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Falling Paper and Flying Business Cards

By David L. Finn

"Every one must have observed that when a slip of paper falls through the air, its motion though undecided and wavering at first, sometimes becomes regular."

—James Clerk Maxwell, 1854, *Maxwell's Collected Works*

Take a business card, hold it out in a horizontal position. Then let go and watch as it falls to the ground.

Now take another business card, hold it out in a vertical position, let go, and watch as it falls to the ground. (See Figures 1 and 2.)

Figure 1. *Business cards held horizontally and vertically.*

The horizontally held card should have fluttered to the ground with a gentle back-and-forth motion, landing roughly below the point from which it was released. The vertically held card should have fallen for a moment, wavering slightly, and then tumbled end over end, landing far from the point directly below the release point. (See Figure 2.)

This simple experiment yields some surprising applications of sophisticated mathematics. Consider the flight times of the two business cards. Which one will hit the ground first? (It is assumed that the cards are released with their centers of mass at the same height.)

This question was posed to a group of freshman science and engineering students on their first day in a multivariate calculus class. Before the experiment, the students correctly predicted the fluttering motion of the horizontally held card. But they predicted a descent straight down for the vertically held card, with the card cutting through the air like a knife. Based on these predictions, the students concluded that the vertically held card would hit the ground first. When the experiment was performed, the students were surprised to see that in most cases the vertically held card tumbled end over end and consequently stayed aloft longer than the horizontally held card, which fluttered down to the ground.

Z. Jane Wang of Cornell University, in a lecture titled "Dragonflies as Airplanes," explained some of the phenomena associated with falling paper and flying business cards in a symposium titled "How Insects Fly" at the February 2006 meeting of the AAAS in St. Louis. Describing the connection between falling paper or business cards and insect flight, Wang says that studying falling paper (the flight of business cards) allows one to "set the wing free." In other words, if it were possible to dissect insect flight into active flight, driven by the motion of the wings, and passive flight, governed by the interaction of the wings with the air, then looking only at the passive interaction of the wing with the air is equivalent to studying falling paper.

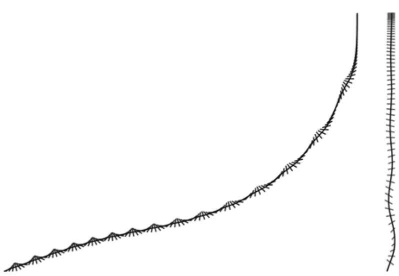

Figure 2. *Simulation of falling business cards, dropped from vertical (left) and horizontal (right) positions.*

This might seem to be a step backward in the study of insect flight, but the physics of an insect wing in flight is not at all well understood. The classical aerodynamics of an airfoil does not explain the motion of insects: According to classical theory, a bumblebee can't fly. The improved aerodynamic models generated in studies of falling paper may eventually help explain the flight of a bumblebee.

In her lecture, Wang considered other aspects of insect flight, such as how insects hover and the efficiency of flapping flight, some of which have benefitted from the models of falling paper. In fact, she says that she studies falling paper to better understand the physics of an insect wing in flight—"a piece of paper experiences the same kind of physics as the dragonfly wing."

In examining the motion of falling paper, Wang and her collaborators, Anders Anderson and Umberto Pesavento (cf. [1–3]), used high-speed digital video to track and accurately measure the trajectories of falling paper and falling aluminum plates in a water tank. Their main experiments with falling aluminum plates were designed to minimize the three-dimensional effects without unduly influencing the motion of the plates. In particular, the experiments avoid adding forces that would constrain the motion of the plates and minimize any interactions of the plates and the walls of the tank. Furthermore, the dimensions of the plates (length $L \approx 19$ cm, width $W \approx 0.8$ cm, and thickness $T \approx 0.12$ cm), along with the release mechanism, ensure that the trajectories are in essence two-dimensional. The controlled experiment differs in this respect from the flying business card experiment described earlier, in which the dimensions are length $L = 8.8$ cm, width $W = 5.0$ cm, and thickness $T = 0.03$ cm. These dimensions and the uncertainty of the horizontal and vertical positions make the three-dimensional effects harder to negate.

The measurements obtained from the high-speed digital video are used in solving the Navier–Stokes equations for the flow around the plate, with a coordinate system adapted to the plate. For computational purposes, the rectangular plate is modeled as an elliptic cylinder; the Navier–Stokes equation can then be modeled by a stream-vorticity function, together with a conformal mapping to account for plate-adapted coordinates:

$$\begin{cases} \dfrac{\partial(S\omega)}{\partial t} + \left(\sqrt{S}\,\mathbf{u}\cdot\nabla\right)\omega = \dfrac{1}{Re}\nabla^2\omega \\ \nabla\cdot\left(\sqrt{S}\,\mathbf{u}\right) = 0, \end{cases}$$

where $\mathbf{u}$ and ω are the velocity field and vorticity fields of the fluid and S is the scaling factor associated to the conformal mapping with plate-adapted coordinates. The observed plate dynamics are incorporated into the Navier–Stokes solver through boundary conditions to reflect the use of plate-adapted coordinates.

The Navier–Stokes solution is then used to compute the pressure and viscous forces on the plate; the result is a fluid force-based model for explaining the phenomena observed in the experiments. This is an improvement on the phenomenological model for the motion of falling paper developed earlier by Tanabe and Kaneko [6] and Mahadevan [3] on the basis of quasi-steady analysis. The phenomenological model consists of the system of coupled differential equations written in plate-adapted coordinates in dimensional form:

$$\begin{aligned} m_1\dot{v}_{x'} &= m_2\dot{\theta}v_{y'} - \rho_f\Gamma v_{y'} - m'g\sin\theta - F_{x'} \\ m_2\dot{v}_{y'} &= -m_1\dot{\theta}v_{x'} + \rho_f\Gamma v_{x'} - m'g\cos\theta - F_{y'} \\ I\ddot{\theta} &= (m_1 - m_2)v_{x'}v_{y'} - \tau, \end{aligned}$$

where $v_{x'}$, $v_{y'}$ are the velocity in the adapted coordinate system and θ is the rotation of the plate from vertical (see Figure 3). In the model, m_1 and m_2 represent the mass of the plate adjusted for added mass effects from the interaction of the plate with the fluid as per inviscid theory and I is the moment of inertia, also adjusted for added mass effects, Γ is the circulation around the plate, and ρ_f is the density of the fluid.

The terms $F_{x'}$, $F_{y'}$, and τ represent the viscous drag force and the dissipative torque. The drag forces $F_{x'}$ and $F_{y'}$ are modeled with standard quadratic models, so that

$$\begin{aligned} F_{x'} &= \rho_f a(C_D(0)\cos^2(\alpha) + C_D(\pi/2)\sin^2(\alpha))\,|\mathbf{v}|\,v_{x'} \\ F_{y'} &= \rho_f a(C_D(0)\cos^2(\alpha) + C_D(\pi/2)\sin^2(\alpha))\,|\mathbf{v}|\,v_{y'}, \end{aligned}$$

where $|\mathbf{v}|$ is the speed of the plate and α is the angle of attack, defined as the angle between the major axis of the ellipse and the velocity vector. The dissipative torque τ is modeled by integrating the normal component of the local drag over the surface of the plate, and is of the general form

$$\tau = \rho_f a^4(\mu_1 + \mu_2|\dot{\theta}|)\dot{\theta},$$

where μ_1 and μ_2 are constants determined in the experiment.

The major improvement in the phenomenological model is the treatment of the circulation Γ of the fluid around the card. By fitting the pressure and force from the Navier–Stokes solution, Wang, Anderson, and Pesavento found that the circulation has a rotational term and a translational term. This means that the circulation can be modeled as

$$\Gamma = C_R a^2\dot{\theta} + C_T a|\mathbf{v}|\sin(2\alpha),$$

where a is the major axis of the elliptic model of the plate. The terms C_R and C_T are constants determining the relative importance of the rotational and the translational terms in determining the lift of the plate. The earlier models of Tanabe and Kaneko [6] and Mahadevan [3] assumed the circulation to be either proportional to the translational velocity or constant.

This improved model for the circulation allows simulations from solving the system of differential equations to better match the observed phenomenon. In particular, it predicts the center of mass elevation that is observed in falling paper (and falling leaves). This elevation explains why, on a still autumn day, leaves sometimes seem to rise as they fall, as if blown by a breeze. Figure 4 shows various trajectories arising from a dimensionless form of the phenomenological model. Some trajectories show a fluttering motion described by alternating gliding at low angles of attack and fast rotational motion at the turning cusp-like points. Other trajectories show a periodic tumbling motion, alternating between short and long gliding motions. In both fluttering and tumbling trajectories, elevation of the center of mass can be observed near the cusp-like points where the card experiences quick rotation. Finally, some trajectories are more chaotic, combining tumbling and fluttering motions.

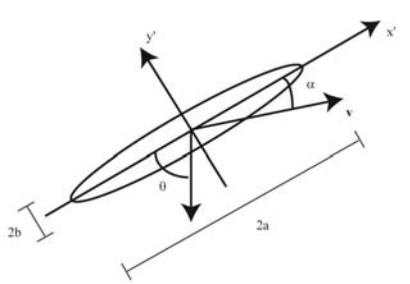

Figure 3. *Adapted coordinates for the phenomenological model.*

The explanation for the more rapid descent of the horizontally held business card comes from an analysis of the phase space for the phenomenological model. The two perfect falling states for business cards stem from the initial states $u = 0$, $v = 0$, $\dot{\theta} = 0$, $\theta = \pi/2$ corresponding to perfect vertical release and $u = 0$, $v = 0$, $\dot{\theta} = 0$, $\theta = 0$ corresponding to perfect horizontal release. The perfect vertical release yields the edge-on steady trajectory $u = \pm U$, $v = 0$, $\theta = 0, \pi$, $\dot{\theta} = 0$; the perfect horizontal release yields the broad side-on trajectory $u = 0$, $v = \pm V$, $\theta = 0, \pi$. The velocity U in the edge-on trajectory is greater than the velocity V in the broad side-on trajectory. This corresponds to the intuition that the edge-on trajectory has less air resistance than the broad side-on trajectory and, consequently, that the vertically held card should hit the ground first. Analysis of the phase space, however, shows that the edge-on steady trajectory is unstable as the $v_{x'}v_{y'}$ term in the expression for $\ddot{\theta}$ acts to rotate the card in the same direction, irrespective of the small perturbation.

Further analysis of the phase space shows that when the parameters dictate an eventual tumbling motion, there are two stable limit cycles in $v_{x'}$-$v_{y'}$-$\dot{\theta}$ space, one with $\dot{\theta} > 0$ and one with $\dot{\theta} < 0$, corresponding to periodic tumbling in different directions. All trajectories except the edge-on fixed point and the broad side-on fixed point approach one of these two stable limit cycles. The important observation for the falling business card experiment is that the broad side-on fixed point is also unstable, but that trajectories near the broad side-on fixed point take longer to approach a stable limit cycle than trajectories near the edge-on fixed point. This corresponds to the observation that the horizontally held card normally hits the ground first.

The work of Anderson, Pesavento, and Wang on the dynamics of falling paper can be viewed essentially as a continuation of the work of Maxwell [4]. The difference is that Maxwell was working before the mathematical techniques and physics of fluid mechanics were well understood, and he lacked the computers needed to solve the Navier–Stokes equations and the corresponding nonlinear ordinary differential equations from quasi-steady analysis. In fact, the original work of Maxwell is qualitative: Not a single equation is presented in his four-page paper. The next step in modeling a falling card will be to account for three-dimensional effects, which will make it possible to model the motion of a falling leaf. Solving the Navier–Stokes equations in three dimensions, however, is daunting at present.

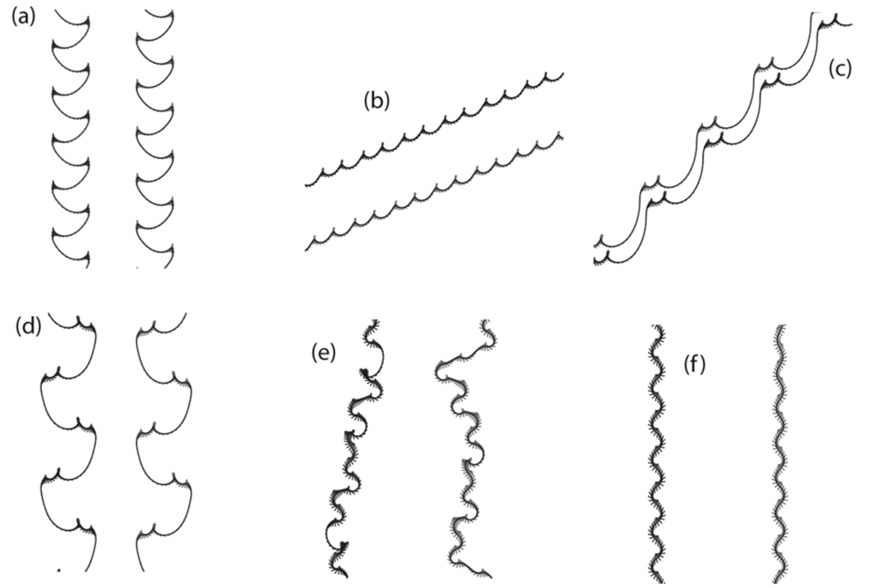

Figure 4. *Various trajectories of falling paper from a dimensionless version of a phenomenological model, depending on the dimensionless quantity $(b\rho_s)/(a\rho_f)$, where ρ_s is the density of the plate, ρ_f is the density of the fluid, and a,b are the semi-major and semi-minor axes of the elliptic model of the plate. (a) fluttering motion, (b) tumbling motion, (c) periodic tumbling with alternating short and long glides separated by cusp-like points of high rotation, (d) periodic fluttering, also with alternating short and long glides separated by cusp-like points of high rotation, (e) chaotic motion, (f) broadside fluttering.*

References

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Z. Jane Wang is scheduled to give an invited talk on the work discussed in this article at the 2007 SIAM Conference on Applications of Dynamical Systems, Snowbird, Utah, May 28–June 1.

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