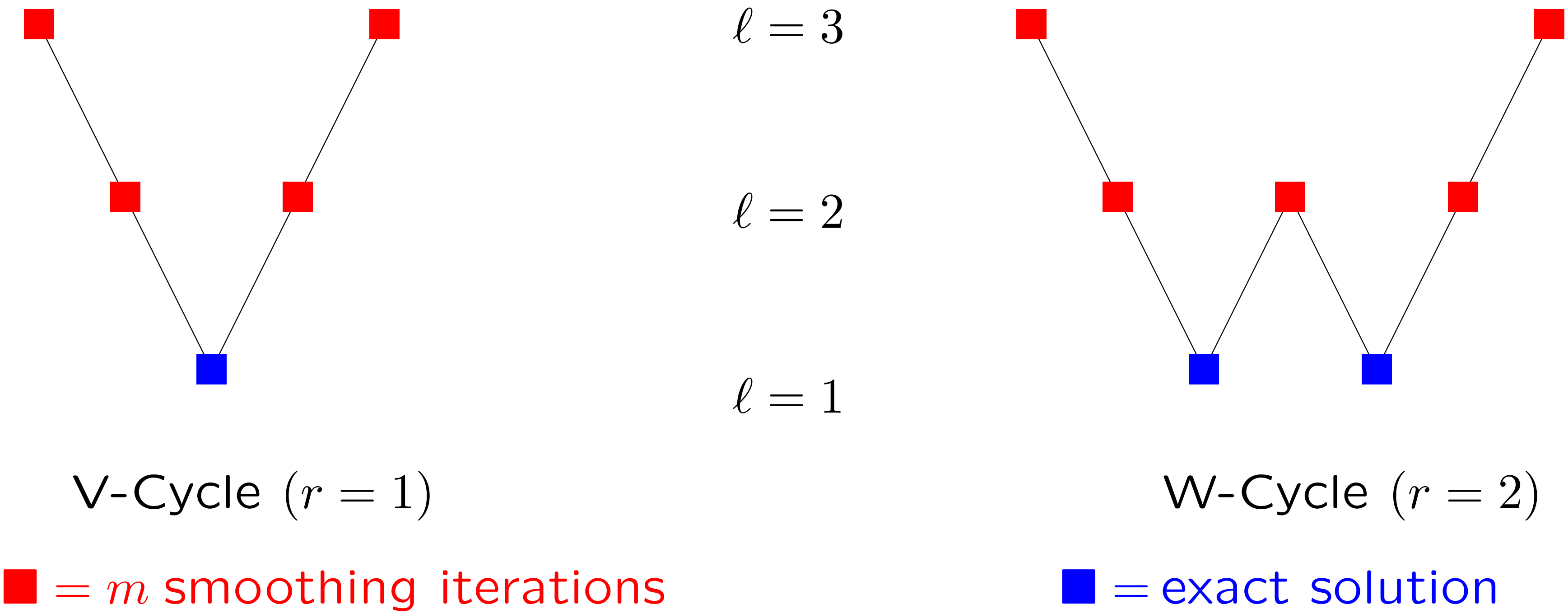


“smooth” $\lambda \approx 0 = O(h^2)$

“rough” $\lambda \approx 1$

Basic Multilevel Idea:

- Smooth rough components of the error on level ℓ
- Project smooth components of the error to coarse level $\ell - 1$
- Use Recursion



Multigrid Iteration (Finite Element Notation):

Level ℓ problem: $a(\delta^{(\ell)}, v) = f(v)$ for all $v \in \mathcal{S}_\ell$

If $\ell = 1$ solve exactly; if $\ell > 1$, let $\delta_0^{(\ell)} = 0$

Pre smoothing: for $0 \leq k \leq m-1$,

$$b(\delta_{k+1}^{(\ell)} - \delta_k^{(\ell)}, v) = f(v) - a(\delta_k^{(\ell)}, v) \qquad \text{for all } v \in \mathcal{S}_\ell$$

Coarse Grid Corrrection: find $\varepsilon^{(\ell-1)} \in \mathcal{S}_{\ell-1}$ such that

$$a(\varepsilon^{(\ell-1)}, v) = f(v) - a(\delta_m^{(\ell)}, v) \equiv \hat{f}(v) \qquad \text{for all } v \in \mathcal{S}_{\ell-1}$$

$$\delta_{m+1}^{(\ell)} = \delta_m^{(\ell)} + \varepsilon^{(\ell-1)}$$

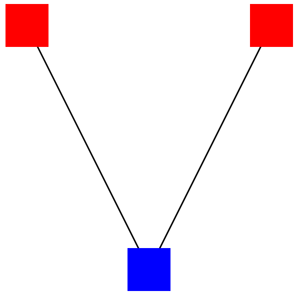
(Approximately) solve by $r = 1, 2$ iterations of $\ell - 1$ level scheme.

Post smoothing: for $m+1 \leq k \leq 2m+1$,

$$b(\delta_{k+1}^{(\ell)} - \delta_k^{(\ell)}, v) = f(v) - a(\delta_k^{(\ell)}, v) \qquad \text{for all } v \in \mathcal{S}_\ell$$

Set $\delta^{(\ell)} = \delta_{2m+1}^{(\ell)}$

A Simple Convergence Theorem (2 Levels):



■ $= m/2$ smoothing iterations

■ $=$ exact solution

Assume

$$\frac{1}{\sqrt{\mathcal{K}}} \le \frac{|||v|||_1}{|||v|||_0} \le 1 \qquad \text{for } v \in \mathcal{S}_\ell \cap \mathcal{S}_{\ell-1}^{\perp}$$

where $\mathcal{K}$ is independent of h. Then $|||e_{m+1}|||_1 \le \gamma |||e_0|||_1$ where

$$\gamma \le \begin{cases} \left(\frac{\mathcal{K}-1}{\mathcal{K}}\right)^m & m \le \mathcal{K}-1 \\ \frac{\mathcal{K}}{m+1}\left(\frac{m}{m+1}\right)^m & m > \mathcal{K}-1 \end{cases}$$

Remark: $|||v|||_1^2 = a(v,v) \Leftrightarrow V^tAV$ and $|||v|||_0^2 = b(v,v) \Leftrightarrow V^tBV$.

Elements of Proof: (including $\mathcal{K}$ independent of h)

- Approximation Properties of $\mathcal{S}_\ell$
- Quasi uniformity, shape regularity of the triangulation
- $\mathcal{H}^2$ (or $\mathcal{H}^{1+\alpha}$) regularity of solution
- The spectral decomposition (generalized eigenvalue problem)

W. Hackbusch

Bank-Dupont

Bramble-Pasciak-*et al*

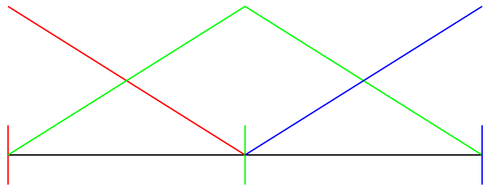
J. Xu

J. Mandel

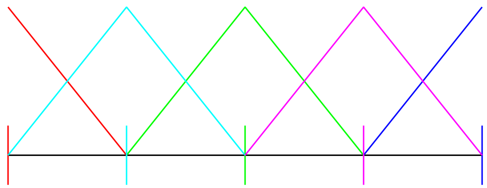
P. Oswald

and many, many others...

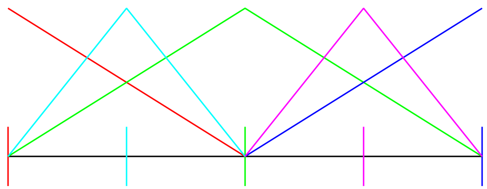
Hierarchical Basis (1-Dimensional Example):



$\mathcal{S}_1$ nodal basis



$\mathcal{S}_2$ nodal basis



$\mathcal{S}_2$ hierarchical basis

Hierarchical Decomposition: $\mathcal{S}_\ell = \mathcal{S}_{\ell-1} \oplus \mathcal{W}_\ell$

Matrix Formulation:

$A_N^{(\ell)}$ is the nodal basis stiffness matrix
$A_H^{(\ell)}$ is the Hierarchical basis stiffness matrix

$$A_N^{(\ell)} = \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}$$

$$A_H^{(\ell)} = \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} A_N^{(\ell)} \begin{pmatrix} I & V^t \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{ff} & A_{fc} + A_{ff}V^t \\ A_{cf} + VA_{ff} & A_N^{(\ell-1)} \end{pmatrix}$$

$$A_N^{(\ell-1)} = A_{cc} + VA_{fc} + A_{cf}V^t + VA_{ff}V^t$$

Remark: $f \Leftrightarrow$ "fine" and $c \Leftrightarrow$ "coarse"

HBMG Preconditioner:

Solve $A_N^{(\ell)}\delta = r$ using 1 iteration of Block Symmetric Gauss-Seidel for $A_H^{(\ell)}$, developed implicitly to exploit sparsity of $A_N^{(\ell)}$

$A_{ff}\hat{\delta}_f = r_f$	solve by "pre smoothing"
$\hat{r}_c = r_c - A_{cf}\hat{\delta}_f + V(r_f - A_{ff}\hat{\delta}_f)$	"restriction"
$A_N^{(\ell-1)}\delta_c = \hat{r}_c$	coarse grid correction
$\hat{\delta}_f \leftarrow \hat{\delta}_f + V^t\delta_c$	"prolongation"
$A_{ff}\bar{\delta}_f = r_f - A_{fc}\delta_c - A_{ff}\hat{\delta}_f$	solve by "post smoothing"
$\delta_f = \bar{\delta}_f + \hat{\delta}_f$	

HBMG Preconditioner (Finite Element Notation):

Level ℓ problem: $a(\delta^{(\ell)}, v) = f(v)$ for all $v \in \mathcal{S}_\ell$

Pre smoothing: find $\hat{\delta}_f \in \mathcal{W}_\ell$ such that

$$a(\delta_f, v) = f(v) \qquad \text{for all } v \in \mathcal{W}_\ell$$

Coarse Grid Corrrection: find $\delta_c \in \mathcal{S}_{\ell-1}$ such that

$$a(\delta_c, v) = f(v) - a(\hat{\delta}_f, v) \qquad \text{for all } v \in \mathcal{S}_{\ell-1}$$

Post smoothing: find $\delta_f \in \mathcal{W}_\ell$ such that

$$a(\delta_f, v) = f(v) - a(\delta_c + \hat{\delta}_f, v) \qquad \text{for all } v \in \mathcal{W}_\ell$$

Set $\delta^{(\ell)} = \delta_f + \delta_c + \hat{\delta}_f$

A Simple Convergence Theorem (2 Levels):

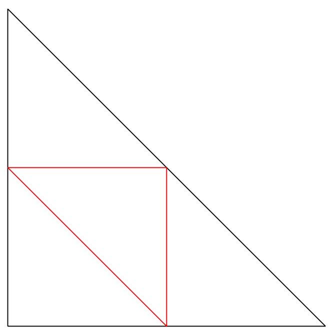
Assume that A_{ff} and $A_N^{(\ell-1)}$ are solved exactly and

$$\sup_{\substack{v \in \mathcal{S}_{\ell-1} \quad |||v||| = 1 \\ w \in \mathcal{W}_\ell \quad |||w||| = 1}} |a(v,w)| \le \gamma < 1$$

where γ is independent of h (strengthened Cauchy Inequality).

Then $|||e_{k+1}||| \le \gamma^2 |||e_k|||$

Remarks: $\gamma = \gamma(m, \ell)$ for m smoothing steps and ℓ levels



$\gamma = 1/2$ for this configuration

HBMG and Classical MG V-Cycle are almost identical

HBMG does not use

- quasiuniformity of the mesh
- approximation properties of $\mathcal{S}_\ell$
- regularity of solution (beyond $\mathcal{H}^1$)
- spectral decomposition

HBMG does use

- local properties of polynomials
- shape regularity of the mesh

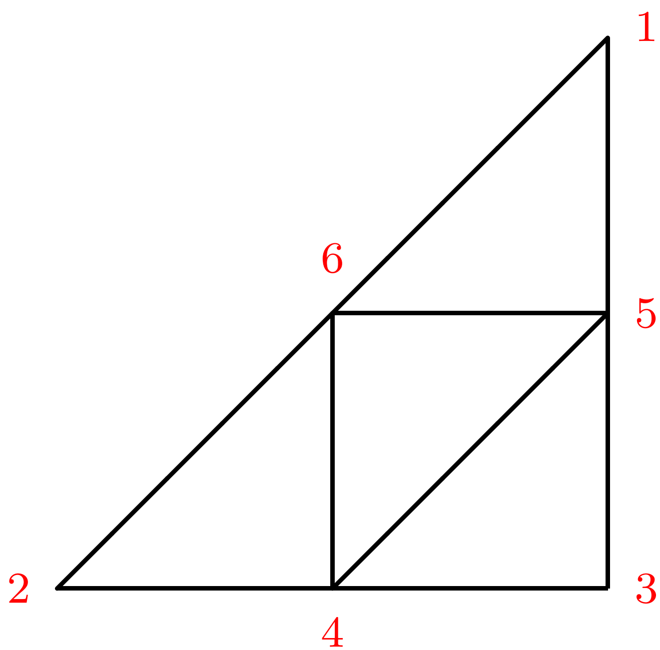
H. Yserentant

Bank-Dupont-Yserentant

M. Griebel

and many others...

The Graph of a Sparse, Symmetric Matrix A:

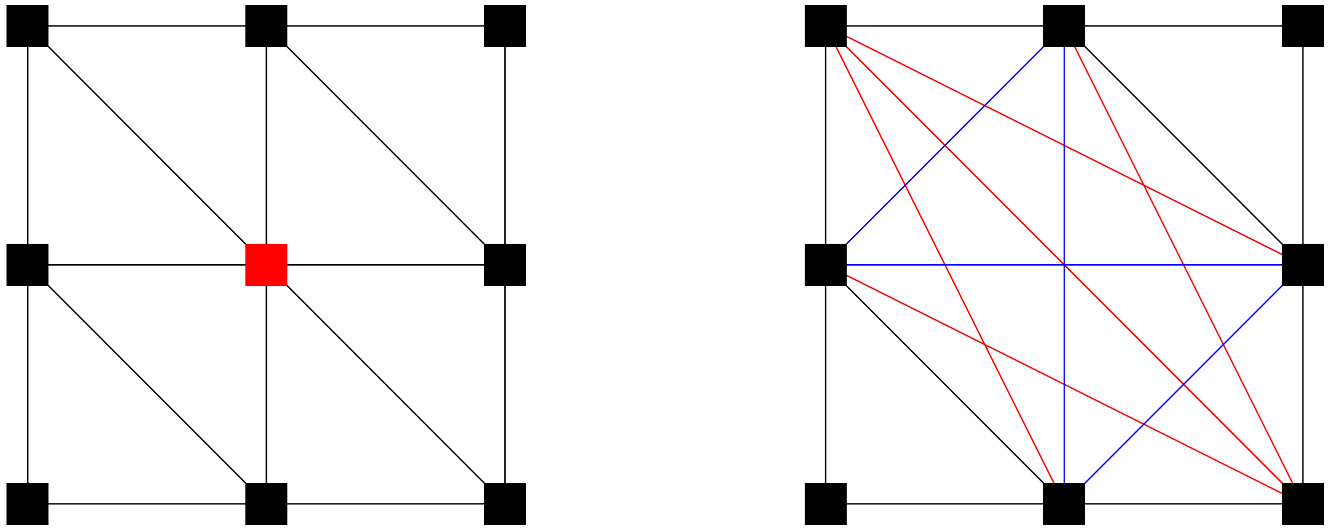


$$A = \begin{pmatrix} \blacksquare & 0 & 0 & 0 & \blacksquare & \blacksquare \\ 0 & \blacksquare & 0 & \blacksquare & 0 & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare & \blacksquare & 0 \\ 0 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 0 & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & 0 & \blacksquare & \blacksquare & \blacksquare \end{pmatrix}$$

$e_{ij} \equiv e_{ji} \in \mathcal{E} \equiv$ edge set $\quad \Leftrightarrow \quad A_{ij} \neq 0$

Graph of $A_N^{(\ell)}$ is level ℓ triangulation (except for Dirichlet b.c.)

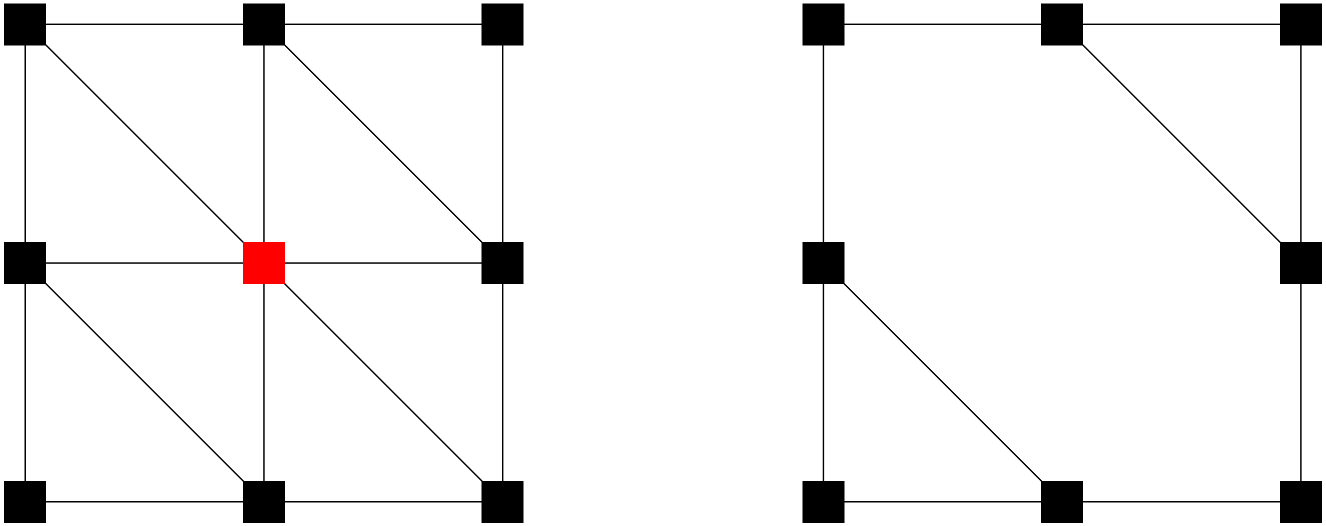
Symbolic Sparse Gaussian Elimination:



To eliminate vertex ■:

- Delete vertex ■ and its incident edges
- Make $adj(■)$ into a clique by adding fill-in edges

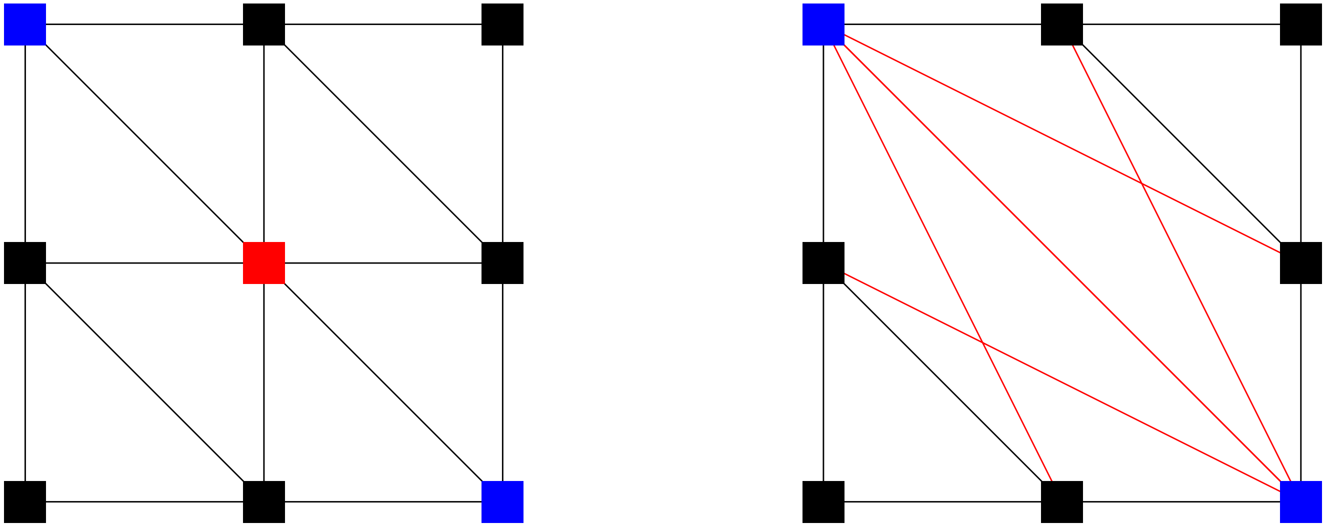
Classical Incomplete LU Decomposition:



To eliminate vertex ■:

- Delete vertex ■ and its incident edges
- Add NO fill-in edges edges

HBMG as an ILU Decomposition:

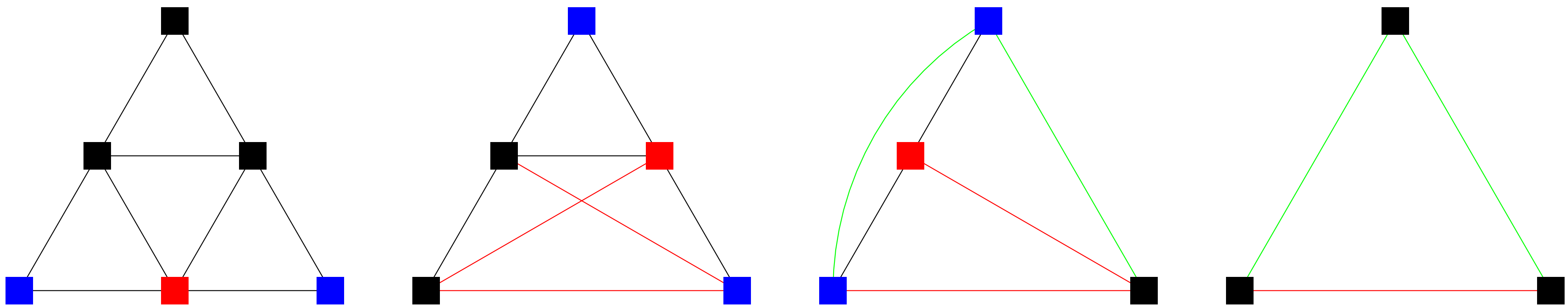


■ are parents of ■; to eliminate vertex ■:

- Delete vertex ■ and its incident edges
- Add fill-in edges for parents (■)

Grid Coarsening and ILU:

As the fine grid nodes are eliminated via the HBMG/ILU scheme, the fine triangulation is transformed to the coarse grid.



Matrix Interpretation:

$$A_N^{(\ell)} = \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}$$

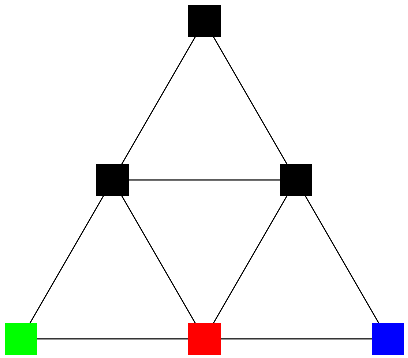
$$\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} A_N^{(\ell)} \begin{pmatrix} I & V^t \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{ff} & A_{fc} + A_{ff} V^t \\ A_{cf} + V A_{ff} & S_{cc} \end{pmatrix}$$

$$S_{cc} = A_{cc} + V A_{fc} + A_{cf} V^t + V A_{ff} V^t$$

Exact Gaussian Elimination: $A_{cf} + V A_{ff} = 0$. Then

$$S_{cc} = A_{cc} - A_{cf} A_{ff}^{-1} A_{fc} \equiv \text{Schur Complement (generally dense)}$$

The Matrix V for the Hierarchical Basis Transformation:



$$u(\blacksquare) = \frac{u(\blacksquare) + u(\blacksquare)}{2}$$

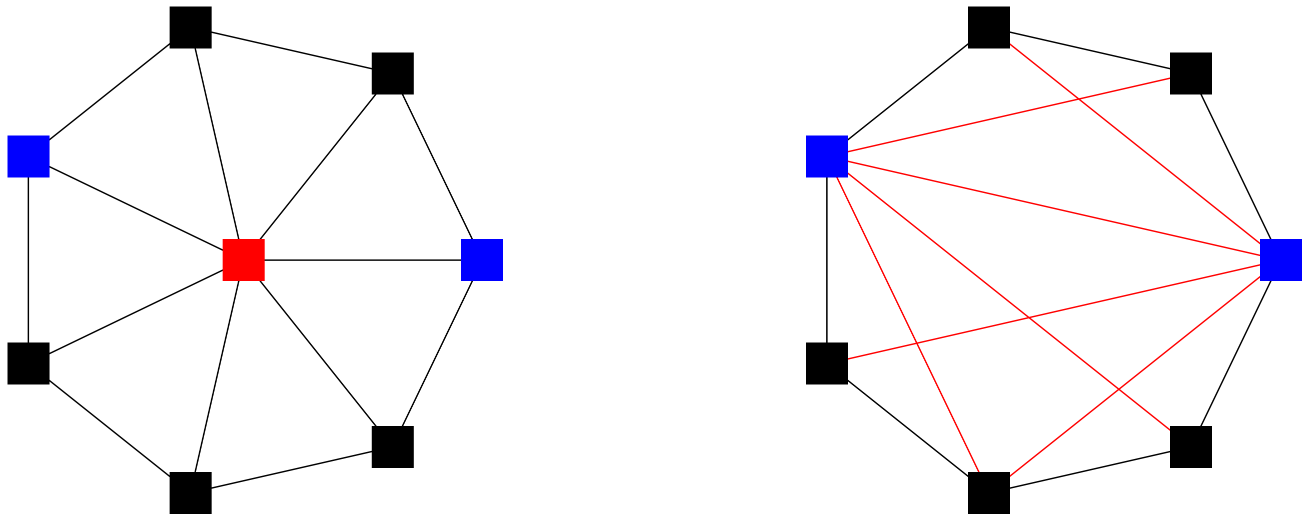
"linear interpolation"

Column $\blacksquare$ of V has nonzero entries $(1/2)$ only in rows $\blacksquare$ and $\blacksquare$

$$A_N^{(\ell-1)} = A_{cc} + V A_{fc} + A_{cf} V^t + V A_{ff} V^t$$

can be viewed a sparse approximate Schur complement

The “Multigraph” Method:



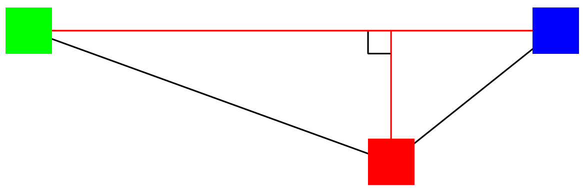
Add fill-in edges only for parents

- For PDE’s, can use geometry of mesh to choose parents
- For general sparse matrix, use max weight / min fill criteria

Remark: can have any number of parents; 1-parent option (“matching”) has some supporting analysis

Choosing Multipliers in V (General case):

- If geometry is available, use "interpolation"



$$u({\color{red}\blacksquare}) = (1-\theta)u({\color{green}\blacksquare}) + \theta u({\color{blue}\blacksquare})$$

- Create as many zeroes as possible in $A_{cf} + {\color{red}V} A_{ff}$ (as in classical ILU algorithms (e.g. $-a_{ij}/a_{ii}$)

Remark: if ${\color{blue}A_N^{(\ell)}}$ is not symmetric, can choose different multipliers for left and right

$$\begin{pmatrix} I & 0 \\ {\color{red}V} & I \end{pmatrix} \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix} \begin{pmatrix} I & {\color{magenta}W^t} \\ 0 & I \end{pmatrix}$$