Semidefinite Programming in Polynomial Optimization

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SIAM Conference on Optimization
Stockholm
2005
Polynomial Optimization Problem (P)

\[ p_{\text{min}} := \inf_{x \in K} p(x) \]

where \( K := \{ x \in \mathbb{R}^n \mid g_1(x) \geq 0 \ldots g_m(x) \geq 0 \} \)

and \( p, g_1, \ldots, g_m \) are polynomials in \( n \) variables
Unconstrained Polynomial Minimization

\[ p_{\text{min}} := \inf_{x \in \mathbb{R}^n} p(x) \]

**Example:** An \( n \times n \) matrix \( M \) is **copositive** if \( p_{\text{min}} \geq 0 \) for \( p(x) := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij} \), i.e., \( p(x) \) is nonnegative on \( \mathbb{R}^n \)

**Example:** A sequence \( a_1, \ldots, a_n \in \mathbb{N} \) can be **partitioned** if \( p_{\text{min}} = 0 \) for \( p(x) = (\sum_{i=1}^{n} a_i x_i)^2 + \sum_{i=1}^{n} (x_i^2 - 1)^2 \)
0/1 Linear Programming

\[
\begin{align*}
\min c^T x & \quad \text{s.t. } Ax \leq b, \ x_i^2 = x_i \ \forall i
\end{align*}
\]

Example: The *stability number* \( \alpha(G) \) of \( G = (V, E) \) can be computed via any of the programs:

\[
\alpha(G) = \max \sum_{i \in V} x_i \quad \text{s.t. } x_i + x_j \leq 1 \ (ij \in E), \ x_i^2 = x_i \ (i \in V)
\]

\[
\frac{1}{\alpha(G)} = \min x^T (I + A_G)x \quad \text{s.t. } \sum_{i \in V} x_i = 1, \ x_i \geq 0 \ (i \in V)
\]

Hence: \( (P) \) is an NP-hard problem
**Strategy**

Approximate (P) by a hierarchy of convex (semidefinite) relaxations


Such relaxations can be constructed using

*representations of nonnegative polynomials as sums of squares of polynomials*

and

*the dual theory of moments*
Underlying paradigm:

Testing whether a polynomial $p$ is nonnegative is hard but one can test whether $p$ is a sum of squares of polynomials efficiently via semidefinite programming.
Some notation about polynomials

- Multivariate polynomial:

\[ p(x) = \sum_{\alpha \in \mathbb{Z}_+^n} p_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{\alpha \in \mathbb{Z}_+^n} p_{\alpha} x^{\alpha} \]

- \( \deg(x^{\alpha}) = |\alpha| := \sum_{i=1}^n \alpha_i \)

- \( \deg(p) = \max \deg(x^{\alpha}) \) s.t. \( p_{\alpha} \neq 0 \)

- We identify a polynomial \( p(x) \) with its sequence of coefficients \( p = (p_{\alpha})_{|\alpha| \leq d} \), if \( \deg(p) = d \)
• \( p(x) \) is a sum of squares of polynomials (SOS) if

\[
p(x) = \sum_{j=1}^{m} [u_j(x)]^2 \quad \text{for some polynomials } u_1, \ldots, u_m
\]

Then, \( \deg(p) \) is even and \( \deg(u_i) \leq \deg(p)/2 \)

**Example:** \( x^2 + y^2 + 2xy + z^6 = (x + y)^2 + (z^3)^2 \) is SOS
Recognizing SOS of pol’s via SDP

\[ p(x) = \sum_{|\alpha| \leq 2d} p_\alpha x^\alpha \]  

is a sum of squares of polynomials  

i.e.,  

\[ p(x) = \sum_j [u_j(x)]^2, \quad \text{where} \quad \deg(u_j) \leq d \]

\[ \uparrow \]

\[ p(x) = z^T \left( \sum_j u_j u_j^T \right) z, \quad \text{setting} \quad z := (x^\beta)_{|\beta| \leq d} \]

\[ \downarrow \]

The following semidefinite program is feasible:

\[ \begin{cases} 
X \succeq 0 \\
\sum_{|\beta|, |\gamma| \leq d, \beta + \gamma = \alpha} X_{\beta,\gamma} = p_\alpha \quad (|\alpha| \leq 2d) 
\end{cases} \]
\( \sim \) system in matrix variable \( X \) of order \( \binom{n+d}{d} \times \binom{n+d}{d} \), with \( \binom{n+2d}{2d} \) equations

\( \sim \) polynomial size fixing either \( n \) or \( d \)
When does a polynomial have a decomposition as a sum of squares?

Hilbert [1888] classified the pairs \((n, d)\) for which every nonnegative polynomial of degree \(d\) in \(n\) variables is a sum of squares of polynomials:

- \(n = 1\)
- \(d = 2\)
- \(n = 2, d = 4\)

E.g., the Motzkin polynomial: \(x^4y^2 + x^2y^4 - 3x^2y^2 + 1\) is nonnegative but not a SOS
Artin [1927] solved Hilbert’s 17th problem [1900]

\[ p \geq 0 \text{ on } \mathbb{R}^n \implies p = \sum_i \left( \frac{p_i}{q_i} \right)^2, \text{ where } p_i, q_i \in \mathbb{R}[x_1, \ldots, x_n] \]

That is,

\[ p \cdot q^2 \text{ is SOS for some } q \in \mathbb{R}[x_1, \ldots, x_n] \]

Sometimes, the common denominator is known:

Pólya [1928] + Reznick [1995]: For \( p \) homogeneous

\[ p > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \implies p \cdot \left( \sum_{i=1}^{n} x_i^2 \right)^r \text{ SOS for some } r \in \mathbb{N} \]
(Dual) SOS approximations for (P) following [Lasserre 2001]

Recall: \( K = \{ x \in \mathbb{R}^n \mid g_\ell(x) \geq 0 \ (\ell = 1, \ldots, m) \} \)

\[
p_{\min} := \inf_{x \in K} p(x) = \sup_{\lambda} p(x) - \lambda \geq 0 \ \forall x \in K
\]

Replacing the nonnegativity condition by a stronger SOS condition: formulate (SOST)

\[
\sigma_t := \sup_{\lambda} p(x) - \lambda = s_0(x) + \sum_{\ell=1}^{m} s_\ell(x)g_\ell(x)
\]

\( s_0, s_\ell \) are SOS
\[ \deg(s_0), \deg(s_\ell g_\ell) \leq 2t \]

for any \( t \geq \lceil \deg(p)/2 \rceil, \lceil \deg(g_\ell)/2 \rceil \)
\[ \sigma_t \leq \sigma_{t+1} \leq p_{\min} \]

**Example:** \( K = \mathbb{R}^n \) (unconstrained case), \( \deg(p) = 2d \)

\( \sigma_t = \sigma_d \leq p_{\min} \), with equality iff \( p(x) - p_{\min} \) is SOS

Equality in the univariate case (Shor [1987], Nesterov [2000])
(Primal) Moment Approximations

\[ p_{\text{min}} = \inf_{x \in K} p(x) = \inf_{\mu \in \mathcal{P}_K} \int p(x)\mu(dx) = \inf_{y \in \mathcal{M}_K} p^T y \]

\[ \int p(x)\mu(dx) = \sum_{\alpha} p_{\alpha} \int x^\alpha \mu(dx) \]

\( \mu \in \mathcal{P}_K \) if \( \mu \) is a probability measure supported by \( K \)

\( y \in \mathcal{M}_K \) if \( y_{\alpha} = \int x^\alpha \mu(dx) \ \forall \alpha \), for some \( \mu \in \mathcal{P}_K \)

moment of order \( \alpha \) of the measure \( \mu \)
Lemma:

If $y$ is the sequence of moments of $\mu \in \mathcal{P}_K$ then

$$y_0 = 1$$

$$M_t(y) := (y_{\alpha+\beta})_{|\alpha|,|\beta| \leq t} \succeq 0 \quad \text{for } t \geq 0$$

*moment matrix of order $t$*

$$M_t(g\ell y) := (\sum_{\gamma} (g\ell)_{\gamma} y_{\alpha+\beta+\gamma})_{|\alpha|,|\beta| \leq t} \succeq 0 \quad \text{for } t \geq 0$$

*localizing matrix*

Moreover: $\text{supp}(\mu) \subseteq \text{Zeros}(p) \quad \forall p \in \text{Ker} M_t(y)$
Proof:

For $p = (p_\alpha)_{|\alpha| \leq t}$,

$$p^T M_t(y)p = \int p(x)^2 \mu(dx) \geq 0$$

$$p^T M_t(g_\ell y)p = \int g_\ell(x)p(x)^2 \mu(dx) \geq 0$$
If we relax the moment condition "\( y \in \mathcal{M}_K \)" by requiring positive semidefiniteness of its moment matrix, we obtain a hierarchy of tractable SDP relaxations:

\[
(MOM_t)
\]

\[
p_t := \inf \ p^T y \quad \text{s.t.} \quad y_0 = 1, \ M_t(y) \succeq 0 \\
M_{t-d_\ell}(g_\ell y) \succeq 0 \ (\ell = 1, \ldots, m)
\]

for any \( t \geq \lceil \deg(p)/2 \rceil, \ d_\ell := \lceil \deg(g_\ell)/2 \rceil \)

\[
p_t \leq p_{t+1} \leq p_{\min}
\]
Lemma:
The semidefinite programs (SOST) and (MOMt) are dual of each other.

Weak duality: \( \sigma_t \leq p_t \)

No duality gap, e.g., if \( K \) has a nonempty interior
Properties of the bounds

\[ \sigma_t \leq p_t \leq p_{\text{min}} \]

- For fixed \( t \), one can compute \( \sigma_t, p_t \) in **polynomial time** (to any fixed precision)

- **Asymptotic (finite) convergence** to \( p_{\text{min}} \)
  via some representation theorems for positive polynomials from real algebraic geometry

- **Optimality certificate**
  via some theorems of Curto and Fialkow about moment matrices

- **Extracting global minimizers**
  via the eigenvalue method for solving systems of polynomial equations
Asymptotic convergence

**Representation theorem:** [Putinar 1993]
(more elementary proof by [Schweighofer 2003])

If \( \exists u_\ell \text{ SOS} \colon \{ x \mid \sum_{\ell} u_\ell(x) g_\ell(x) \geq 0 \} \) is compact \((\ast)\)

then \( p > 0 \) on \( K \implies p = s_0 + \sum_\ell s_\ell g_\ell \) with \( s_0, s_\ell \) SOS

**Convergence theorem:** [Lasserre 2001]

If \((\ast)\) then \( \lim_{t \to \infty} \sigma_t = \lim_{t \to \infty} p_t = p_{\min} \)
Note: (*) holds, e.g.,

- if \( \{ x \mid g_\ell(x) \geq 0 \} \) is compact for some \( \ell \)
- if the equations \( x_i^2 = x_i \) are present in the description of \( K \) (0/1 case)
- if the radius \( R \) of a ball containing \( K \) is known, then add the (redundant) constraint \( R^2 - \sum_i x_i^2 \geq 0 \) to the description of \( K \)

Note: A representation result valid for “\( p \geq 0 \) on \( K \)” would give a finite convergence result
Finite Convergence: Finite Variety

Assume the equations: \( h_1(x) = 0 \ldots h_k(x) = 0 \) are present in the description of \( K \) and

\[ V := \{ x \in \mathbb{C}^n \mid h_1(x) = 0 \ldots h_k(x) = 0 \} \text{ is finite} \]

Then: \( p_t = p_{\min} \) for \( t \) large enough \[ \text{[Laurent 2002]} \]

Moreover: \( \sigma_t = p_t = p_{\min} \) for \( t \) large enough

if \( h_1, \ldots, h_k \) form a Groebner basis \[ \text{[Laurent 2002]} \]
or if they generate a radical ideal \[ \text{[Parrilo 2002]} \]

**Representation result in radical case:** \[ \text{[Parrilo 2002]} \]
\( p \geq 0 \) on \( K \implies p = s_0 + \sum_\ell s_\ell g_\ell (s_0, s_\ell \text{ SOS}) \)
In the 0/1 case

\[ p_t = p_{\text{min}} \quad \text{for } t \geq n + d \]

where \( d := \max_{\ell} \left\lceil \frac{\deg(g_{\ell})}{2} \right\rceil \) \[\text{[Lasserre 2001]}\]

The hierarchy of bounds \( p_t \) refines other combinatorial hierarchies, e.g., by Lovász-Schrijver [1991] (cf. [Laurent 2003])

**Note:** One can use the equations \( h_1(x) = 0 \ldots h_k(x) = 0 \) to *reduce the number of variables* in the moment relaxations (MOMt)
Optimality Certificate

**Theorem:** [Henrion-Lasserre 2005]
If $y$ is an optimum solution to the program (MOMt) satisfying:

$$(RC) \quad \text{rank} M_t(y) = \text{rank} M_{t-d}(y),$$

where $d = \max_{\ell} \lceil \deg(g_{\ell})/2 \rceil$

then: $p_t = p_{\text{min}}$
Proof

By a theorem of [Curto-Fialkow 1996] (cf. [Laurent 2004])

\( y \) is the sequence of moments of an \( r \)-atomic nonnegative measure \( \mu \)

\[
\mu = \sum_{i=1}^{r} \lambda_i \delta_{x_i}, \text{ where } \lambda_i > 0, \sum_i \lambda_i = 1, x_i \in K
\]

\[\Rightarrow y = \sum_{i=1}^{r} \lambda_i (x_i^\alpha) |\alpha| \leq 2t\]

\[\Rightarrow p_{\text{min}} \geq p_t = p^T y = \sum_{i=1}^{r} \lambda_i p(x_i) \geq p_{\text{min}}\]

Moreover: \( x_1, \ldots, x_r \) are global minimizers over \( K \)
Extracting global minimizers

Then, \( \text{supp}(\mu) = \{x_1, \ldots, x_r\} \subseteq \{\text{global minimizers}\} \)

Moreover,

\[
\text{supp}(\mu) = \bigcap_{p \in \text{Ker}M_t(y)} \text{Zeros}(p)
\]

**Therefore:** One can find the global minimizers \( x_1, \ldots, x_r \) by computing the common zeros to a system of polynomial equations

\( \rightsquigarrow \) Use the ‘eigenvalue method’

Based on computing the eigenvalues of the \( r \times r \) multiplication matrices in the polynomial ring \( \mathbb{R}[x_1, \ldots, x_n]/I \), where \( I \) is the ideal generated by the kernel of \( M_t(y) \)
Implementations of the SOS/moment relaxation method

**GloptiPoly** by Henrion, Lasserre

**SOSTOOLS** by Prajna, Papachristodoulou, Parrilo
Example 1

\[
\begin{align*}
\min \ p &= -25(x_1 - 2)^2 - (x_2 - 2)^2 - (x_3 - 1)^2 \\
&\quad - (x_4 - 4)^2 - (x_5 - 1)^2 - (x_6 - 4)^2 \\
\text{s.t.} \quad (x_3 - 3)^2 + x_4 &\geq 4, \quad (x_5 - 3)^2 + x_6 \geq 4 \\
&\quad x_1 - 3x_2 \leq 2, \quad -x_1 + x_2 \leq 2, \quad x_1 + x_2 \leq 6, \\
&\quad x_1 + x_2 \geq 2, \quad 1 \leq x_3 \leq 5, \quad 0 \leq x_4 \leq 6, \\
&\quad 1 \leq x_5 \leq 5, \quad 0 \leq x_6 \leq 10, \quad x_1, x_2 \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>order $t$</th>
<th>bound $p_t$</th>
<th>solution extracted</th>
</tr>
</thead>
<tbody>
<tr>
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<td>unbounded</td>
<td>none</td>
</tr>
<tr>
<td>2</td>
<td>-310</td>
<td>(5,1,5,0,5,10)</td>
</tr>
</tbody>
</table>

The global minimum is found at the relaxation of order $t = 2$
Example 2

\[
\begin{align*}
\text{min} & \quad p = -x_1 - x_2 \\
\text{s.t.} & \quad x_2 \leq 2x_1^4 - 8x_1^3 + 8x_1^2 + 2 \\
& \quad x_2 \leq 4x_1^4 - 32x_1^3 + 88x_1^2 - 96x_1 + 36 \\
& \quad 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 4
\end{align*}
\]

<table>
<thead>
<tr>
<th>order $t$</th>
<th>bound $p_t$</th>
<th>solution extracted</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>-7</td>
<td>none</td>
</tr>
<tr>
<td>3</td>
<td>-6.6667</td>
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</tr>
<tr>
<td>4</td>
<td>-5.5080</td>
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</table>

The global minimum is found at the relaxation of order $t = 4$.
Unconstrained polynomial minimization

As $K = \mathbb{R}^n$, the relaxation scheme just gives one bound:
$\sigma_d = p_d \leq p_{\text{min}}$, with equality iff $p(x) - p_{\text{min}}$ is SOS

**Idea:**
Get better bounds by transforming the unconstrained problem into a constrained problem

If $p$ has a minimum:

$$p_{\text{min}} = p_{\text{grad}} := \inf_{x \in V_{\text{grad}}^\mathbb{R}} p(x)$$

where
$$V_{\text{grad}}^\mathbb{R} := \{ x \in \mathbb{R}^n \mid \frac{\partial p}{\partial x_i} = 0 \ (i = 1, \ldots, n) \}$$
If, moreover, a bound $R$ is known on the norm of a global minimizer:

$$p_{\min} = p_{\text{ball}} := \inf_{R^2 - \sum_i x_i^2 \geq 0} p(x)$$
When $p$ attains its minimum:

The ‘ball approach’:

- $p_{\text{ball}}$ can be approximated via Lasserre’s relaxation scheme (asymptotic convergence, as Putinar’s assumption holds!)
- seems to work well only if the radius $R$ of the ball is not too large ...

The ‘gradient variety’ approach:

**Representation result:** [Demmel, Nie, Sturmfels 2004]

\[
p > 0 \text{ on } V_{\frac{\partial p}{\partial x}} \implies p = s_0 + \sum_{i=1}^{n} u_i \frac{\partial p}{\partial x_i}
\]

where $s_0 \text{ SOS, } u_i \in \mathbb{R}[x_1, \ldots, x_n]$
Convergence result: [Demmel, Nie, Sturmfels 2004] There is asymptotic convergence of the SOS and moment bounds to $p_{\text{grad}}$

Moreover: When $\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}$ generate a radical ideal, the representation result holds for $p \geq 0$, implying the finite convergence
What if $p$ is not known to have a minimum?

**Strategy:** Perturb the polynomial $p$

[Hanzon-Jibetean 2003] [Jibetean-Laurent 2004]

$$p_\epsilon(x) := p(x) + \epsilon \left( \sum_{i=1}^{n} x_i^{2d+2} \right)$$

for small $\epsilon > 0$

- $p_\epsilon$ has a minimum and
  $$\lim_{\epsilon \to 0} (p_\epsilon)_{\min} = p_{\min}$$

- the gradient variety of $p_\epsilon$ is finite

$\leadsto$ one can compute $(p_\epsilon)_{\min}$ via the ‘gradient variety’ approach

$\leadsto$ finite convergence of the moment bounds to $(p_\epsilon)_{\min}$

(in 2nd steps)
• one can use the gradient equations:

\[
\frac{\partial p_\epsilon}{\partial x_i} = (2d + 2)x_i^{2d+1} + \frac{\partial p}{\partial x_i} = 0 \quad \forall i
\]


to reduce # variables (\(\leadsto\) only variables \(y_\alpha\) with \(\alpha_i \leq 2d \forall i\)) and to express \((p_\epsilon)_{\min}\) via a SDP with a single LMI

• the limits of global minimizers of \(p_\epsilon\) (as \(\epsilon \rightarrow 0\)) are global minimizers of \(p\) (if any)
Examples

**Motzkin polynomial:** \( p = x^2 y^2 (x^2 + y^2 - 3) + 1 \)

Then, \( p_{\text{min}} = 0 \), attained at \((\pm 1, \pm 1)\), and \( p \) is not SOS

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>order ( t )</th>
<th>((r_t, r_{t-1}, r_{t-2}))</th>
<th>((p_\epsilon)_t)</th>
<th>extracted solutions</th>
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<td>(5.0177 \times 10^{-4})</td>
<td>(\pm 0.9935, \pm 0.9935)</td>
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<td>(5.2951 \times 10^{-6})</td>
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<tr>
<td></td>
<td>5</td>
<td>(4, 4, 4)</td>
<td>(5.3170 \times 10^{-8})</td>
<td>(\pm 0.9999, \pm 0.9999)</td>
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</table>
Example:

\[ p = (xy - 1)^2 + y^2 \]

Then, \( p_{\min} = 0 \) is not attained

<table>
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<th>order ( t )</th>
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<th>( (p_\epsilon)_t )</th>
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<td>(7,4,2)</td>
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Testing copositivity of a matrix $M$

$M$ copositive iff $p_{\min} = 0$ for $p(x) := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij}$

Example 1:

$M = \begin{pmatrix}
1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1
\end{pmatrix}$

$M$ is copositive, e.g., since $(\sum_i x_i^2)p$ is SOS [Parrilo 2000]
<table>
<thead>
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<tr>
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<td>$(\pm0.7058, 0, 0, \pm0.7058)$</td>
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<tr>
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<td>4</td>
<td>(9,7,5)</td>
<td>$1.5544 \times 10^{-7}$</td>
<td>none</td>
</tr>
</tbody>
</table>
Example 2:

\[ M = \begin{pmatrix}
1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1
\end{pmatrix} \]

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>order ( t )</th>
<th>( (r_t, r_{t-1}, r_{t-2}) )</th>
<th>( (p_\epsilon)_t )</th>
<th>extracted solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>(8,7,4)</td>
<td>-1.3333</td>
<td>none</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(8,8,7)</td>
<td>-1.3333</td>
<td>±(0.8165, 0.8165, 0.8165, 0, 0)</td>
</tr>
<tr>
<td>( 10^{-1} )</td>
<td>3</td>
<td>(8,7,4)</td>
<td>(-133.3333)</td>
<td>none</td>
</tr>
<tr>
<td>( 10^{-1} )</td>
<td>4</td>
<td>(8,8,7)</td>
<td>(-133.3333)</td>
<td>±(2.5820, 2.5820, 2.5820, 0, 0)</td>
</tr>
<tr>
<td>( 10^{-2} )</td>
<td>3</td>
<td>(8,7,4)</td>
<td>(-1.3333 \times 10^4)</td>
<td>none</td>
</tr>
</tbody>
</table>

\( \leadsto \) certificate that \( M \) is not copositive
Approximating the stability number via matrix copositivity and Pólya’s representation result

[de Klerk-Pasechnik 2002] [Parrilo 2000]

\[ M \text{ strictly copositive}, \ i.e., \ p_M := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij} > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \]

[Polya] \[\sum_{i} x_i^2 \] \(r\) is SOS for some \(r \in \mathbb{N}\)

\[\implies M \text{ copositive} \]

[Motzkin-Straus 1965]

\[\alpha(G) = \min t \text{ s.t. } t(I + A_G) - J \text{ is copositive}\]
Define: \[ \vartheta^{(r)}(G) := \min t \text{ s.t. } t(I + A_G) - J \in \mathcal{K}(r) \]

where \[ \mathcal{K}(r) := \{ M \mid (\sum_i x_i^2)^r p_M \text{ is SOS} \} \]

**Theorem:** [dKP] \[ \lfloor \vartheta^{(r)}(G) \rfloor = \alpha(G) \text{ if } r \geq \alpha(G)^2 \]

**Conjecture:** [dKP] \[ \vartheta^{(r)}(G) = \alpha(G) \text{ if } r \geq \alpha(G) - 1 \]

**Motivation:** this rate of convergence holds for other hierarchies, e.g., by Lovász-Schrijver, Lasserre

**Theorem:** [Gvozdenović-L 2004] True for \( \alpha(G) \leq 8 \)

Lasserre’s hierarchy refines the hierarchy \( \vartheta^{(r)}(G) \)

Other results by [Peña-Vera-Zuluaga 2005]
Yet another representation result

**Theorem:** [Lasserre 2004]  
\[ p \geq 0 \text{ on } \mathbb{R}^n \quad \Rightarrow \quad \forall \epsilon > 0 \ \exists r \in \mathbb{N} \text{ for which} \]

\[ p_{r,\epsilon} := p + \epsilon \left( \sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!} \right) \text{ is SOS} \]

**Note:** \[ \|p_{r,\epsilon} - p\|_1 \rightarrow 0 \text{ as } \epsilon \downarrow 0 \]

Although there are *much more nonnegative polynomials than SOS polynomials* (at fixed degree, when \( n \rightarrow \infty \)) [Blekherman 2003]
new relaxation scheme for $p_{\text{min}} = \inf_{x \in \mathbb{R}^n} p(x)$

$$\inf y^T p_{r,\epsilon} \quad \text{s.t. } M_r(y) \succeq 0, \ y_0 = 1$$

extension to the constrained case
Many other interesting results unfortunately not covered here ....

- **SDP versus LP relaxations** [Sherali-Adams] [Lasserre]

- **Exploiting the symmetry to reduce the size of the SDP**
  
  [Parrilo-Gaterman] [Schrijver] [Dukanovic-Rendl]
  [de Klerk-Pasechnik-Schrijver] [Laurent] ...

\(\hookrightarrow\) invariant theory, block-diagonalization of \(C^*\)-algebras

- **Exploiting the sparsity of the polynomials to reduce the size of the SDP**
  
  [Reznick 1978] [Kim-Kojima-Muramatsu-Waki] ...

- **Extension to matrix polynomial optimization**
  
  [Kojima] [Hol-Scherer] [Henrion-Lasserre] ...