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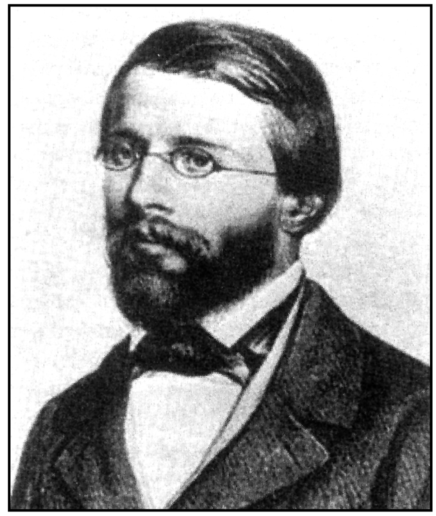
Bringing the Riemann Zeta Function to the World's Attention

The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics. *By Marcus du Sautoy, Harper Collins, New York, 2003, 352 pages, $24.95.*

At least three books about the Riemann hypothesis appeared during 2003. The dust jacket of the one under review identifies the author as a professor of mathematics at the University of Oxford, a research fellow at the Royal Society, and a frequent contributor on mathematics to *The Times* and to BBC radio.* The early chapters of his book provide—in aggressively nontechnical terms—background information about the prime numbers, and the modern quest for information about their distribution along the real number line. His account begins rather slowly, but the pace quickens as the focus shifts from ancient to more recent history.

BOOK REVIEW

By James Case



Not previously known as a gifted or tireless calculator, Riemann devised a method for calculating the zeros of his zeta function that truly came into its own only with the birth of the modern computer. Currently, the first 6.3 billion zeros of ζ are known to lie on the Riemann line Λ.

In the course of convincing the world of the value of mathematical proof, Euclid demonstrated that there are infinitely many primes, and that every nonprime has a unique factorization into primes. Eratosthenes, the librarian at Alexandria, was apparently the first to tabulate these "building blocks" of the natural number system. Using the "sieve method" for which he is still remembered, he reputedly identified several thousand primes. Because his table did not survive the fire that destroyed the great library of the ancient world, however, nobody knows for sure how many of them he actually found. Thereafter, the subject lay dormant until the likes of Fermat and Mersenne—armed with Arabic numerals and the techniques of modern arithmetic—revived it during the 17th century. Euler later published many of the proofs those two challenged one another to discover during their decades of correspondence, including one of Fermat's little theorem: $x^p \equiv x$ (modulo p). Euler also calculated the first 100,000 primes, demonstrated that the Mersenne number $2^{32} - 1$ is not a prime, and discovered that $x^2 + x + 41$ is prime for $x = 0, \ldots, 39$. If a simple polynomial could generate so many primes, might not a more complicated function generate all?

Unable to produce a formula for the nth prime, Gauss focused on a simpler problem. By examining a table of the first million primes, he observed that the number $\pi(N)$ of primes smaller than N is well approximated by $N/\ln(N)$, which is roughly the number that would be expected if primality were a property assigned to individual integers by means of a random device that, when activated for the nth time, gives an affirmative response with probability $1/\ln(n)$. The expected number of primes among the first N positive integers would then be $1/\ln(2) + \ldots + 1/\ln(N) \approx \int_1^N dx/\ln(x) = \mathrm{Li}(N)$. Legendre later improved on Gauss's original estimate by subtracting 1.80366 from the denominator. Yet neither could prove that his approximation remains accurate as N increases without bound.

Not until 1851 did Chebyshev contrive, using more or less elementary methods, to prove that $ax/\ln(x) \le \pi(x) \le bx/\ln(x)$ for numbers $a < 1 < b$ not too different from unity. Then in 1859, Riemann published a ten-page paper in the *Notices* of the Berlin Academy—the only one he was ever to write on number theory—in which he not only produced what seemed to be the most accurate approximation yet of $\pi(N)$, but calculated the first three complex zeros of the function $\zeta(z) = \sum_n 1/n^z$, and explained how his approximation might be rendered exact if (as he seemed to suspect) all the remaining complex zeros should turn out to lie on the "Riemann line" Λ: Re $(z) = ½$ in the complex plane. By so doing, he brought "the Riemann zeta function" to the attention of the entire mathematical world. Hilbert refocused it there in 1900, when he placed a proof of "The Riemann Hypothesis" first on his list of 23 challenging problems for the new century.

Riemann almost certainly learned of the zeta function from Dirichlet, who left mathematically active Berlin for sleepy medieval Göttingen to assume the chair vacated on Gauss's death; Dirichlet had already exploited the properties of ζ as a real-valued function of a real variable to prove that any arithmetic sequence $\{a + rn\}_{n=0}^{\infty}$ contains infinitely many primes, provided only that the integers a and r have no common divisor. Riemann noted that ζ's domain of definition extends to the entire complex plain, and embarked on a detailed study of its properties. In particular, he described the surface (sketched in Figure 1) whose altitude is the modulus of $\zeta(z)$.

It then took Hadamard and de la Vallée Poussin (working independently) until 1896 to turn Gauss's prime number conjecture into Gauss's prime number theorem, by showing that a and b can be replaced by $1 - \varepsilon$ and $1 + \varepsilon$, respectively, for any small

*Du Sautoy is also the mathematician included in *Science, not Art: Ten Scientists' Diaries*, reviewed by Philip Davis in the December 2003 issue of *SIAM News* (www.siam.org/siamnews/12-03/tocdec03.htm).

positive ε. Each did this by proving that the zeta function has no zeros on the line $\mathrm{Re}(z) = 1$. It is here, with 20th-century efforts to discern the consequences of Riemann's hypothesis, and to settle it one way or another, that du Sautoy's story really begins. It is a remarkable tale in which he manages, without deviating too far from his main story line, to involve an enormous number of noteworthy mathematicians.

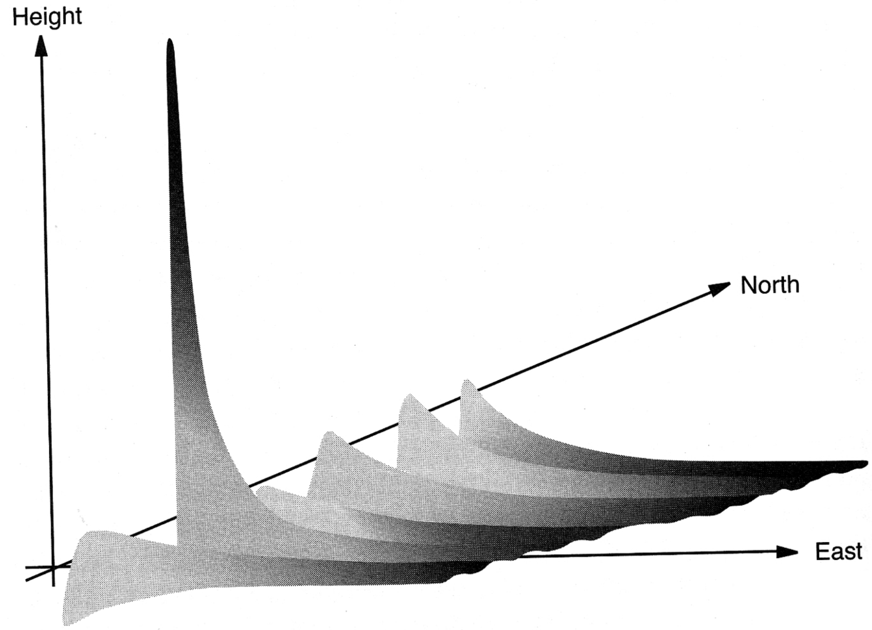


Figure 1. *The zeta landscape. Riemann discovered how to continue this picture into a new land to the west. From* The Music of the Primes.

Du Sautoy quotes G.H. Hardy (often from *A Mathematician's Apology*) with particular relish and frequency. He does so in part because he is anxious to portray the ingrown and provincial nature of mathematics in England at the end of the 19th century—a time when the works of even the most prominent continental authors were seldom read. So out of touch had English mathematicians become that Ernest Barnes, one of J.E. Littlewood's undergraduate "tutors" at Cambridge (and a future Bishop of Birmingham) suggested the Riemann conjecture as a problem Littlewood might care to solve during his summer vacation. While Littlewood had no luck locating the zeros, he did figure out that knowledge of their whereabouts would disclose a wealth of information concerning the prime numbers, and hastened to write up (in September 1907, as a thesis in support of his application for a fellowship at Trinity College, Cambridge) what he thought was a new connection between these apparently unrelated branches of mathematics. That he believed his discovery to be original merely underlines the isolation in which British mathematicians then labored. It also helps to explain why, as late as the 1940s and 1950s, U.S. graduate students in mathematics were forced to rely mainly on texts written in French and German.

One of Hardy's lifelong collaborators was Harald Bohr, brother of physicist Niels and a celebrated member of the Danish soccer team that won a silver medal at the 1908 Olympics. Shortly after the Olympics, Bohr proved a theorem—in collaboration with Edmund Landau—to the effect that most of the Riemann zeta function's infinitely many zeros lie in a strip of width ε to the right of the (vertical) line Λ. That didn't mean that any of the zeros are actually on Λ, but it did make the Riemann hypothesis seem marginally more plausible. By this time 71 zeros had been shown (by numerical means) to lie on the line. Then, in 1914, under the influence of Landau's 1909 *Handbook of the Theory of the Distribution of Prime Numbers*, Hardy was able to prove that infinitely many of the zeros actually do lie on Λ, and set out to verify Riemann's unsupported claim that "most of the zeros" can be shown to do likewise. Hardy's own methods could identify no positive fraction, and to this day less than 50% of the zeros are known to lie on Λ.

Among Littlewood's many contributions to the subject was his demonstration that Gauss had been wrong in his belief that $\mathrm{Li}(N)$ would always exceed the true number $\pi(N)$ of primes smaller than N. Littlewood proved that, to the contrary, there would be infinitely many N-intervals in which $\pi(N)$ would exceed $\mathrm{Li}(N)$. But it remained for a graduate student of Littlewood's named Stanley Skewes to prove, in 1933, that one would have to look no further than $N = 10^{10^{10^{34}}}$ to observe an instance in which Gauss's estimate $\mathrm{Li}(N)$ underestimates rather than overestimates $\pi(N)$. Hardy later declared the so-called Skewes number to be the largest ever contemplated in a mathematical proof. Skewes's demonstration was also memorable in being among the first to employ the Riemann hypothesis as a hypothesis. Not until 1955 did Skewes produce an even larger number that would suffice even if the Riemann hypothesis were false. Ramanujan, Hardy's most important collaborator, had relatively little to say about the distribution of the primes, although he did discover an asymptotic formula that, when rounded off to the nearest integer, gives the number $p(n)$ of partitions of n. Had he been able to repeat this feat for $\pi(n)$, interest in the Riemann hypothesis would of course have dwindled.

An interesting twist on the practice of predicating other results on the Riemann hypothesis turned up when Max Deuring, Louis Mordell, and Hans Heilbronn showed that a certain conjecture of Gauss concerning the factorization of his "complex integers" would be true if Riemann's hypothesis could be proved false. Imagine their surprise to learn that, some years previously, one Erich Hecke had proved that the same conjecture would be true if the Riemann hypothesis could be proved true!

Together, Hardy and Littlewood developed a method based on the Euler–MacLaurin formula for locating the first few zeros of the zeta function. By the late 1920s, they and their students had located 138 such zeros, without finding any that contradicted Riemann's prediction. But their method was clearly approaching the limits of its utility. So it was fortunate indeed that C.L. Siegel arranged for the remnants of Riemann's papers (many of which had been burned by his housekeeper at the time of his death) to be sent to him in Frankfurt via interlibrary loan. From them, he learned that Riemann—not previously known as a tireless or gifted calculator—had developed a method more powerful than that of Hardy and Littlewood for evaluating the zeta function and calculating its zeros. Using the better method, Hardy's students in Cambridge were soon able to confirm that the first 1041 zeros do indeed lie on Λ. But it was not until the birth of the modern electronic computer that Riemann's method would truly come into its own. At present, the first 6.3 billion zeros of ζ are known to lie on Λ.

The development of the electronic computer, with its relevance for the Rie-mann hypothesis, furnishes du Sautoy with an excuse to relate all manner of personal-interest items about Alan Turing, his activities at Bletchley Park during World War II, and his various mathematical contemporaries. It also permits a digression on the RSA method of public-key cryptography. While much of this material will be new and interesting to the nontechnical audience for whom the book is written, most of it will be familiar to readers of *SIAM News*. Yet the presentation is not without food for thought, including the (here unsupported) claim that, if the Riemann hypothesis is true, there must exist an efficient algorithm for factoring large nonprimes.

In 1971, hoping that the results would help him to dispose of some of Gauss's other unanswered questions concerning the factorization of complex integers, Hugh Montgomery set out to determine the distribution of gaps between the zeros of the Riemann zeta function. Basing his analysis on a conjecture by Hardy and Littlewood concerning the frequency of twin primes, Montgomery was surprised to conclude that the likelihood of a gap of length x is proportional to $1 - (\sin(\pi x)/\pi x)^2$, which is all but independent of x unless x is small. He had expected the zeros to occur in bursts, as do the arrival times of a Poisson process, and was surprised to learn that they "repel one another" instead. He was even more surprised to learn, quite by chance—from Freeman Dyson, during a social visit to the Institute for Advanced Study in Princeton—that the gaps between pairs of eigenvalues of random Hermitian matrices are likewise distributed. Such eigenvalues have been studied at length in connection with energy levels in the nucleus of a heavy atom under bombardment by low-energy neutrons.

Montgomery naturally wondered about the accuracy of his prediction concerning the gaps between zeros of the Riemann zeta function. Based as it is on a mere conjecture, his theory is hardly definitive. He soon discovered that the gaps between the first few hundred zeros do not conform particularly well to his predictions, but remained hopeful that the goodness of fit would improve as additional zeros were calculated. Fortunately, in the late 1970s, Andrew Odlyzko of Bell Labs, using AT&T's recently purchased Cray supercomputer, was able to perform just such calculations. By the mid-1980s, he furnished Persi Diaconis with some 50,000 zeros of magnitude 10^{20} or more. Diaconis reportedly "tested the hell out of" the gaps between those zeros without finding anything like a significant discrepancy between Montgomery's theory and fact.

In his final chapter, du Sautoy describes the post-World War II careers of André Weil, Atle Selberg, Alexandre Grothendieck, John Nash, Paul Cohen, Alain Connes, Enrico Bombieri, N. Bourbaki, and any number of others. In doing so, he at least suggests that the descent into madness of both Nash and Grothendieck was in part a consequence of their inability to settle the Riemann conjecture. While wishing that du Sautoy had included a few more technical details, one is forced to share his wonder at the connections these and others have documented between the prime numbers and a host of apparently unrelated branches of math and physics.

James Case writes from Baltimore, Maryland.