Image Analysis and PDE 's

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Book :

Download a working version at :

http://www.ceremade.dauphine.fr/~fguichar/Research.html

Plan

- Goals of image processing
- Linear theory and the heat equation
- Non-linear diffusions
- Invariant image analysis
- Invariant PDE 's and applications

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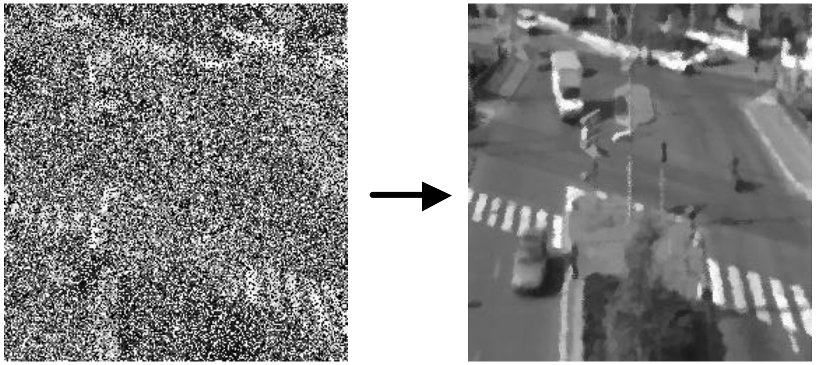
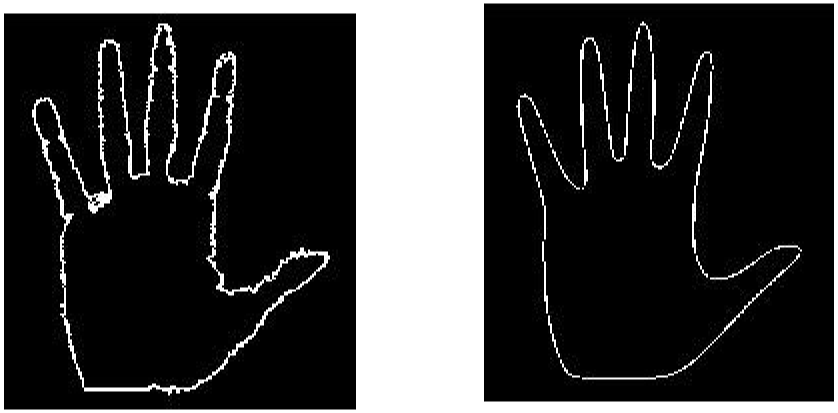
1. Objectives of image processing

•Sampling, compression

•Restoration, (debluring - denoising)



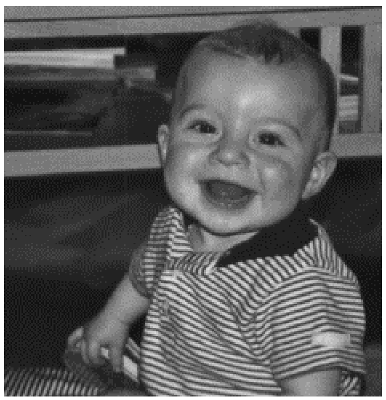
•Analysis :
extraction of meaningful parts



2. Linear theory and the heat equation

Image generation = a convolution ($u = k * u_0$) followed by a sampling

Shannon, 1948



Original image

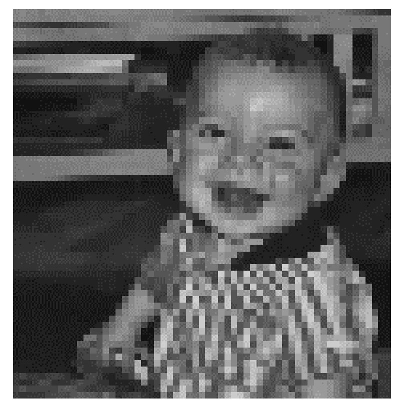
smoothing

Smoothed image

subsampling

Sampled version

subsampling



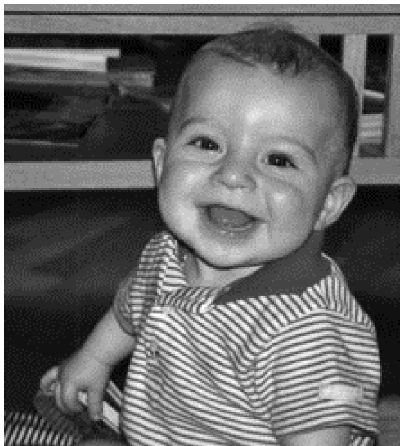
Sampling without smoothing creates aliasing

Gabor (1960) remarked that: $u - u_0 = k^* u_0 - u_0 \approx C \Delta u$

Gabor enhancement filter : $u_{restored} = u_{observed} - \varepsilon \Delta u_{observed}$

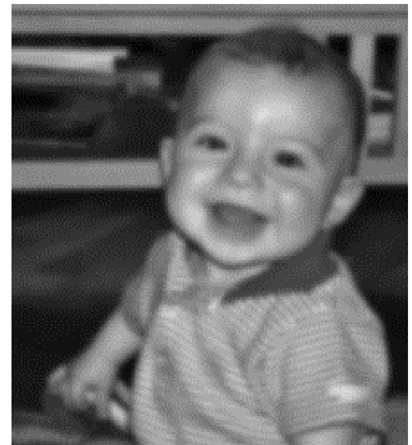
Thus : image deblurring means inverting the heat equation

Original



$u_{n+1}=u_n+ \varepsilon \Delta u_n$

Blur

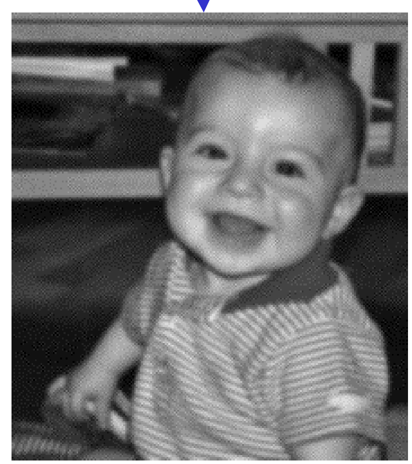


$u_{n+1}=u_n- \varepsilon \Delta u_n$

"Enhanced"

$u_{n+1}=u_n- \varepsilon \Delta u_n$

"Deblur"



Two directions

1. Image restoration:

   - improving inverse heat equation

2. Image multi-scale representations

   - simulating iterated convolution-sampling process:
     wavelet theory (Burt, Adelson (1983), Morlet, Grossmann (1984), Mallat, Meyer (1986), …

     Application: image compression - but sampling implies loss of invariance

   - simulating iterated convolution without sampling:
     heat equation and other nonlinear diffusions

     Application: image analysis

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Improving inverse heat equation

Rudin (1987),
Osher, Rudin (1990)
Shock filter

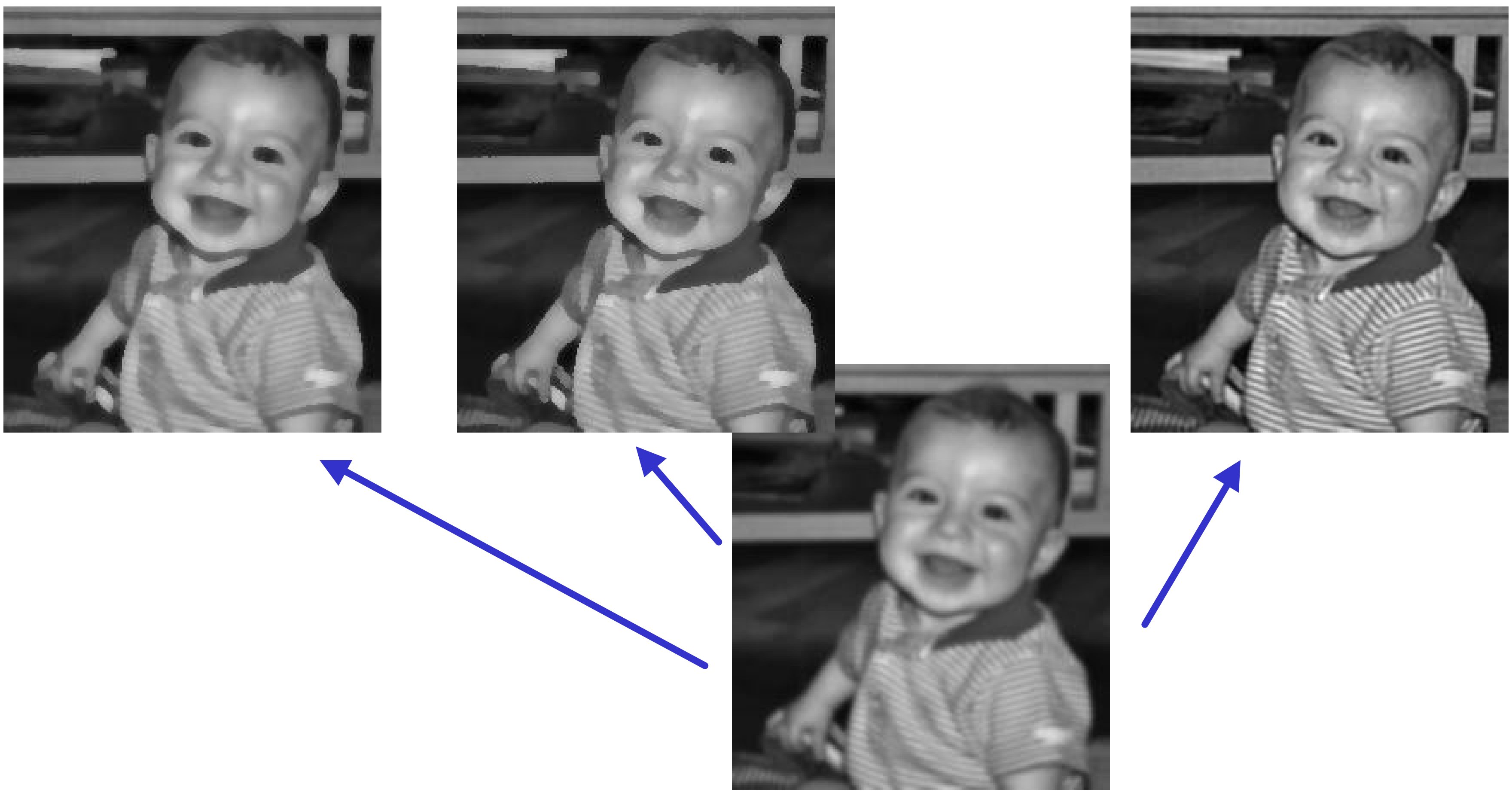
$$\frac{\partial u}{\partial t} = -sign(\Delta u)|Du|$$

Asymptotic Kramer's filter

$$\frac{\partial u}{\partial t} = -sign(D^2 u(Du, Du))|Du|$$

Rudin-Osher-Fatemi
(1992)

$$u = \arg\min\left(\int |Du| + l\,(k * u - u_0)^2\right)$$



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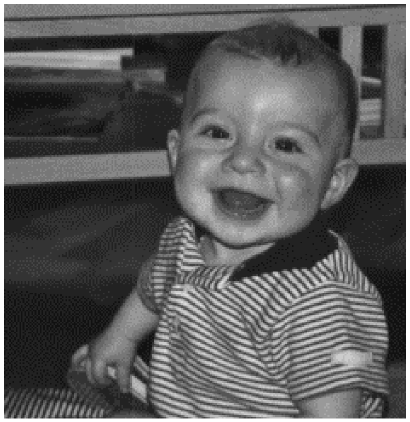
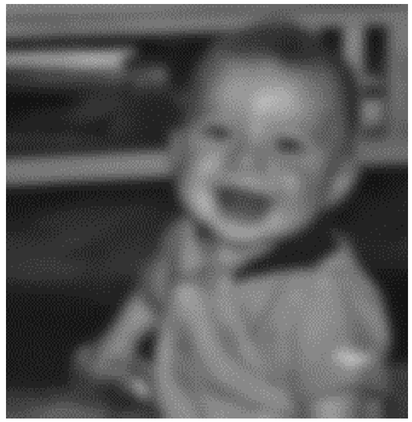
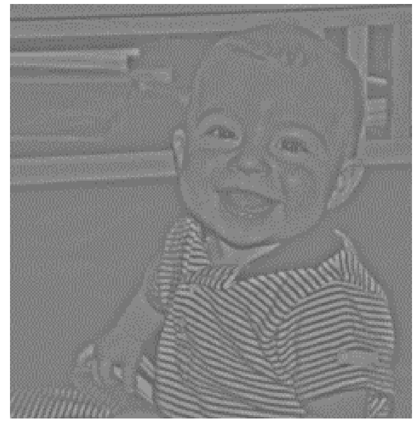
Wavelets and the Laplacian pyramid



u_0



$u_1 = k * u_0$



$v_1 = u_1 - u_0$ » $e D u_0$

Large coefficients

Laplacian pyramid of u_0
(Burt, Adelson 1983)

(Each vertical arrow ↓ indicates a subsampling)

$u_1 = k * u_0$ ⟶ $v_1 = u_1 - u_0$ » $e D u_0$

↓

$u_2 = k * u_1$ ⟶ $v_2 = u_2 - u_1$ » $e D u_1$

↓

...

↓

$u_{n+1} = k * u_n$ ⟶ $v_{n+1} = u_{n+1} - u_n$ » $e D u_n$

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- improving inverse heat equation

2. Image multi-scale representations

- simulating iterated convolution-sampling process:
  wavelet theory (Burt, Adelson (1983), Morlet, Grossmann (1984), Mallat, Meyer (1986), …

  Application: image compression - but sampling implies loss of invariance

→ • **simulating iterated convolution without sampling:**
  heat equation and other nonlinear diffusions

  Application: image analysis

Heat equation and feature extraction

Marr, Hildreth (1980), Canny (1983), Witkin (1983), Koenderink (1984),

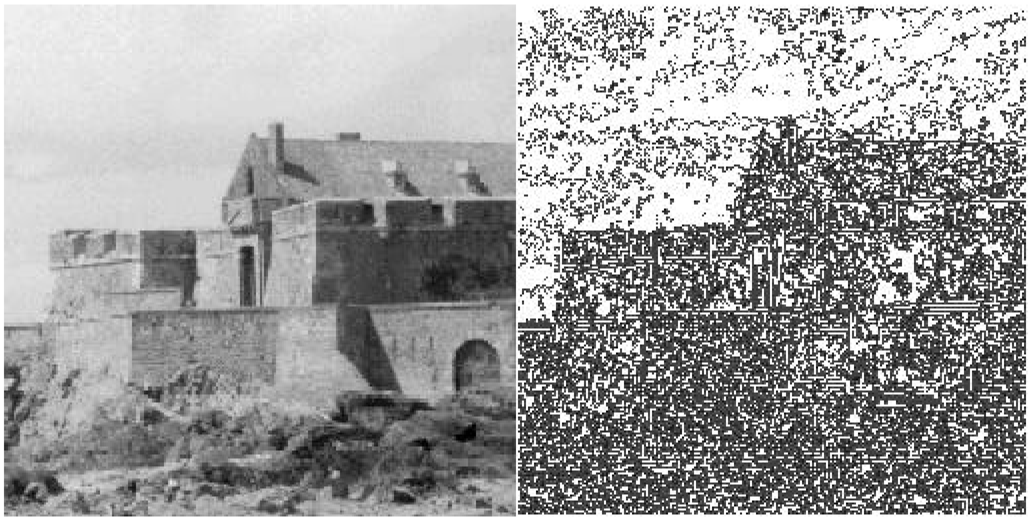

Image boundaries are defined as the points
where $D^2u(Du,Du)$ changes sign and $|Du|$ is large
(large extrema of $|Du|$)

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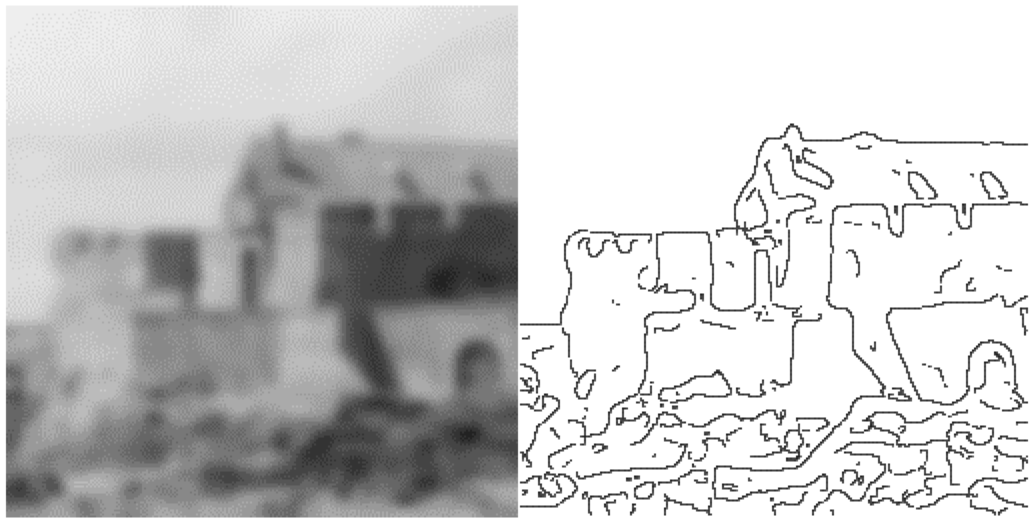

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3. Non Linear Diffusions

Perona-Malik equation (1987)

It aims at enhancing edges and smoothing regions

$$\frac{\partial u}{\partial t} = div\left(g\left(|Du|^2\right)Du\right) \quad \text{with} \quad g\left(|Du|^2\right) = \frac{1}{1 + l^2 |Du|^2}$$

Intrinsic coordinates : h = coordinate in the direction of Du

x = orthogonal direction

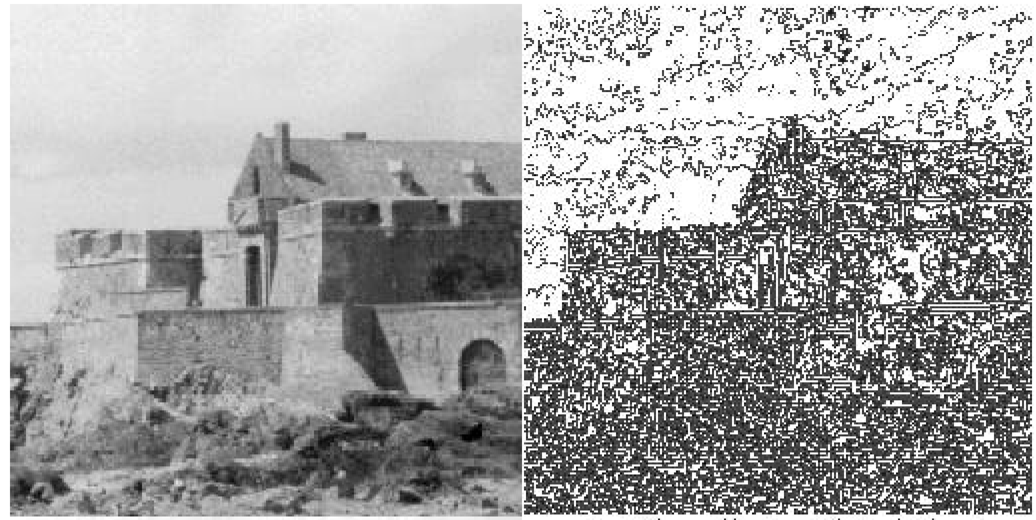
$Du = u_{xx} + u_{hh}$ $\quad u_{xx} = |Du|\ curv(u)$ $\quad u_{hh} = D^2u(Du, Du)/|Du|^2$

$$\frac{\partial u}{\partial t} = \frac{\left(1 - l^2 |Du|^2\right)}{\left(1 + l^2 |Du|^2\right)^2} u_{hh} + \frac{1}{\left(1 + l^2 |Du|^2\right)} u_{xx}$$

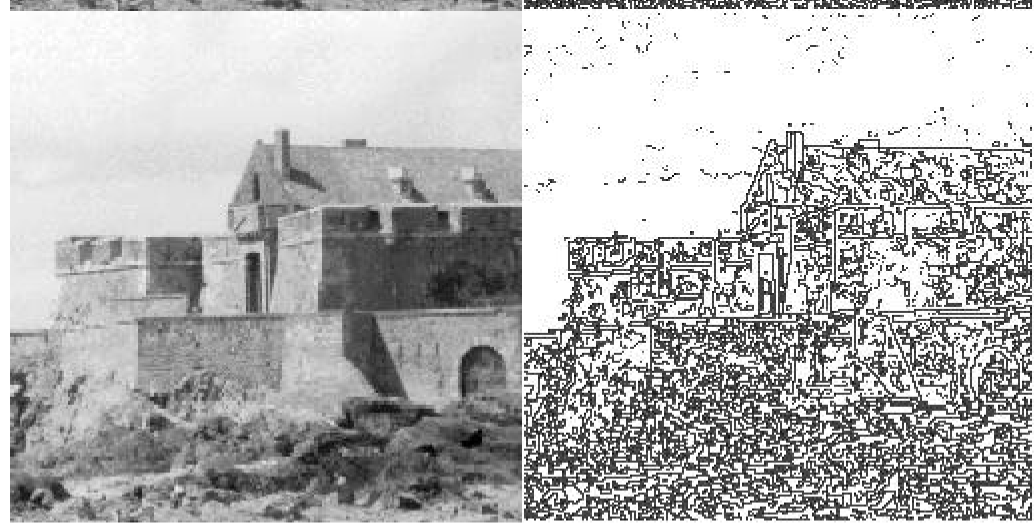
This equation behaves like a diffusion at points where $|Du| < l^{-1}$.
If instead, $|Du| > l^{-1}$, the first term becomes a reverse heat term.

Comparing Perona-Malik and heat equations

Heat equation and edges



Perona-Malik diffusion and edges



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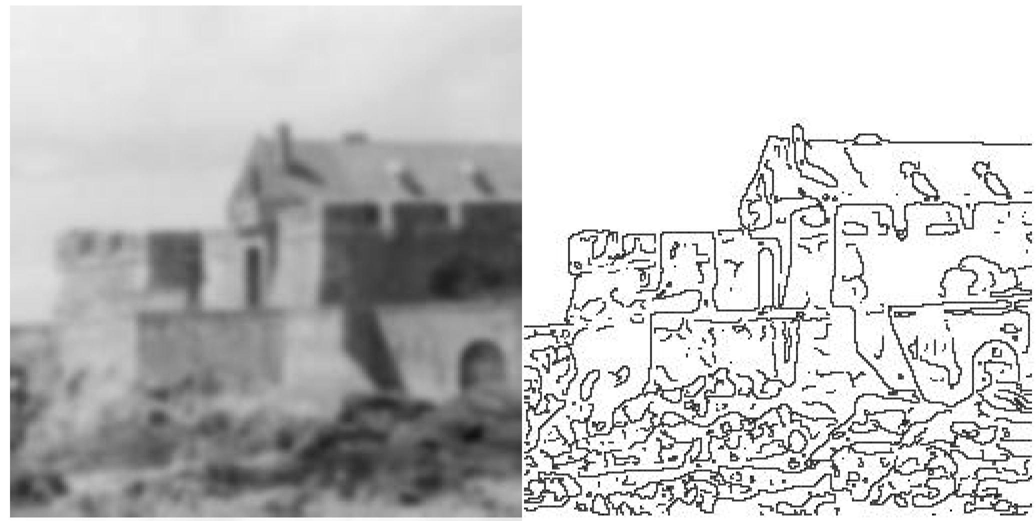
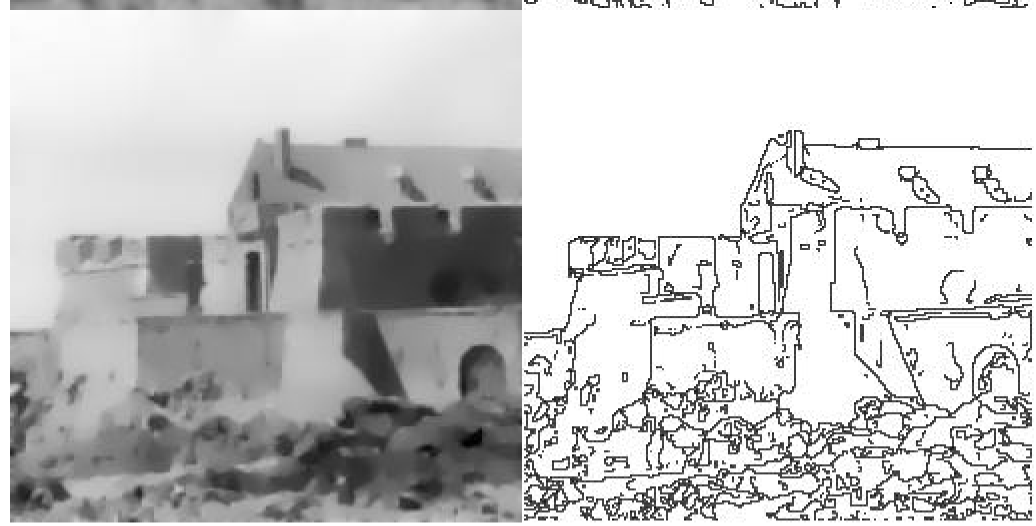


Perona-Malik diffusion
and edges



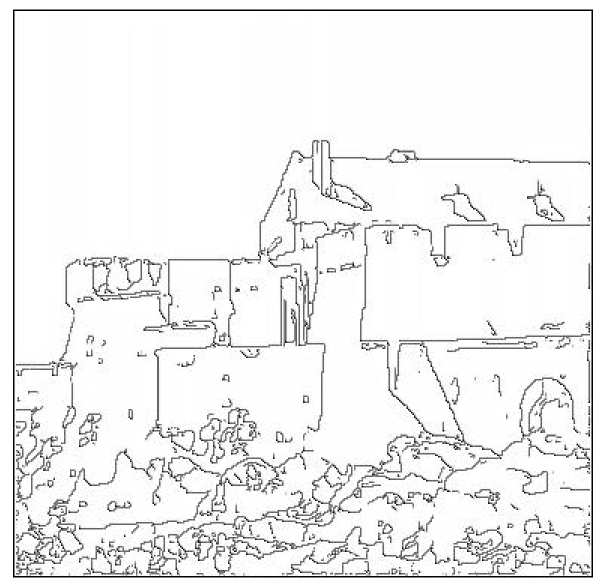
Drawbacks of Perona-Malik equation :

it mixes restoration, analysis and feature extraction with two parameters :

- *the time (scale)*
- *the contrast parameter* l.

No existence-uniqueness theory

(despite some partial results by Kichenassamy, Weickert 1998)



Consequence : A proliferation of nonlinear diffusions

Original

Perona, Malik, 1987

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla u|^2 + \lambda^2}\right)$$

Rudin, Osher, Fatemi, 1992

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla u|}\right)$$

Alvarez, Lions, 1992

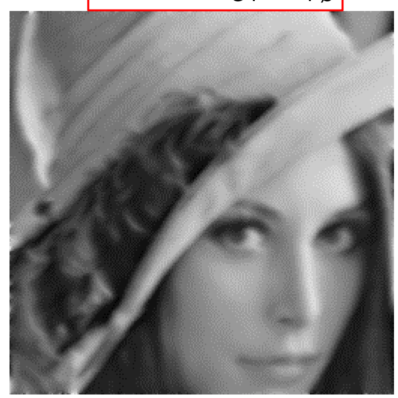
$$\frac{\partial u}{\partial t} = \frac{|\nabla u|}{|k * \nabla u|}\, div\left(\frac{\nabla u}{|\nabla u|}\right)$$

Weickert, 1994

$$\frac{\partial u}{\partial t} = D^2u(d,d)$$
$$d = \Sigma Eigen(k * (\nabla u \otimes \nabla u))$$

Caselles, Sbert, 1997

$$\frac{\partial u}{\partial t} = \frac{1}{|\nabla u|^2} D^2u(\nabla u, \nabla u)$$



Zhong Carmona, 1998

$$\frac{\partial u}{\partial t} = D^2u(d,d)$$
$$d = \Sigma Eigen(D^2u)$$

Sochen, Kimmel, Malladi,1998

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + 1}}\right)$$

4. Invariant Image analysis

- Contrast invariance : u and v are equivalent
  if there is a continuous increasing function g (contrast change)
  such that $v = g(u)$

- Affine invariance : u and v are equivalent
  if there is an affine map A, such that $v(x) = u(Ax)$

*(Affine invariance means invariance
with respect to plane chinese perspective)*

All image analysis operators should be defined on these equivalence classes.

Contrast invariant representations

(Wertheimer 1923, Matheron 1975)

$u \rightarrow X_\lambda u = \{x \; ; u(x) \geq \lambda \}$ upper level set, contrast invariant.

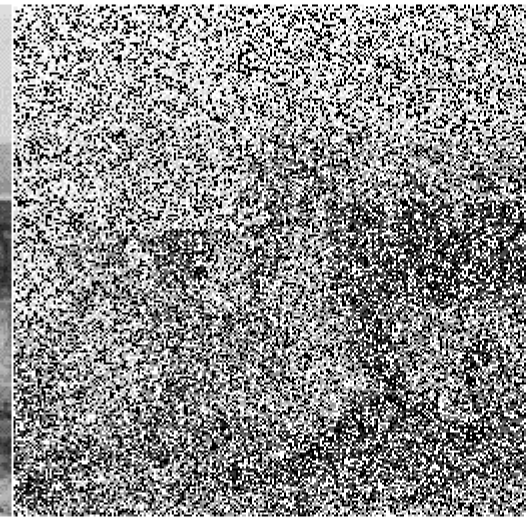
Level set 140

Matheron principle : shape information is contained in the bunch of level sets

Application: Extrema Killer (Vincent, Serra, 1993)

Removes all connected components of upper or lower level sets with area smaller than some scale.

Initial Image



Noisy image
75% Salt&Pepper

Scale 80 :
(level sets smaller than 80 pixels are removed)

Original processed

Noisy processed

Topographic Map: a more local constrast invariant representation

Caselles, Coll 1996

u $\rightarrow$ boundaries of upper and lower level sets.
It is a set of Jordan curves if the image is BV : the “level lines”.

Proposition :These Jordan curves are ordered by inclusion into an “inclusion tree”.
(Kronrod 1950 in continuous case, Monasse 1998 in semi-continuous case).

Then, image contrast invariant smoothing boils down to
- an independent smoothing of each level curve
- a smoothing that preserves curve inclusion.

Level lines with level 183.

Contrast invariant smoothing

Theorem (Chen, Giga, Goto 1991, Alvarez, Lions et al. 1992)
If the image analysis is local, isotropic, contrast invariant and satisfies an inclusion principle, then it satisfies

$$\frac{\partial u}{\partial t} = |Du| F(curv(u), t)$$

Where *curv(u)* is the curvature of the level curve.

(Alvarez, Lions et al. 1992)
If in addition it is affine invariant then the only possible equation is

$$\frac{\partial u}{\partial t} = |Du| curv(u)^{\frac{1}{3}} \qquad \text{(AMSS)}$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and *u(t,x)* is solution of AMSS, then *u(det(A) t, Ax)* also is a solution.

Curve motion

$$\frac{\partial x}{\partial t} = (curv(x))n,$$ (curve shortening)
where n is the normal vector to the curve

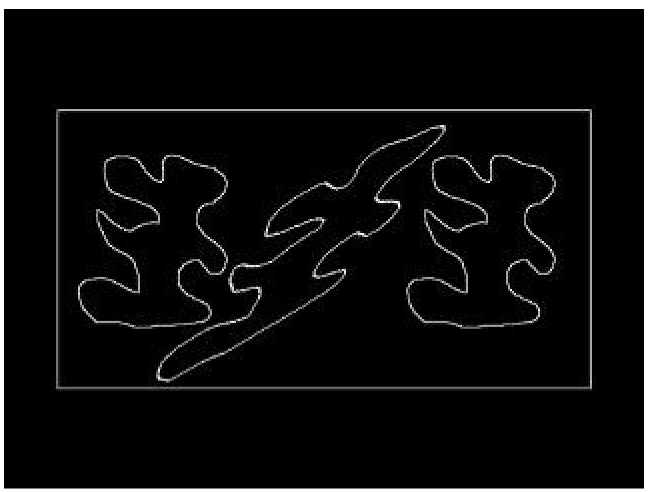
Gage, Hamilton, Grayson (1984-1987) proved existence, uniqueness and analycity of the solution.

Angenent, Sapiro, Tannenbaum (1998) prove the same result for the affine curve shortening

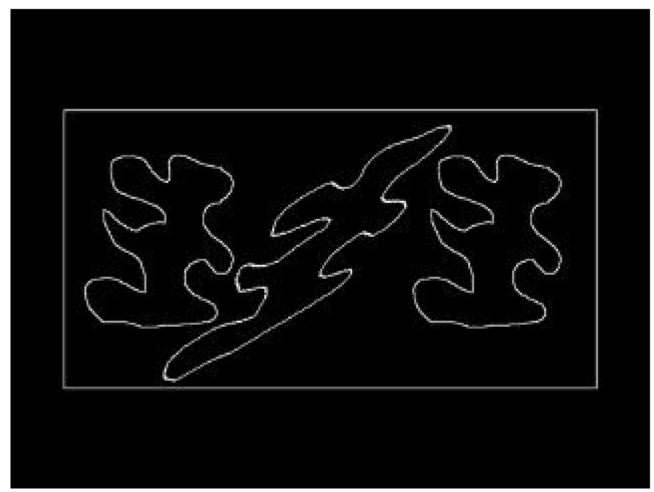
$$\frac{\partial x}{\partial t} = (curv(x))^{\frac{1}{3}} n.$$

Moisan's fast affine invariant algorithm

AMSS : affine invariant



MCM : not affine invariant



Curve motion

$$\frac{\partial \mathrm{x}}{\partial t} = \left(curv(\mathrm{x})\right)\mathrm{n},$$ (curve shortening)
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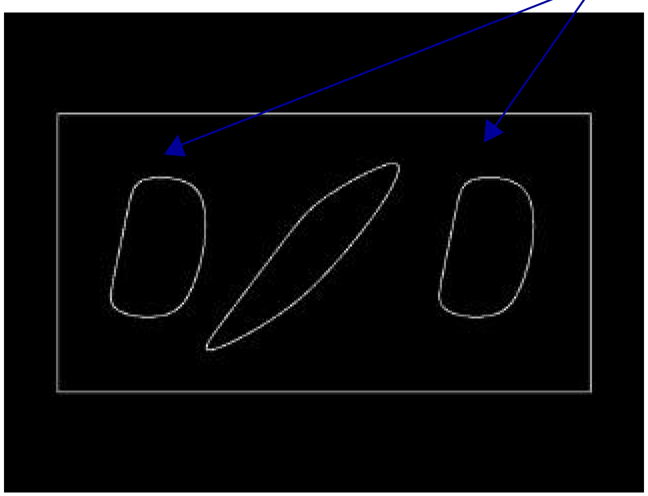
AMSS : affine invariant

Moisan's fast affine invariant algorithm



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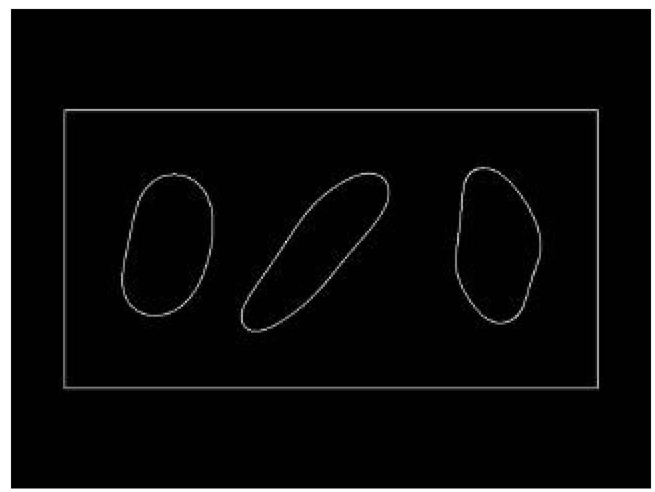


Theorem. Evans, Spruck (1991), Chen, Giga, Goto (1991)

A continuous image moves by curvature motion (viscosity sense - Crandall, Lions)

$$\frac{\partial u}{\partial t} = |Du| curv(u)$$

if and only if, almost all of its level lines move by curve shortening (in classical sense).

$$\frac{\partial \mathrm{x}}{\partial t} = (curv(\mathrm{x}))\mathrm{n},$$

This result justifies the Osher, Sethian level set algorithm for curvature motion (1985)

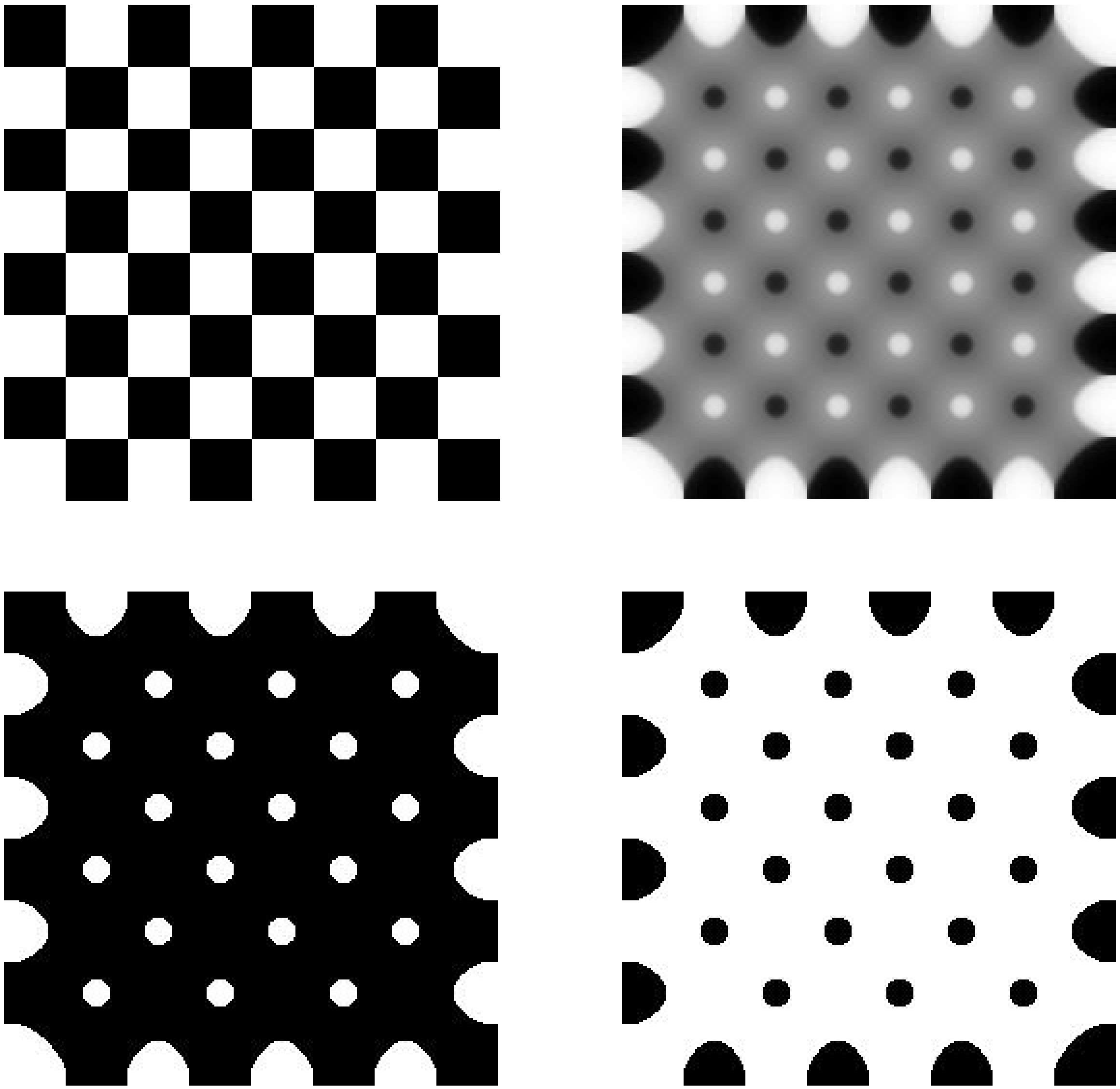
Same result holds with affine shortening

Extension to discontinuous images is possible,
but raises the figure/background problem.

Original image

Not contrast invariant

(Creation of new levels)



lsc solution

usc solution

Curvature Motion and AMSS

Associated level lines
(16 levels are displayed)

$$\frac{\partial u}{\partial t} = |Du| curv(u)^{\frac{1}{3}}$$

u(t)=  

u(0)= 

u(t)=  

$$\frac{\partial u}{\partial t} = |Du| curv(u)$$

Curvature Motion and AMSS

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(16 levels are displayed)

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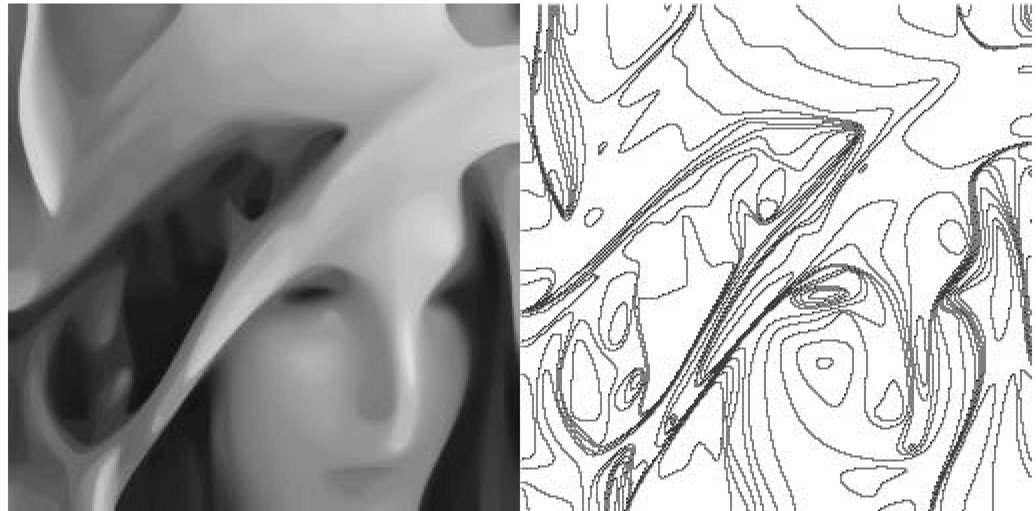
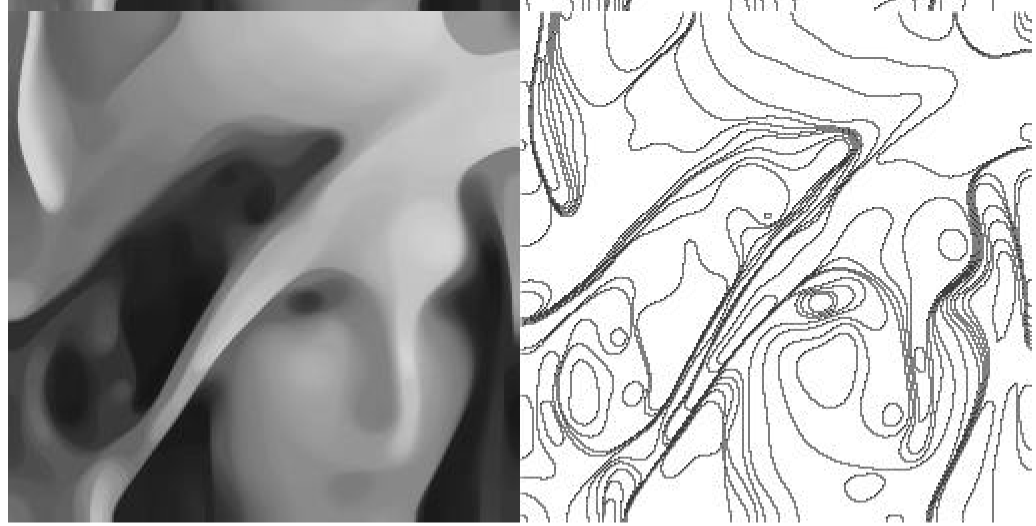
u(t)=



u(0)=

u(t)=



$$\frac{\partial u}{\partial t} = |Du|\, curv(u)$$

Application of contrast invariance : contour selection by geodesic snake method

Caselles, Kimmel, Sapiro (1997), Malladi, Sethian (1997)

u_0 original image, $g(x) = \dfrac{1}{1+|Du_0(x)|^2}$ its edge map, (vanishing on edges)

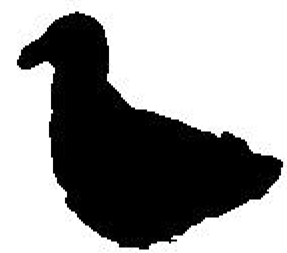
v analysing image, whose zero level set approximates some desired contour, by moving v by the equation

$$\frac{\partial v}{\partial t} = g\ curv(v)|Dv| - Dv.Dg,$$

the zero level curve γ of v, tends to minimize

$$\int_g g(x(s))ds,$$

where $x(s)$ is an arc length parameterization of γ.



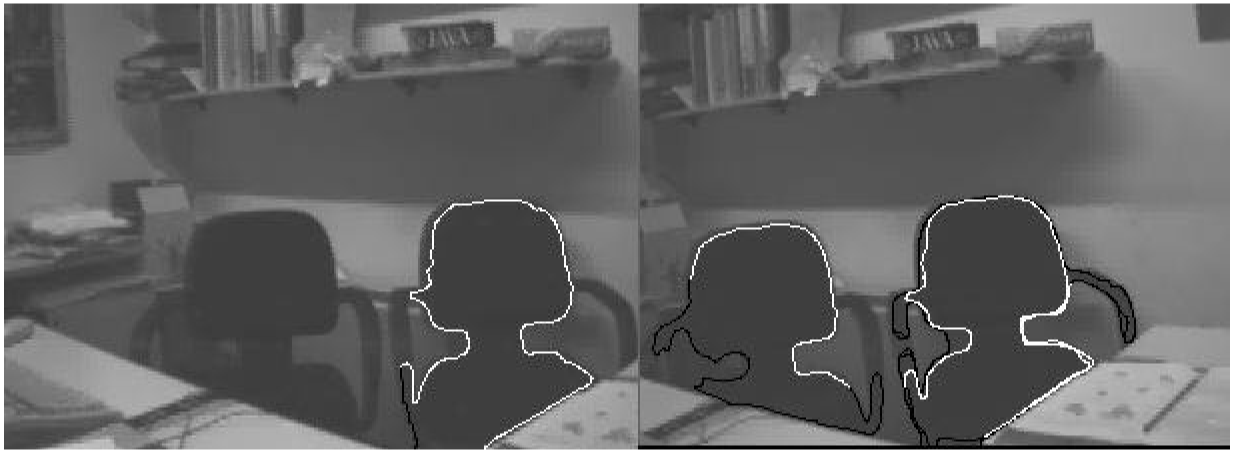
Application: Massive Shape Recognition Algorithms
image parsers

Idea : to smooth all level lines of each image by affine shortening until the average code (given e.g. by inflexion points) of each shape is small enough and encode them into a dictionary of shapes.

Fast and massive image comparison is possible

Example: image comparison, detection of repeated shapes.

Lisani & al
shape parser



One shape (filtered level curve) Shapes found similar

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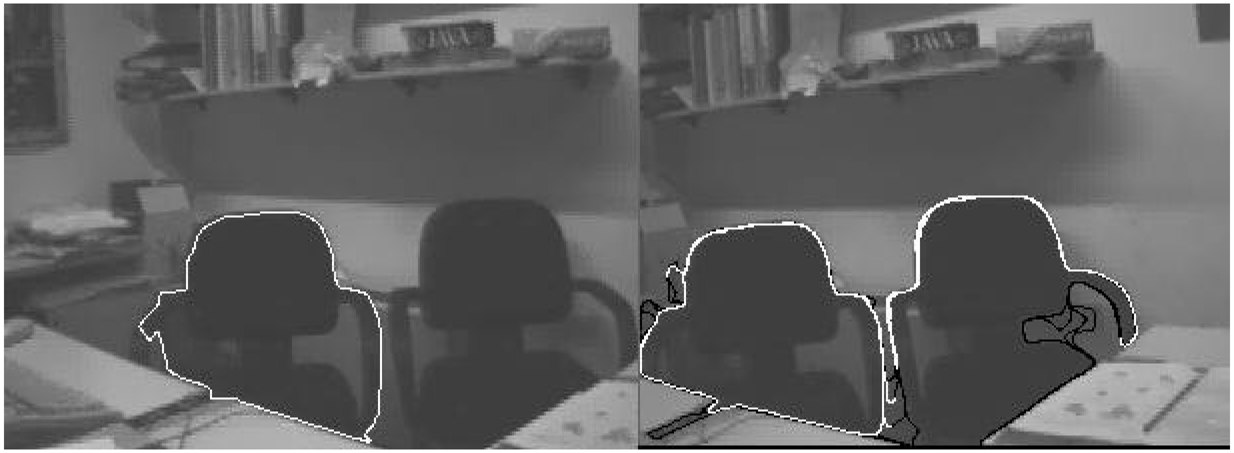

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Lisani & al
shape parser

all matching shapes

Conclusion

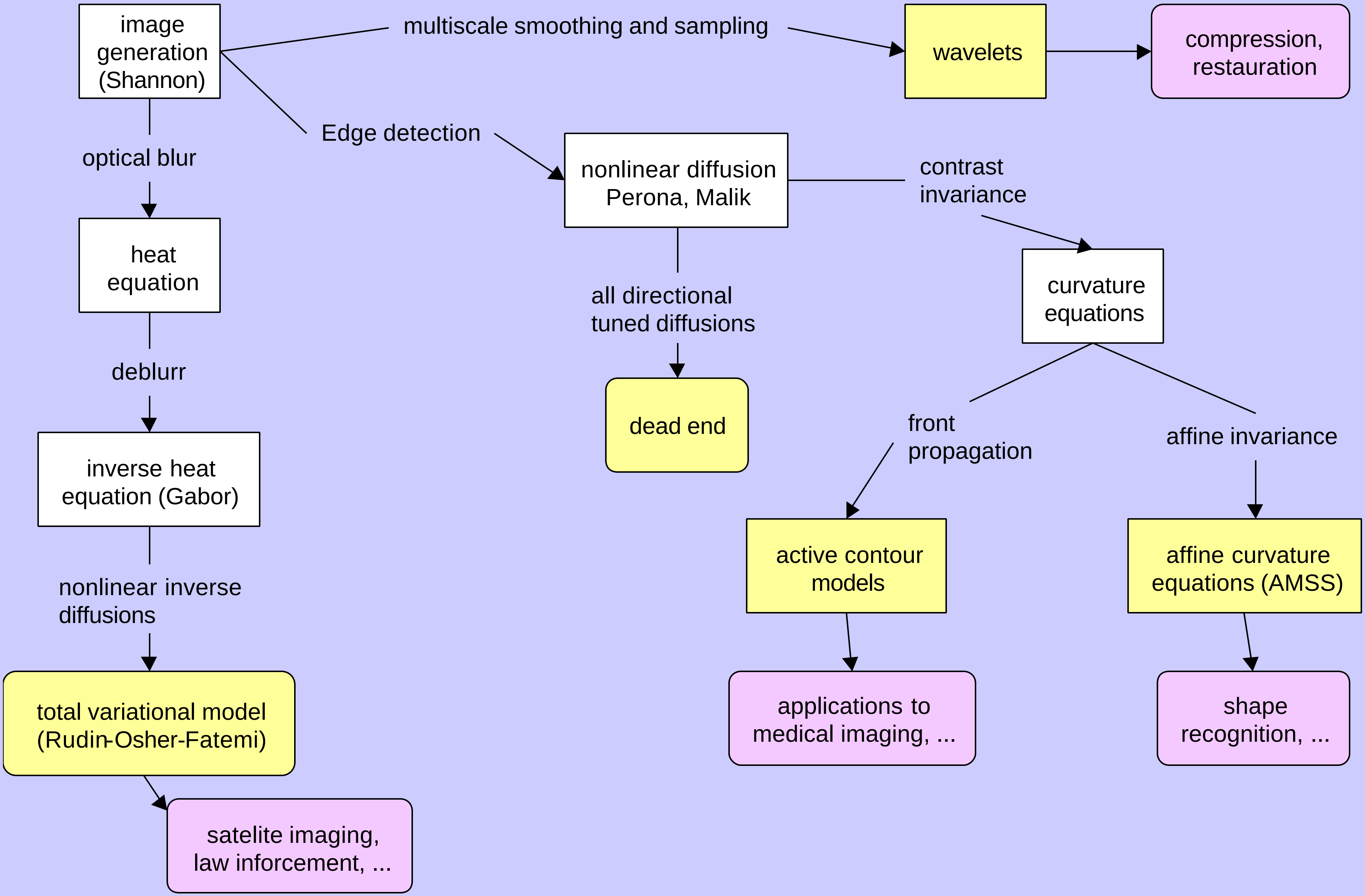
image generation (Shannon)
multiscale smoothing and sampling
wavelets
compression, restauration
Edge detection
nonlinear diffusion Perona, Malik
contrast invariance
optical blur
heat equation
all directional tuned diffusions
curvature equations
dead end
deblurr
inverse heat equation (Gabor)
front propagation
affine invariance
active contour models
affine curvature equations (AMSS)
nonlinear inverse diffusions
total variational model (Rudin-Osher-Fatemi)
applications to medical imaging, ...
shape recognition, ...
satelite imaging, law inforcement, ...