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Are Medical Students Meeting Their (Best Possible) Match?

By Sara Robinson

Every March, hundreds of graduating medical students put themselves at the mercy of a mathematical algorithm that pairs them with the teaching hospital they'll go to for their residency training.

Participating students and hospitals submit rank-ordered lists of their preferences; on Match Day, a computer comes up with a matching of students to programs that all participants must accept. The system is based on an elegant mathematical theory dating from the 1960s, yet medical students, unversed in game theory, have trouble believing that it's fair.

It's not surprising, then, that three disgruntled physicians, fed up with the low wages and long hours that characterize medical residencies, are blaming the algorithm for their woes. In their words (or their lawyers'), the National Resident Matching Program and its sponsoring hospitals:

"contract, combine and conspire to restrain competition in the recruitment, hiring, employment and compensation of resident physicians through the NRMP matching program, a mechanism that eliminates a free and competitive market and substitutes a centralized, anticompetitive allocation system assigning prospective resident physicians ("applicants") to a single, specific and mandatory residency position."

Some of the rancor of the former residents might be attributed to the illusion of unfairness that comes from having a computer decide one's future. Still, the residents do have a point: Their salaries are indeed uniform and low (about $35,000), and their working hours are uniformly long (starting next year, hospitals will be able by law to work their residents a maximum monthly average of "only" 80 hours a week). But is the algorithm really to blame for the ills of the residency system?

On the surface, the matching system does seem to deviate from what might be expected in a free market. Salaries for residency positions are posted in advance, for instance, and all the residents in a particular residency program receive the same salary, with no room for negotiation. In a free market, the most desirable residents might be expected to command higher salaries than their less attractive counterparts, and the more pres-tigious programs might get away with paying less to work their residents harder.

On the other hand, as discussed below, use of the NRMP algorithm has many advantages. The algorithm has addressed some of the other problems that once plagued the residency market, and it produces a match that's guaranteed to be, in some sense, optimal.

To win their suit, says Mark Lemley, a professor specializing in antitrust law at the University of California, Berkeley, the physicians will likely have to demonstrate that harm caused by the match outweighs any benefits. Asked what would happen if someone were to offer the hospitals an algorithm with all the benefits of the current system but none of the harm, Lemley replies that the defendants wouldn't look so good if they turned down such a proposal.

Thus, the NRMP lawsuit raises (at least) two interesting (somewhat) mathematical questions: Is the algorithm really responsible for the poor working conditions of medical residents? And is there a way to adjust the current residency-matching algorithm that would fix these problems, rescuing the NRMP and its affiliates and improving the lot of hundreds of doctors-in-training?

Addressing the first question requires a glimpse of what the market looked like before the introduction of the matching algorithm.

Evolution of the Matching System

The centralized matching system now in place evolved from decades of struggle with some of the unwelcome features of free market. Alvin Roth, a Harvard University economist and a leading expert on two-sided matching theory, has described this evolution in a paper titled "The Evolution of the Labor Market for Interns and Residents: A Case Study in Game Theory."

Medical internships were introduced around the turn of the last century as an optional form of postgraduate education. Because interns were a source of cheap labor for hospitals, Roth explains, slots soon outnumbered applicants and competition for interns was fierce.

Salaries and working conditions of the internships were specified ahead of time, even then, and there was no negotiation in the process. Thus, the competition manifested itself in timing rather than price: Hospitals began to insist that their offers be finalized before those of their competitors.

The average date for finalizing an internship gradually crept from the end to the beginning of the senior year. By the mid-1940s, internships were being finalized at the beginning of the junior year of medical school, and some inquiries came even during the sophomore year.

Recognizing that the situation was out of hand, the Association of American Medical Colleges adopted a resolution prohibiting medical schools from disseminating student transcripts or reference letters before a certain date during the senior year. This fixed the advancing-date problem but created a new one: Students tended to hold onto offers as long as possible, hoping for offers from better schools.

The hospitals were unhappy with this situation; if a student rejected an offer at the last moment, the hospital might have trouble filling the slot with a desirable applicant. So the hospitals, still battling fiercely for residents, passed a series of resolutions shortening the time a student could sit on an offer. At one point, hospitals had telegrams offering residency positions delivered at precisely 12:01 A.M. on the earliest allowed day; the students were forced to accept or reject these offers within hours.

In response to this clearly unsustainable situation, all parties decided, in the early 1950s, that it was time for drastic changes in the procedure. The centralized system created as a result had students and hospitals communicating with each other as before, but it replaced the rounds of offers by ranking lists, submitted by both sides to a central authority. Following a standard procedure, the central authority then matched students with residency programs. Such was the inception of the residency-matching algorithm, although it took some fiddling (trial and error) to get a procedure that worked.

Stable Marriage

A decade after this ad hoc residency-matching algorithm was put in place, two game theorists published a paper titled "College Admissions and the Stability of Marriage" in the *American Mathematical Monthly.* The mathematicians, David Gale and Lloyd Shapley, imagined a situation in which each of n boys and n girls was armed with a preference list in rank-order of the n people of the opposite sex. The goal was to find a matching of the boys to the girls that was "stable," in the sense that there was no boy and girl who preferred each other to the people they were paired with. (Otherwise, they'd run off together and break up two marriages.)

Gale and Shapley were able to show that such a matching always exists, and their paper provides algorithms for finding one. It also shows that the matchings found by their algorithms have some interesting characteristics.

In the Gale–Shapley language of boys and girls and marriage, the algorithm goes as follows:

Assume that each boy has a list of the n girls in order of preference, and vice versa. The algorithm then proceeds in rounds.

In round one, each boy goes to his first-choice girl and says, "Will you marry me?" Each girl, after considering the boys proposing to her (if any), answers "maybe" (gets engaged) to the one who's highest on her preference list and "no" to the rest.

In successive rounds, each boy rejected in the previous round goes to his next-choice girl and proposes to her. Each girl, once again, looks at everyone proposing to her (including her engagement from the previous round) and says "maybe" to her most preferred boy and "no" to the rest.

Finally, when every boy is engaged to some girl, the algorithm terminates and the girls turn to their partners and say "yes."

To see that this point is eventually reached, notice that no boy can be rejected by all the girls, because there are exactly n of each sex and a girl rejects an offer only when she is engaged to someone else; once engaged, moreover, she stays engaged. In addition, because each round involves at least one proposal and no boy proposes to the same girl more than once, the algorithm must halt after at most n^2 rounds.

It's also easy to see that the match produced satisfies the stability criterion: Suppose that a boy from couple A and a girl from couple B prefer each other to the people they're paired with. Then girl B is higher on boy A's preference list than girl A, so he must have proposed to her in an earlier round and she must have said "no" because a boy she preferred proposed to her. But since the girls consistently march up their preference lists with each successive "no," she must prefer boy B to boy A, which is a contradiction.

Gale and Shapley also showed that the match achieved in this manner has a remarkable property: It is "boy-optimal" and "girl-pessimal," meaning that each boy is matched to the best girl he can get in any stable matching, while each girl ends up with the worst possible guy. (I leave this as an easy exercise for the reader's morning commute.) Of course, the corresponding algorithm that has the girls proposing achieves the opposite, prompting some reflection on real-life dating conventions.

Another exercise is to show that it's possible for those on the side that's not proposing to "game the system." By lying about her preferences, a girl can do better in the male-proposing algorithm than she would otherwise.

The Gale–Shapley paper goes on to consider the roommate-matching problem (in which couples don't have to be of the opposite sex) and shows that in some situations there is no stable matching. The paper launched a flurry of related research that continues today. Hundreds of papers on the subject have appeared, along with a couple of books—on the "stable marriage algorithm" (Knuth, and Gusfield and Irving) and on the general theory of two-sided matching (Roth and Sotomayor).

Gale–Shapley and Medical Residents

Remarkably, as Alvin Roth eventually showed, the Gale–Shapley algorithm is essentially the same as the residency-matching algorithm that had been developed nearly ten years earlier.

The Gale–Shapley result showed that the NRMP version of the algorithm, because it had the hospitals proposing to the residents, produced a match that was hospital-optimal and resident-pessimal. This meant that, in theory, the residents had the opportunity to get a better matching by lying about their preferences; in practice, however, the residents didn't have enough information about the preferences of their fellow residents to pull this off.

In the mid-1990s, following extensive discussions of some of the imperfections of the algorithm, the NRMP board hired Roth to adjust it. One of the changes he made was to have the residents do the proposing; he showed empirically, however, that it makes almost no difference which side proposes—only one resident in a thousand would end up at a different place. Roth also adjusted the way the algorithm treats couples, who are allowed to enter the match as a unit.

Alternative Algorithms

Last May, not long after reading about the NRMP lawsuit, I attended a theoretical computer science conference where discussion over lunch one day turned to the NRMP algorithm and the anti-competitiveness issue. The group decided that while the residency-matching algorithm, as is, does a good job of finding an equitable matching of hospitals and residents, it seems likely that the system has some effect on residents' salary level. Some of us proposed simple adjustments to the system that would allow for price competition.

Dana Randall of the Georgia Institute of Technology and I suggested an easy adjustment: allowing schools to divide each residency into sets of slots at different salary levels. Residencies offered could include, say, Harvard radiology at $33,000, Harvard radiology at $34,000, and so forth, each with some fraction of the total number of Harvard radiology slots. These residencies paired with price levels could then be ranked in the match as if they were separate schools.

This variation has the advantage of allowing schools to raise some salaries to attract better students without incurring the expense of across-the-board salary increases. The disadvantage of this residency–price system is that while schools may be willing to bid higher to attract a specific individual, they might not be willing to bid without knowing whom they'll get.

Peter Winkler of Bell Labs suggested a combinatorial matching in which students would submit their rankings of schools, together with the price differentials that would make the schools equivalent. A student might rank UCLA above Johns Hopkins, for instance, if UCLA's salary were $10,000 higher. The schools, for their part, would submit their lists of students, together with a maximum salary they're willing to pay for each. Winkler's algorithm then works in rounds, starting with the schools bidding the lowest possible salary for the students they want the most.

Winkler's algorithm has the advantages of allowing pricing to be a factor and letting hospitals bid for specific students, but it also has drawbacks: It doesn't enable the hospitals to stay within a given budget, for instance, nor does it recognize a bargain in a student who's not a top-choice candidate.

Generally, we decided, a desirable algorithm for residency matching will have the following properties: In addition to having the stability criterion, it will force hospitals to pay the free-market value of each student it gets, allow the schools to stay within a budget and to bid higher for specific students, and motivate all parties to be honest about their preferences.

It turns out, of course, that economists have already addressed the price-competition issue. In a 1982 paper, Alexander Kelso and Vincent Crawford described a matching of workers to firms, by way of an auction, that satisfies most but not all of the above properties. And just this year, Paul Milgrom of Stanford University has put the auction-style theory and the matching theory into one over-arching framework.

As the economists have shown, it's possible to devise an algorithm that results in stable matchings, and allows hospitals to bid higher for specific students. The matching won't be stable, however, if the hospitals have to stay within a certain budget, and there are always opportunities for "gaming the system."

The Real World

Sadly, advances in theory don't always translate into viable advances in practice. While it's easy for economists and mathematicians to imagine having residents assign price differentials to their residency choices, Milgrom says, it would be very difficult for them to do so in practice.

Milgrom also points out that nothing about the NRMP algorithm precludes having residents negotiate conditional salaries with prospective hospitals in advance of the match. ("If you end up matched to my program, I promise to pay you at least this much.") The rules set up by the NRMP don't allow this type of contract, however. (This should be the real issue when the case goes to court.)

It may also be that opportunities for price competition won't be taken advantage of in practice. Roth is skeptical, for instance, that a change in the algorithm or rules would make any real difference: "It's not clear that this sort of market naturally has price competition," he says, pointing to the situation before the match was implemented. There are other markets, such as the competition among law students for judicial clerk-ships, where competition manifests itself in timing rather than prices. (Although who knows what would happen if the timing incentives were removed without limits on the possibilities for price competition.)

Even if the market does admit salary competition, other factors could be stifling it. For instance, the federal government pays a subsidy to teaching hospitals, the Graduate Medical Education supplement, out of Medicare funds. Because the subsidies favor teaching hospitals that have large numbers of residents and treat many Medicare patients, the total number of residency slots in the U.S. has increased dramatically over the last 20 years, although the number of medical school slots has remained more or less constant. The difference has been made up by an influx of doctors trained in medical schools outside the U.S.

The net effect of the subsidy is thus to blur the distinctions between the top and bottom medical school graduates, which would seem to quell salary competition. This issue is not raised in the NRMP lawsuit, maybe because the plaintiffs would rather blame the problems on a computer than on the government.

In the end, mathematics may have some of the answers to the problems of the residents, but it cannot solve all of them. Other real-world situations, however, such as judicial clerkships and high school admissions, could benefit from clever matching algorithms. Stay tuned for discussions of such opportunities, and some of the recent research in economics, in upcoming issues of *SIAM News*.

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