

Partial Fillup and Search Time in LC Tries*

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Abstract

Andersson and Nilsson introduced in 1993 a *level-compressed trie* (in short: LC trie) in which a full subtree of a node is compressed to a single node of degree being the size of the subtree. Recent experimental results indicated a “dramatic improvement” when full subtrees are replaced by “partially filled subtrees”. In this paper, we provide a theoretical justification of these experimental results showing, among others, a rather moderate improvement of the search time over the original LC tries. For such an analysis, we assume that n strings are generated independently by a binary memoryless source (a generalization to Markov sources is possible) with p denoting the probability of emitting a “1” (and $q = 1 - p$). We first prove that the so called α -fillup $F_n(\alpha)$ (i.e., the largest level in a trie with α fraction of nodes present at this level) is concentrated on two values whp (with high probability); either $F_n(\alpha) = k_n$ or $F_n(\alpha) = k_n + 1$ where $k_n = \log_{\frac{1}{\sqrt{pq}}} n - \frac{|\ln(p/q)|}{2 \ln^{3/2}(1/\sqrt{pq})} \Phi^{-1}(\alpha) \sqrt{\ln n} + O(1)$ is an integer and $\Phi(x)$ denotes the normal distribution function. This result directly yields the typical depth (search time) $D_n(\alpha)$ in the α -LC tries with $p \neq 1/2$, namely we show that whp $D_n(\alpha) \approx C_1 \log \log n$ where $C_1 = 1/|\log(1 - h/\log(1/\sqrt{pq}))|$ and $h = -p \log p - q \log q$ is the Shannon entropy rate. This should be compared with recently found typical depth in the original LC tries which is $C_2 \log \log n$ where $C_2 = 1/|\log(1 - h/\log(1/\min\{p, 1-p\}))|$. In conclusion, we observe that α affects only the lower term of the α -fillup level $F_n(\alpha)$, and the search time in α -LC tries is of the same order as in the original LC tries.

1 Introduction

Tries and suffix trees are the most popular data structures on words [7]. A *trie* is a digital tree built over, say n , strings (the reader is referred to [12, 14, 24] for an in

depth discussion of digital trees.) A string is stored in an external node of a trie and the path length to such a node is the shortest prefix of the string that is not a prefix of any other strings (cf. Figure 1). Throughout, we assume a binary alphabet. Then each branching node in a trie is a binary node. A special case of a trie structure is a *suffix trie* (tree) which is a trie built over suffixes of a *single* string.

Since 1960 tries were used in many computer science applications such as searching and sorting, dynamic hashing, conflict resolution algorithms, leader election algorithms, IP addresses lookup, coding, polynomial factorization, Lempel-Ziv compression schemes, and molecular biology. For example, in the internet IP addresses lookup problem [15, 22] one needs a fast algorithm that directs an incoming packet with a given IP address to its destination. As a matter of fact, this is the *longest matching prefix* problem, and standard tries are well suited for it. However, the search time is too large. If there are n IP addresses in the database, the search time is $O(\log n)$, and this is not acceptable. In order to improve the search time, Nilsson [15] introduced a novel data structure called the *level compressed trie* or in short LC trie (cf. Figure 1). In the LC trie we replace the root with a node of degree equal to the size of the largest *full subtree* emanating from the root (the depth of such a subtree is called the *fillup level*). This is further carried on recursively throughout the whole trie.

Some recent experimental results reported in [8, 18, 17] indicated a “dramatic improvement” in the search time when full subtrees are replaced by “partially fillup subtrees”. In this paper, we provide a theoretical justification of these experimental results by considering α -LC tries in which one replaces a subtree with the last level only $\alpha\%$ filled by a node of degree equal to the size of such a subtree (and we continue recursively). In order to understand theoretically the α -LC trie behavior, we study here the so called α -fillup level $F_n(\alpha)$ and the *typical depth* or the search time $D_n(\alpha)$. The α -fillup level is the last level in a trie that is $\alpha\%$ filled up (e.g., in a binary trie level k is $\alpha\%$ filled if it contains $\alpha 2^k$ nodes). The typical depth is the length of a path from the root to a randomly selected external node; thus it represents the typical search time.

*The work was supported by NSF Grants CCR-0208709 and DMS-02-02815, NIH Grant R01 GM068959-01, and NSA Grant MDA 904-03-1-0036

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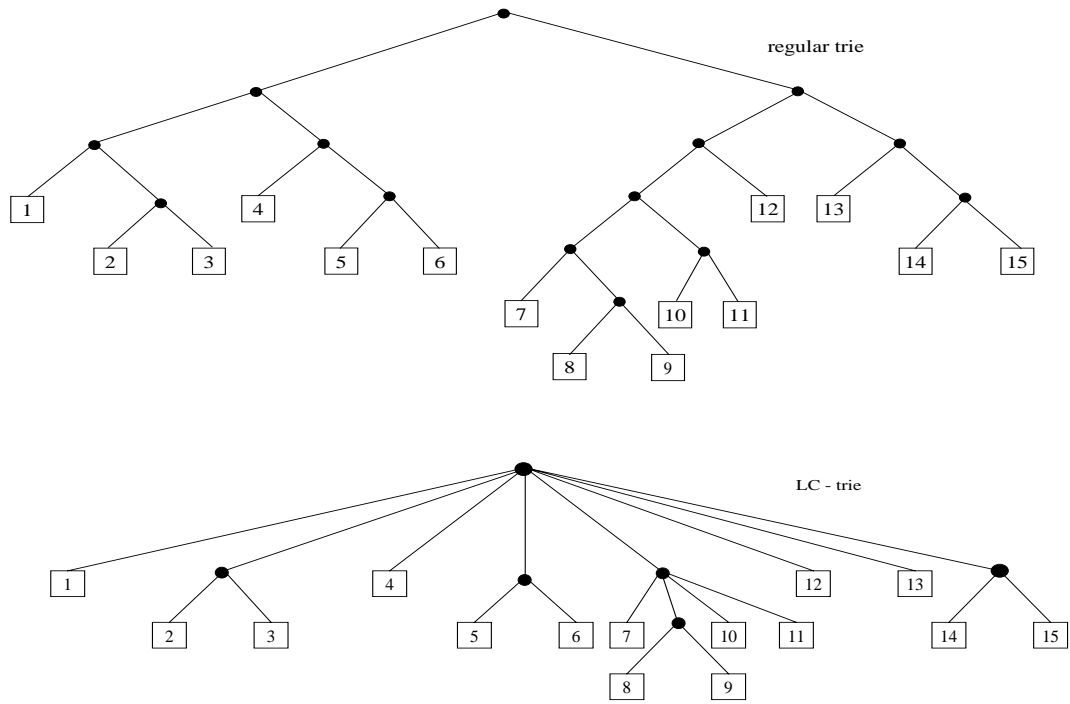


Figure 1: A trie and its associated full LC trie.

In this paper we analyze the α -fillup and the typical depth in an α -LC trie in a probabilistic framework when all strings are generated by a memoryless source with $\mathbb{P}(1) = p$ and $\mathbb{P}(0) = q := 1 - p$. Among other results, we prove that the α -LC trie shows a rather moderate improvement over the original LC tries. We shall quantify this statement below.

Tries were analyzed over the last thirty years for memoryless and Markov sources (cf. [2, 9, 11, 12, 14, 19, 20, 23, 24]). Pittel [19, 20] found the typical value of the fillup level F_n (i.e., $\alpha = 1$) in a trie built over n strings generated by mixing sources; for memoryless sources with high probability (whp)

$$F_n \stackrel{p}{\sim} \frac{\log n}{\log(1/p_{\min})} = \frac{\log n}{h_{-\infty}}$$

where $p_{\min} = \min\{p, 1 - p\}$ is the smallest probability of generating a symbol and $h_{-\infty} = \log(1/p_{\min})$ is the Rényi entropy of infinite order (cf. [24]). We let $\log := \log_2$.

This was further extended by Devroye [2], and Knessl and Szpankowski [11] who, among other results, proved that the fillup F_n is concentrated on two points k_n and $k_n + 1$, where k_n is an integer

$$(1.1) \quad \frac{1}{\log p_{\min}^{-1}} (\log n - \log \log \log n) + O(1)$$

for $p \neq 1/2$. The depth in regular tries was analyzed by many authors who proved that whp the depth is about $(1/h) \log n$ (where $h = -p \log p - (1 - p) \log(1 - p)$ is the Shannon entropy rate of the source) and that it is normally distributed when $p \neq 1/2$ [20, 24].

The original LC tries were analyzed by Andersson and Nilsson [1] for unbiased memoryless source and by Devroye [3] for memoryless sources (cf. also [21]). The typical depth (search time) for regular LC tries was only studied recently by Devroye and Szpankowski [4] who proved that for memoryless sources with $p \neq 1/2$

$$D_n \stackrel{p}{\sim} \frac{\log \log n}{-\log(1 - h/h_{-\infty})}.$$

In this paper we shall prove some rather surprising results. First of all, for $0 < \alpha < 1$ we show that the α -fillup $F_n(\alpha)$ is whp equal either to k_n or $k_n + 1$ where

$$(1.2) \quad k_n = \log_{\frac{1}{\sqrt{pq}}} n - \frac{|\ln(p/q)|}{2 \ln^{3/2}(1/\sqrt{pq})} \Phi^{-1}(\alpha) \sqrt{\ln n} + O(1).$$

As a consequence, we find that if $p \neq 1/2$, the depth $D_n(\alpha)$ of the α -LC is for large n typically about

$$\frac{\log \log n}{-\log(1 - h/\log(1/\sqrt{pq}))}.$$

The (full) 1-fillup F_n shown in (1.1) should be compared to the α -fillup $F_n(\alpha)$ presented in (1.2). Observe that the leading term of $F_n(\alpha)$ is *not* the same as the leading term of F_n when $p \neq 1/2$. Furthermore, α contributes only to the second term asymptotics. When comparing the typical depths D_n and $D_n(\alpha)$ we conclude that both grow like $\log \log n$ with two constants that do not differ by much. This comparison led us to a statement in the abstract that the improvement of α -LC tries over the regular LC tries is rather moderate. We may add that for relatively slowly growing functions such as $\log \log n$ the constants in front of them do matter (even for large values of n) and perhaps this led the authors of [8, 17, 18] to their statements.

The paper is organized as follows. In the next section we present our main results which are proved in the next two sections. We first consider a poissonized version of the problem for which we establish our findings. Then we show how to depoissonize our results completing our proof.

2 Main Results

Consider tries created by inserting n random strings of 0 and 1. We will always assume that the strings are (potentially) infinite and that the bits in the strings are independent random bits, with $\mathbb{P}(1) = p$ and thus $\mathbb{P}(0) = q := 1 - p$; moreover we assume that different strings are independent.

We let $X_k := \#\{\text{internal nodes filled at level } k\}$ and $\bar{X}_k := X_k/2^k$, i.e. the proportion of nodes filled at level k . Note that X_k may both increase and decrease as k grows, while

$$1 \geq \bar{X}_k \geq \bar{X}_{k+1} \geq 0.$$

Recall that the fillup level of the trie is defined as the last full level, i.e. $\max\{k : \bar{X}_k = 1\}$, (and for example the height is the last level with any nodes at all, i.e. $\max\{k : \bar{X}_k > 0\}$). Similarly, if $0 < \alpha \leq 1$, the α -fillup level is the last level where at least a proportion α of the nodes are filled, i.e. $\max\{k : \bar{X}_k \geq \alpha\}$.

We will in this paper study the α -fillup level for a given α with $0 < \alpha < 1$ and a given p with $0 < p < 1$.

We have the following result, where whp means with probability tending to 1 as $n \rightarrow \infty$, and Φ denotes the normal distribution function. Theorem 2.1 is proved in Section 4, after first considering a Poissonized version in Section 3.

THEOREM 2.1. *Let α and p be fixed with $0 < \alpha < 1$ and $0 < p < 1$, and let $F_n(\alpha)$ be the α -fillup level for the trie formed by n random strings as above. Then, for each n*

there is an integer

$$k_n = \log_{\frac{1}{\sqrt{pq}}} n - \frac{|\ln(p/q)|}{2 \ln^{3/2}(1/\sqrt{pq})} \Phi^{-1}(\alpha) \sqrt{\ln n} + O(1)$$

such that whp $F_n(\alpha) = k_n$ or $k_n + 1$. Moreover, $\mathbf{E} \bar{X}_{k_n} = \alpha + O(1/\sqrt{\log n})$ for $p \neq 1/2$.

Thus the α -fillup $F_n(\alpha)$ is concentrated on at most two values; as in many similar situations (cf. [11]), it is easily seen from the proof that in fact for most n it is concentrated on a single value k_n , but there are transitional regimes, close to the values of n where k_n changes, where $F_n(\alpha)$ takes two values with comparable probabilities.

Note that when $p = 1/2$, the second term on the right hand side disappears, and thus simply $k_n = \log n + O(1)$; in particular, two different values of $\alpha \in (0,1)$ have their corresponding k_n differing by $O(1)$ only. When $p \neq 1/2$, changing α means shifting k_n by $\Theta(\log^{1/2} n)$. By Theorem 2.1, whp $F_n(\alpha)$ is shifted by the same amounts.

To the first order, we thus have the following simple result.

COROLLARY 2.1. *For any fixed α and p with $0 < \alpha < 1$ and $0 < p < 1$,*

$$F_n(\alpha) = \log_{\frac{1}{\sqrt{pq}}} n + O_p(\sqrt{\ln n});$$

in particular, $F_n(\alpha) / \log_{1/\sqrt{pq}} n \xrightarrow{P} 1$ as $n \rightarrow \infty$.

Surprisingly enough, the fill up level for $\alpha = 1$ and $\alpha < 1$ are quantitatively different for $p \neq 1/2$. It is well known, as explained in the introduction, that the regular fillup F_n is concentrated on two points around $\log n / \log(1/p_{\min})$, while the partial fillup $F_n(\alpha)$ concentrates around $k_n \sim \log n / \log(1/\sqrt{pq})$. Secondly, the leading term of $F_n(\alpha)$ does not depend on α and the second term is proportional to $\sqrt{\log n}$, while for the regular fillup F_n the second term is of order $\log \log n$.

Theorem 2.1 yields several consequences for the behavior of α -LC tries. In particular, it implies the typical behavior of the depth, that is, the search time. Below we formulate our main second result concerning the depth for α -LC tries delaying the formal proof (that will follow the footsteps of [4]) till the final (journal) version of the paper. However, after the statement of theorem we provide a brief heuristics justification.

THEOREM 2.2. *For any fixed $0 < \alpha < 1$ and $p \neq 1/2$ we have*

$$(2.3) \quad D_n(\alpha) \stackrel{P}{\approx} \frac{\log \log n}{-\log \left(1 - \frac{h}{\log(1/\sqrt{pq})} \right)}$$

as $n \rightarrow \infty$ where $h = -p \log p - (1-p) \log(1-p)$ is the entropy rate of the source.

First, let us explain heuristically our estimate for $D_n(\alpha)$. By the Asymptotic Equipartition Property (cf. [24]) at level k_n there are about $n2^{-hk_n}$ strings where h is the entropy. That is, $n2^{-hk_n} \approx n^{1-h/b}$ where for simplicity $b = \log(1/\sqrt{pq})$. In the next level, we shall have about $n^{(1-h/b)^2}$ nodes, and so on. In particular, at level $D_n(\alpha)$ we have approximately

$$n^{(1-h/b)^{D_n(\alpha)}} = O(1)$$

nodes. This leads to our estimate (2.3) of Theorem 2.2.

As a direct consequence of Theorem 2.2 we can numerically quantify experimental results observed by Nilsson and Karlsson who reported in [18] a ‘‘dramatic improvement’’ in the search time of α -LC tries over the regular LC tries. In a regular LC trie the search time is $O(\log \log n)$ with the constant in front of $\log \log n$ being $1/\log(1-h/\log(1/p_{\min}))^{-1}$ [4]. For α -LC tries this constant decreases to $1/\log(1-h/\log(1/\sqrt{pq}))^{-1}$. While it is hardly a ‘‘dramatic improvement’’, the fact that we deal with a slowly growing leading term $\log \log n$, may indeed lead to experimentally observed significant changes in the search time.

3 Poissonization

In this section we consider a Poissonized version of the problem, where there are $\text{Po}(\lambda)$ strings inserted in the trie. We let $\tilde{F}_\lambda(\alpha)$ denote the α -fillup level of this trie.

THEOREM 3.1. *Let α and p be fixed with $0 < \alpha < 1$ and $0 < p < 1$, and let $\tilde{F}_\lambda(\alpha)$ be the α -fillup level for the trie formed by $\text{Po}(\lambda)$ random strings as above. Then, for each $\lambda > 0$ there is an integer*

$$(3.4) \quad k_\lambda = \log_{\frac{1}{\sqrt{pq}}} \lambda - \frac{|\ln(p/q)|}{2 \ln^{3/2}(1/\sqrt{pq})} \Phi^{-1}(\alpha) \sqrt{\ln \lambda} + O(1)$$

such that whp (as $\lambda \rightarrow \infty$) $\tilde{F}_\lambda(\alpha) = k_\lambda$ or $k_\lambda + 1$.

We shall prove Theorem 3.1 through a series of lemmas. Observe first that a node at level k can be labelled by a binary string of length k , and that the node is filled if and only if at least two of the inserted strings begin with this label. For $r \in \{0,1\}^k$, let $N_1(r)$ be the number of ones in r , and let $P(r) = p^{N_1(r)} q^{k-N_1(r)}$ be the probability that a random string begins with r . Then, in the Poissonized version, the number of inserted strings beginning with $r \in \{0,1\}^k$ has a Poisson distribution $\text{Po}(\lambda P(r))$, and these numbers are independent for different strings r of the same length.

Consequently,

$$(3.5) \quad X_k = \sum_{r \in \{0,1\}^k} I_r$$

where I_r are independent indicators with

$$(3.6) \quad \mathbb{P}(I_r = 1) = \mathbb{P}(\text{Po}(\lambda P(r)) \geq 2) = 1 - (1 + \lambda P(r))e^{-\lambda P(r)}.$$

Hence,

$$\mathbf{Var}(X_k) = \sum_{r \in \{0,1\}^k} P(I_r = 1)(1 - P(I_r = 1)) < 2^k$$

so $\mathbf{Var}(\bar{X}_k) < 2^{-k}$ and, by Chebyshev's inequality,

$$(3.7) \quad \mathbb{P}(|\bar{X}_k - \mathbf{E}\bar{X}_k| > 2^{-k/3}) \rightarrow 0.$$

Consequently, \bar{X}_k is sharply concentrated, and it is enough to study its expectation. (It is straightforward to calculate $\mathbf{Var}(X_k)$ more precisely, and to obtain a normal limit theorem for X_k , but we do not need that.)

Assume first $p > 1/2$.

LEMMA 3.1. *If $p > 1/2$ and*

$$(3.8) \quad k = \log_{\frac{1}{\sqrt{pq}}} \lambda - \frac{\ln(p/q)}{2 \ln^{3/2}(1/\sqrt{pq})} \Phi^{-1}(\alpha) \sqrt{\ln \lambda} + O(1),$$

then $\mathbf{E}\bar{X}_k = \alpha + O(k^{-1/2})$.

Proof. Let $\rho = p/q > 1$ and define γ by $\lambda p^\gamma q^{k-\gamma} = 1$, i.e.,

$$\rho^\gamma = \left(\frac{p}{q}\right)^\gamma = \lambda^{-1} q^{-k},$$

which leads to

$$(3.9) \quad \gamma = \frac{k \ln(1/q) - \ln \lambda}{\ln(p/q)}.$$

Let $\mu_j = \lambda p^j q^{k-j} = \rho^{j-\gamma}$. By (3.5) and (3.6),

$$(3.10) \quad \mathbf{E}\bar{X}_k = 2^{-k} \sum_{j=0}^k \binom{k}{j} \mathbb{P}(\text{Po}(\mu_j) \geq 2).$$

If $j < \gamma$, then $\mu_j < 1$ and

$$\mathbb{P}(\text{Po}(\mu_j) \geq 2) < \mu_j^2 < \mu_j.$$

If $j \geq \gamma$, then $\mu_j \geq 1$ and

$$1 - \mathbb{P}(\text{Po}(\mu_j) \geq 2) = (1 + \mu_j)e^{-\mu_j} \leq 2\mu_j e^{-\mu_j} < 4\mu_j^{-1}.$$

Hence (3.10) yields, using $\binom{k}{j} \leq \binom{k}{\lfloor k/2 \rfloor} = O(2^k k^{-1/2})$,

$$(3.11) \quad \begin{aligned} \mathbf{E}\bar{X}_k &= 2^{-k} \sum_{j < \gamma} \binom{k}{j} O(\mu_j) + 2^{-k} \sum_{j \geq \gamma} \binom{k}{j} (1 - O(\mu_j^{-1})) \\ &= 2^{-k} \sum_{j \geq \gamma} \binom{k}{j} + 2^{-k} \sum_{j=0}^k \binom{k}{j} O(\rho^{-|j-\gamma|}) \\ &= \mathbb{P}(\text{Bi}(k, 1/2) \geq \gamma) + O(k^{-1/2}). \end{aligned}$$

By the Berry–Esseen theorem [6, Theorem XVI.5.1],

$$(3.12) \quad \mathbb{P}(\text{Bi}(k, 1/2) \geq \gamma) = 1 - \Phi\left(\frac{\gamma - k/2}{\sqrt{k/4}}\right) + O(k^{-1/2}).$$

By (3.9) and the assumption (3.8),

$$(3.13) \quad \begin{aligned} \gamma - \frac{k}{2} &= \frac{1}{\ln(p/q)} \left(k \ln \frac{1}{q} - \ln \lambda - \frac{k}{2} \ln \frac{p}{q} \right) \\ &= \frac{1}{\ln(p/q)} \left(k \ln \frac{1}{\sqrt{pq}} - \ln \lambda \right) \\ &= \frac{\ln(1/\sqrt{pq})}{\ln(p/q)} \left(k - \log_{1/\sqrt{pq}} \lambda \right) \\ &= -\frac{1}{2} (\ln(1/\sqrt{pq}))^{-1/2} \Phi^{-1}(\alpha) \sqrt{\ln \lambda} + O(1) \\ &= -\frac{1}{2} \Phi^{-1}(\alpha) k^{1/2} + O(1). \end{aligned}$$

This finally implies

$$\begin{aligned} 1 - \Phi\left(\frac{\gamma - k/2}{\sqrt{k/4}}\right) &= 1 - \Phi(-\Phi^{-1}(\alpha)) + O(k^{-1/2}) = \\ &= \alpha + O(k^{-1/2}), \end{aligned}$$

and the lemma follows by (3.11) and (3.12).

LEMMA 3.2. *Fix $p > 1/2$. For every $A > 0$, there exists $c > 0$ such that if $|k - \log_{1/\sqrt{pq}} \lambda| \leq Ak^{1/2}$, then $\mathbf{E}\bar{X}_k - \mathbf{E}\bar{X}_{k+1} > ck^{-1/2}$.*

Proof. A string $r \in \{0,1\}^k$ has two extensions $r0$ and $r1$ in $\{0,1\}^{k+1}$. Clearly, $I_{r0}, I_{r1} \leq I_r$, and if there are exactly 2 (or 3) of the inserted strings beginning with r , then $I_{r0} + I_{r1} \leq 1 < 2I_r$. Hence

$$(3.14) \quad \begin{aligned} \mathbf{E}(2X_k - X_{k+1}) &= \sum_{r \in \{0,1\}^k} \mathbf{E}(2I_r - I_{r0} - I_{r1}) \\ &\geq \sum_{r \in \{0,1\}^k} \mathbb{P}(\text{Po}(\lambda P(r)) = 2). \end{aligned}$$

Let ρ and γ be as in the proof of Lemma 3.1, and let $j = \lceil \gamma \rceil$. Then $\mu_j = \rho^{j-\gamma} \in [1, \rho]$ and thus

$\mathbb{P}(\text{Po}(\mu_j) = 2) \geq \frac{1}{2}e^{-\rho}$. Moreover, by (3.13) and the assumption,

$$|j - k/2| \leq \frac{\ln(1/\sqrt{pq})}{\ln(p/q)} Ak^{1/2} + 1 = O(k^{1/2}).$$

Thus, if k is large enough, we have by the standard normal approximation of the binomial probabilities (which follows easily from Stirling's formula, as found already by de Moivre [5])

$$2^{-k} \binom{k}{j} = \frac{1 + o(1)}{\sqrt{2\pi k/4}} e^{-2(j-k/2)^2/k} \geq c_1 k^{-1/2}$$

for some $c_1 > 0$. Hence, by (3.14),

$$\begin{aligned} \mathbf{E} \bar{X}_k - \mathbf{E} \bar{X}_{k+1} &= 2^{-k-1} \mathbf{E} (2X_k - X_{k+1}) \\ &\geq 2^{-k-1} \binom{k}{j} \mathbb{P}(\text{Po}(\mu_j) = 2) \\ &\geq \frac{c_1 e^{-\rho}}{4} k^{-1/2} \end{aligned}$$

as needed.

Now assume $p > 1/2$. Starting with any k as in (3.8), we can by Lemmas 3.1 and 3.2 shift k up or down $O(1)$ steps and find k_λ as in (3.4) such that, for a suitable $c > 0$, $\mathbf{E} \bar{X}_{k_\lambda} \geq \alpha + \frac{1}{2}ck_\lambda^{-1/2} > \mathbf{E} \bar{X}_{k_\lambda+1}$ and $\mathbf{E} \bar{X}_{k_\lambda+2} \leq \mathbf{E} \bar{X}_{k_\lambda+1} - ck_\lambda^{-1/2} < \alpha - \frac{1}{2}ck_\lambda^{-1/2}$. It follows by (3.7) that whp $\bar{X}_{k_\lambda} \geq \alpha$ and $\bar{X}_{k_\lambda+2} < \alpha$, and hence $\tilde{F}_\lambda(\alpha) = k_\lambda$ or $k_\lambda + 1$.

This proves Theorem 3.1 in the case $p > 1/2$. The case $p < 1/2$ follows by symmetry, interchanging p and q .

In the remaining case $p = 1/2$, all $P(r) = 2^{-k}$ are equal. Thus, by (3.5) and (3.6),

$$(3.15) \quad \mathbf{E} \bar{X}_k = \mathbb{P}(\text{Po}(\lambda 2^{-k}) \geq 2).$$

Given $\alpha \in (0, 1)$, there is a $\mu > 0$ such that $\mathbb{P}(\text{Po}(\mu) \geq 2) = \alpha$. We take $k_\lambda = \lfloor \log(\lambda/\mu) - 1/2 \rfloor$. Then, $\lambda 2^{-k_\lambda} \geq 2^{1/2}\mu$ and thus $\mathbf{E} \bar{X}_{k_\lambda} \geq \alpha_+$ for some $\alpha_+ > \alpha$. Similarly, $\mathbf{E} \bar{X}_{k_\lambda+2} \leq \alpha_-$ for some $\alpha_- < \alpha$, and the result follows in this case too.

4 Depoissonization

To complete the proof of Theorem 2.1 we must depoissonize the results obtained in Theorem 3.1, which we do in this section.

Proof. [Proof of Theorem 2.1] Given an integer n , let k_n be as in the proof of Theorem 3.1 with $\lambda = n$, and let $\lambda_\pm = n \pm n^{2/3}$. Then $\mathbb{P}(\text{Po}(\lambda_-) \leq n) \rightarrow 1$ and $\mathbb{P}(\text{Po}(\lambda_+) \geq n) \rightarrow 1$ as $n \rightarrow \infty$. By monotonicity,

we thus have whp $\tilde{F}_{\lambda_-}(\alpha) \leq F_n(\alpha) \leq \tilde{F}_{\lambda_+}(\alpha)$, and by Theorem 3.1 it remains only to show that we can take $k_{\lambda_-} = k_{\lambda_+} = k_n$.

Let us now write $X_k(\lambda)$ and $\bar{X}_k(\lambda)$, since we are working with several λ .

LEMMA 4.1. *Assume $p \neq 1/2$. Then, for every k ,*

$$\frac{d}{d\lambda} \mathbf{E} \bar{X}_k(\lambda) = O(\lambda^{-1} k^{-1/2}).$$

Proof. We have

$$\frac{d}{d\mu} \mathbb{P}(\text{Po}(\mu) \geq 2) = \frac{d}{d\mu} ((1 - (1 + \mu)e^{-\mu}) = \mu e^{-\mu}$$

and thus, by (3.10) and the argument in (3.11),

$$\begin{aligned} \frac{d}{d\lambda} \mathbf{E} \bar{X}_k(\lambda) &= 2^{-k} \sum_{j=0}^k \binom{k}{j} \mu_j e^{-\mu_j} \frac{d\mu_j}{d\lambda} \\ &= \lambda^{-1} 2^{-k} \sum_{j=0}^k \binom{k}{j} \mu_j^2 e^{-\mu_j} \\ &= O\left(\lambda^{-1} \sum_{j=0}^k 2^{-k} \binom{k}{j} \min(\mu_j, \mu_j^{-1})\right) \\ &= O(\lambda^{-1} k^{-1/2}) \end{aligned}$$

which completes the proof.

By Lemma 4.1, $|\mathbf{E} \bar{X}_k(\lambda_\pm) - \mathbf{E} \bar{X}_k(n)| = O(n^{-1/3} k^{-1/2}) = o(k^{-1/2})$. Hence, by the proof of Theorem 3.1, for large n , $\mathbf{E} \bar{X}_{k_n}(\lambda_\pm) \geq \alpha + \frac{1}{3}ck_n^{-1/2}$ and $\mathbf{E} \bar{X}_{k_n+2}(\lambda_\pm) < \alpha - \frac{1}{3}ck_n^{-1/2}$, and thus whp $\tilde{F}_{\lambda_\pm}(\alpha) = k_n$ or $k_n + 1$. Moreover, the estimate $\mathbf{E} \bar{X}_{k_n} = \alpha + O(1/\sqrt{\log n})$ follows easily from the similar estimate for the Poisson version in Lemma 3.1; we omit the details. This completes the proof of Theorem 2.1 for $p > 1/2$. The case $p < 1/2$ is again the same by symmetry. The proof when $p = 1/2$ is similar, now using (3.15).

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