The Asymptotic Number of Spanning Trees in Circulant Graphs (Extended Abstract)*

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Abstract

Let $T(G)$ be the number of spanning trees in graph G. In this note we explore the asymptotics of $T(G)$ for circulant graphs.

The circulant graph $C_n^{s_1,s_2,\cdots,s_k}$ is the $2k$ regular graph with n vertices labelled $0,1,2,\cdots,n-1$, where node i has the $2k$ neighbors, $(0 \le i \le n-1)$ adjacent to vertices $i+s_1, i+s_2, \cdots, i+s_k \bmod n$. In this note we give a closed formula for the asymptotic limit $\lim_{n\to\infty} T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}}$ as a function of $s_1, s_2, \ldots, s_n$. We then extend this by permitting the s_i to be linear functions of n, i.e., we give a closed formula for

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k,\lfloor \frac{n}{d_1}\rfloor+e_1,\lfloor \frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor \frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}}$$

where the d_i and e_i are arbitrary integers. One consequence of our derivation is that if we let the s_i go to infinity then

$$\lim_{s_1,s_2,\cdots,s_k\to\infty} \lim_{m\to\infty} T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}} = 4^k \exp\left[\int_0^1 \cdots \int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi x_i\right) dx_1 dx_2 \cdots x_k\right].$$

Interestingly, this value is same as the asymptotic number of spanning trees in the k-dimensional square lattice obtained in Garcia, Noy and Tejel in [7].

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1 Introduction

Let G and D denote, respectively, a (undirected) graph and a digraph (directed graph). Throughout this note, we allow graphs and digraphs to contain multiple edges (arcs) and self-loops unless otherwise specified. *A spanning tree* in G is a tree having the same vertex set as G. *An oriented spanning tree* in a digraph D is a rooted tree with the same vertex set as D, that is, there is a node specified as the root and from it there is a path to any vertex of D. The study of the number of (oriented) spanning trees in a (digraph) graph has a very long history. The evaluation of this number is not only interesting from a combinatorial perspective but also arises in practical applications, e.g., analyzing the reliability of a network and designing electrical circuits etc. ([6]). Given the adjacency matrix of G (D) *Kirchoff's matrix tree theorem* ([11]) gives a closed formula for calculating $T(G)$. The real problem, then, is to calculate the number of spanning trees of graphs in particular parameterized classes, as a function of the parameters. A well studied class, which we will be further analyzing in this paper, is the *Circulant Graphs.*

We start by formally defining the graphs and the values to be counted. Let $s_1, s_2, \cdots, s_k$ be positive integers. The circulant graph with n vertices and jumps $s_1, s_2, \ldots, s_k$ is defined by

$$C_n^{s_1,s_2,\cdots,s_k} = (V, E)$$

where

$$V = \{0, 1, 2, \cdots, n-1\}$$

and

$$E = \bigcup_{i=0}^{n-1} \{ (i, i \pm s_1), (i, i \pm s_2), \ldots (i, i \pm s_k) \}$$

where all of the additions are done $\pmod n$. That is, each node is connected to the nodes that are jumps $\pm s_j$ away from it, for $j = 1, 2, \ldots, k$. Similarly the directed circulant graph, $\vec{C}_n^{s_1, s_2, \cdots, s_k}$, has the same edge set but

$$E = \bigcup_{i=0}^{n-1} \{ (i, i + s_1), (i, i + s_2), \ldots (i, i + s_k) \}$$

i.e, there is an edge directed from each i to the nodes s_j ahead of it, for $j = 1, 2, \ldots, k$. We emphasize that, since we are allowing multiple edges in our graphs, $C_n^{s_1, s_2, \cdots, s_k}$ is *always* a $2k$-regular graph and $\vec{C}_n^{s_1, s_2, \cdots, s_k}$ is always k-regular one. Figure 1 gives examples of four circulant graphs.

Throughout this note we will use $T(G)$ to denote the number of spanning trees in a directed or undirected graph G. For those notations and terminologies not defined here, we refer to [17] and [19]. It was shown in [17] that, for *directed circulant graphs*,

$$\lim_{n \to \infty} \frac{T(\vec{C}_{n+1}^{s_1, s_2, \cdots, s_k})}{T(\vec{C}_n^{s_1, s_2, \cdots, s_k})} = k$$

where k is the degree of the vertices of $\vec{C}_{n+1}^{s_1, s_2, \cdots, s_k}$. One might hope that similar asymptotic behavior, i.e., a limit dependent only upon k but independent of the actual values of the s_i, would also be true for *undirected circulant graphs*. Unfortunately, as seen in the asymptotic (numerical) results presented in Table 1 of [19], this is not the case; the asymptotic limits do seem somehow dependent on the s_i. We therefore, in that paper, posed as an open question the analysis of the asymptotics as a function of the s_i.

The problem of calculating the asymptotic *maximum* number of spanning trees in a circulant graph with k jumps was treated in [13], but their technique doesn't seem to permit analyzing the number of spanning trees for any *given* fixed jumps. Asymptotic limits for grids and tori (which turn out to be equal) were obtained in [5], [7]. More recently, while examining the structure of non-constant jump circulant graphs, [9] conjectured that the asymptotics of the number of spanning trees of the $m \times n$ tori and grids and the circulant graphs $C_{mn}^{1,n}$ would be equivalent.

The main result of this paper is the derivation in Section 2 of closed formulas for the first order asymptotics of the number of spanning trees in circulant graphs, both for fixed jump circulants and linear-jump ones (in which the jump sizes can depend linearly upon n).[1] A secondary result is that, as described in Section 3, this will permit us to show that the k-dimensional tori and the circulant graphs $C_n^{s_1, s_2, \cdots, s_k}$ have the same asymptotic value when $n, s_1, s_2, \cdots, s_k$ tend to infinity.

We note that, a-priori, it is slightly surprising that the limit

$$\lim_{n \to \infty} T \left(C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l} \right)^{\frac{1}{n}}$$

exists. Consider, for example, the simple case of $C_n^{1, \lfloor \frac{n}{3} \rfloor}$ and separate out the cases where $n = 3k$ for some k and where $n \neq 3k$. If $n = 3k$ then $C_n^{1, \lfloor \frac{n}{3} \rfloor}$ is the union of $k + 1$ disjoint cycles; one of size $3k = n$ and $k = n/3$ of size 3. On the other hand, if $n = 3k + 1$ then $C_n^{1, \lfloor \frac{n}{3} \rfloor}$ is the union of exactly two disjoint cycles, each of size n. Yet, in the limit, the result in the next section implies that the number of spanning trees for these two cases is the same.

Most studies of the number of spanning trees in circulants start with the following facts. It is known ([6], [12]) that the formulas for the number of spanning trees in a d-regular graph G and the number of oriented spanning trees in a d-regular digraph D can be expressed, respectively, as

$$(1.1)\quad T(G) = \frac{1}{n} \prod_{j=1}^{n-1} (d - \lambda_j), \qquad T(D) = \prod_{j=1}^{n-1} (d - \lambda_j),$$

where $\lambda_0 = d, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. Because the adjacency matrices are circulant, from [2] we have

$$\lambda_j = \varepsilon^{s_1 j} + \varepsilon^{s_2 j} + \cdots + \varepsilon^{s_k j}, \qquad j = 0,\ 1,\ \cdots,\ n-1,$$

where $\varepsilon = e^{\frac{2\pi\sqrt{-1}}{n}}$. These facts directly imply the *known* results:

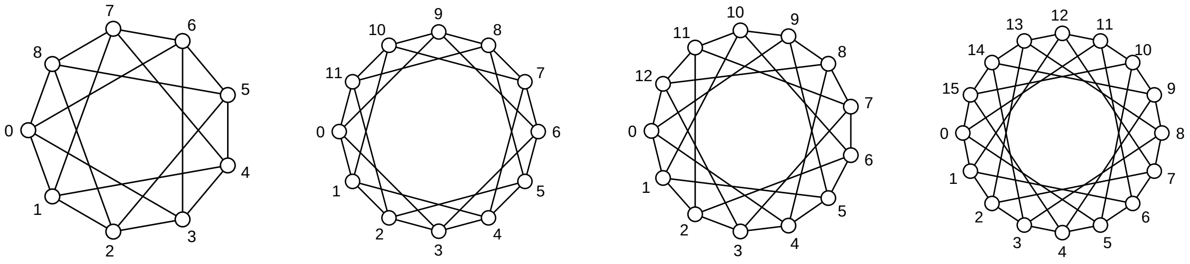
$$(1.2)\quad T(C_n^{s_1, s_2, \cdots, s_k}) = \frac{1}{n} \prod_{j=1}^{n-1} (2k - 2 \sum_{i=1}^{k} cos \frac{2\pi s_i j}{n})$$

and

$$(1.3)\qquad T(\vec{C}_n^{s_1, s_2, \cdots, s_k}) = \prod_{j=1}^{n-1} (k - \sum_{i=1}^{k} \varepsilon^{s_i j}).$$

Starting from this, [15] and [19] proved

[1] We note that, recently, Lyons [14] has developed general techniques for deriving the asymptotics of the spanning trees of large graphs. His techniques can be used to derive the asymptotics of fixed-jump circulants (our Theorem 2) but do not seem to be usable to derive the asymptotics when the jumps are not fixed.



(a) $C_9^{1,3}$ (b) $C_{12}^{1,3}$ (c) $C_{13}^{1,4}$ (d) $C_{16}^{1,4}$

Figure 1: 4 circulant graphs. (b) and (d) are $C_{2n+1}^{1,4n+8}$ for $n = 1, 2$. (a) and (c) are $C_{4n+1}^{1,n+1}$ for $n = 2, 3$.

LEMMA 1. *For any* $1 \le s_1 < s_2 < \cdots < s_k$,

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2, \qquad T(\vec{C}_n^{s_1,s_2,\cdots,s_k}) = na'_n$$

where a_n *and* a'_n *satisfy recurrence relations for* $\forall n \ge 2^{s_k-1}$ *and the initial values can be calculated directly by the Matrix Tree Theorem ([19]). Furthermore, the largest eigenvalues (in modulus) of* a_n *and* a'_n *are unique.*

(This lemma is actually a combination of Lemma 4, the "Note" following that Lemma and Lemma 5 from [19].)

Equation (1.2) and the formulas in Lemma 1 will be crucial for our later analyses.

The number of spanning trees in *non-fixed jump* circulant graphs, $T\left(C_{pq+r}^{a_1n+b_1,a_2n+b_2,\ldots,a_kn+b_k}\right)$ was later also shown ([10]) to satisfy a linear fixed order recurrence relation but no theorem as strong as Lemma 1 is known for the non-fixed jump cases

Lemma 1 actually provides an *algorithmic* way of determining the asymptotics of

$$\begin{aligned}\lim_{n\to\infty} T(C_n^{s_1,s_2,\cdots,s_k})^{1/n} &= \lim_{n\to\infty}\left(na_n^2\right)^{1/n} \\ &= \lim_{n\to\infty}\left(a_n^2\right)^{1/n}.\end{aligned}$$

For any given $s_1, s_2, \ldots, s_n$ just algorithmically derive the coefficients of the recurrence relation and then solve for a_n. Similarly, in the non-fixed jump case, use the results in [10] to derive the recurrence relation, solve to get a formula for the number of spanning trees and then calculate the asymptotics from that.

As a simple example, consider the *square cycle* $C_n^{1,2}$. Using Lemma 1 it is not hard to derive that

$$T(C_n^{1,2}) = nF_n^2,$$

(this was originally conjectured by [1] and [4] and variously proven by [3, 16, 19]) where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $n = 2, 3, \cdots$, is the *Fibonacci* sequence. This implies that

$$\begin{aligned}\lim_{n\to\infty} T(C_n^{1,2})^{1/n} &= \lim_{n\to\infty}\frac{T(C_{n+1}^{1,2})}{T(C_n^{1,2})} \\ &= \lim_{n\to\infty}\frac{F_{n+1}^2}{F_n^2} = \frac{3+\sqrt{5}}{2}.\end{aligned}$$

Note though, that we *did not* calculate $\frac{3+\sqrt{5}}{2}$, by plugging $s_1 = 1$, $s_2 = 2$ into a closed formula. Instead, we essentially used the fact that we knew that $T(C_n^{1,2})$ satisfied a recurrence relation to then derive the recurrence relation and then plugged in the asymptotics of the solution to the recurrence relation. In this paper, we show the existence of a simple formula in the s_i that yields the asymptotics.

2 Spanning Trees in Circulant Graphs

The final goal of this section is to analyze the following quantity,

$$(2.4)\quad \lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_r,\lfloor\frac{n}{d_1}\rfloor+e_1,\lfloor\frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor\frac{n}{d_k}\rfloor+e_k}\right)^{\frac{1}{n}}$$

as a function of nonnegative integers s_i, d_i and e_i. We will do this in stages.

Before starting, we need to note an important caveat, which is that *all limits will be over* **non-zero** *values.* More specifically, note that, if $gcd(n, s_1, \ldots, s_k) > 1$, then $C_n^{s_1,s_2,\cdots,s_k}$ is unconnected so it has no spanning trees. This makes it impossible for us to define a limit. For example, when n is even, $C_n^{2,4}$ partitions into two unconnected components, so no spanning tree exists and $T(C_n^{2,4}) = 0$. On the other hand when n is not even, $T(C_n^{2,4}) > 0$ and we can show the existence of $c > 0$ such that $\lim_{m\to\infty} T(C_{2m+1}^{2,4})^{1/(2m+1)} = c$. Thus, technically, $\lim_{n\to\infty} T(C_n^{2,4})$ doesn't exist. But, as mentioned, we will take all of our limits to be over **non-zero** values, so we will write $\lim_{n\to\infty} T(C_n^{2,4})^{1/n} = c$.

We first start by analyzing (2.4) when all of the jumps are constant, i.e., the $d_i = \infty$, and prove the following lemma. We should point out that, as mentioned, Lyons's [14] recent results imply the following lemma. The reason we are giving an alternative proof of it is that we will apply the same technique later in the paper to derive the formulas for the nonfixed jump circulants cases.

LEMMA 2. *For any fixed integers $1 \le s_1 < s_2 < \cdots < s_k$,*

$$\lim_{n\to\infty} T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{T(C_{n+1}^{s_1,s_2,\cdots,s_k})}{T(C_n^{s_1,s_2,\cdots,s_k})} = 4^k \exp\left(\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_i x\right) dx\right).$$

Proof. We will write T_n for $T(C_n^{s_1,s_2,\cdots,s_k})$. From Lemma 1 we know that $T_n = na_n^2$ where $a_n = \alpha^n(1 + O(\epsilon^n))$ for some $\alpha > 1$ and $\epsilon < 1$. Set $\beta = \alpha^2$. Then

$$T_n = na_n^2 = n\alpha^{2n}\left(1 + O(\epsilon^n)\right)^2 = n\beta^n(1 + O(\epsilon^n))$$

so

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \beta = \lim_{n\to\infty} T_n^{1/n}$$

Now note that, from (1.2) and the fact that $1 - \cos(2x) = 2\sin^2 x$ we get

$$\begin{aligned}
&\lim_{n\to\infty} T_n{}^{\frac{1}{n}} \\
=& \lim_{n\to\infty} \left[\frac{1}{n}\prod_{j=1}^{n-1}\left(2k - 2\sum_{i=1}^k \cos\frac{2\pi s_i j}{n}\right)\right]^{\frac{1}{n}} \\
=& \lim_{n\to\infty} \exp\left[\ln\left(\prod_{j=1}^{n-1}(2k - 2\sum_{i=1}^k \cos\frac{2\pi s_i j}{n})\right) \times \frac{1}{n}\right] \\
=& 4^k \lim_{n\to\infty} \exp\left[\ln\left(\prod_{j=1}^{n-1}(\sum_{i=1}^k \sin^2\frac{\pi s_i j}{n})\right) \times \frac{1}{n}\right] \\
=& 4^k \lim_{n\to\infty} \exp\left[\sum_{j=1}^{n-1}\ln\left(\sum_{i=1}^k \sin^2\frac{\pi s_i j}{n}\right) \times \frac{1}{n}\right]
\end{aligned}$$

Now we use the fact that if $f(x)$ is a continuous non-negative real function defined on $(0, 1]$ such that $\int_0^1 \ln(f(x))dx$ exists, then

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n-1}\ln\left(f\left(\frac{j}{n}\right)\right) \times \frac{1}{n}\right) = \int_0^1 \ln(f(x))dx.$$

to get

$$\lim_{n\to\infty} T_n{}^{\frac{1}{n}} = 4^k \exp\left[\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_i x\right) dx\right],$$

proving the lemma.

Starting from (1.3), a similar proof yields the following statement about *directed* circulants

COROLLARY 1. *For any $1 \le s_1 < s_2 < \cdots < s_k$,*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{T(\vec{C}_{n+1}^{s_1,s_2,\cdots,s_k})}{T(\vec{C}_n^{s_1,s_2,\cdots,s_k})} \\
=& \lim_{n\to\infty} T(\vec{C}_{n+1}^{s_1,s_2,\cdots,s_k})^{1/n} \\
=& \exp\left[\int_0^1 \ln\left(k - \sum_{t=1}^k e^{2\pi s_t ix}\right) dx\right] \\
=& k,
\end{aligned}$$

where $i = \sqrt{-1}$.

which is the result previously referenced from [17].

We now derive the asymptotics of the simplest nonconstant jump case:

THEOREM 3. *Let $1 \le s_1 < \cdots < s_k$, p and $a_1 \le \cdots \le a_l < p$ be positive integers. Then*

$$\begin{aligned}
&\lim_{n\to\infty} T(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln})^{\frac{1}{n}} \\
&= 4^k \exp\left[\sum_{t=1}^p \int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_i x + \frac{1}{2}(l - \sum_{i=1}^l \cos\frac{2t\pi a_i}{p})\right) dx\right].
\end{aligned}$$

Proof. Splitting the formulas, we have

$$\begin{aligned}
&T(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}) \\
=& \frac{1}{pn}\prod_{j=1}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^k \cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^l \cos\frac{2\pi a_i nj}{pn}\right] \\
=& \frac{1}{pn}\prod_{\substack{j=1\\(j \bmod p)\neq 0}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^k \cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^l \cos\frac{2\pi a_i nj}{pn}\right] \\
&\prod_{\substack{j=1\\(j \bmod p)= 0}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^k \cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^l \cos\frac{2\pi a_i nj}{pn}\right] \\
=& \frac{1}{pn}\prod_{q=1}^{p-1}\left(\prod_{\substack{j=1\\(j \bmod p)= q}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^k \cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^l \cos\frac{2\pi a_i j}{p}\right]\right) \\
&\prod_{j'=1}^{n-1}\left[2k - 2\sum_{i=1}^k \cos\frac{2\pi s_i j'}{n}\right].
\end{aligned}$$

Now, let $0 < q < p$. If $j = pj' + q$ then $\frac{j'}{n} < \frac{j'}{n} + \frac{q}{pn} < \frac{j'+1}{n}$. Replacing each index j with the appropriate j' in the above formula and taking limits as in the proof of Lemma 2, we obtain the following

$$\begin{aligned}
&\lim_{n\to\infty} T(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln})^{\frac{1}{n}} \\
= &\exp\Bigg[\sum_{q=1}^{p-1}\lim_{n\to\infty}\ln\Bigg(\prod_{j'=0}^{n-1}\Bigg[2(k+l-\sum_{i=1}^{l}\cos\frac{2q\pi a_i}{p}) \\
&\qquad -2\sum_{i=1}^{k}\cos 2\pi s_i(\frac{j'}{n}+\frac{q}{pn})\Bigg]\times\frac{1}{n}\Bigg) \\
&\qquad +\int_0^1 \ln\left(2k-2\sum_{i=1}^{k}\cos 2\pi s_i x\right)dx\Bigg] \\
= &\exp\Bigg[\sum_{q=1}^{p-1}\int_0^1\ln\Bigg(2\left[k+l-\sum_{i=1}^{l}\cos\frac{2q\pi a_i}{p}\right] \\
&\qquad -2\sum_{i=1}^{k}\cos 2\pi s_i x\Bigg)dx \\
&\qquad +\int_0^1\ln\left(2k-2\sum_{i=1}^{k}\cos 2\pi s_i x\right)dx\Bigg] \\
= &4^k\exp\Bigg[\sum_{t=1}^{p}\int_0^1\ln\Bigg(\sum_{i=1}^{k}\sin^2\pi s_i x + \\
&\qquad +\frac{1}{2}\left[l-\sum_{i=1}^{l}\cos\frac{2t\pi a_i}{p}\right]\Bigg)dx\Bigg].
\end{aligned}$$

We can now extend this to the case where the number of vertices in the graph is no longer an exact multiple of p.

Corollary 2. *Let $1 \le s_1 < \cdots < s_k$ and $a_1 \le \cdots \le a_l < p$ be positive integers and q an integer such that $0 < |q| < p$. Then*

$$\begin{aligned}
&\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}\right)^{\frac{1}{n}} \\
&\quad = \lim_{n\to\infty} T\left(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}\right)^{\frac{1}{n}}
\end{aligned}$$

Proof. For all n, j set

$$\begin{aligned}
x_{n,j} &= \ln\Bigg(2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}\frac{pn}{pn+q} \\
&\qquad -2\sum_{i=1}^{l}\cos\frac{2\pi a_i n}{pn}\frac{pn}{pn+q}\Bigg) \\
y_{n,j} &= \ln\Bigg(2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} \\
&\qquad -2\sum_{i=1}^{l}\cos\frac{2\pi a_i n}{pn}\Bigg).
\end{aligned}$$

Note that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\ln\left[T\left(C_{pn+q}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}\right)\right]/n}{\ln\left[T\left(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}\right)\right]/n} \\
= &\lim_{n\to\infty}\frac{\sum_{j=1}^{pn+q-1}x_{n,j}/n}{\sum_{j=1}^{pn-1}y_{n,j}/n} \\
= &\lim_{n\to\infty}\frac{\sum_{j=1}^{pn-1}x_{n,j}/n+O(\frac{1}{n})}{\sum_{j=1}^{pn-1}y_{n,j}/n}.
\end{aligned}$$

Since $p > a_i$ we can prove that $\lim_{n\to\infty}\frac{x_{n,j}}{y_{n,j}} = 1$, where the convergence is *uniform,* i.e., not dependent upon j. This is because even though j can run to infinity, we can (use the mean-value theorem to) bound the error in terms of the residue class of (j mod p). That is, for any $\epsilon > 0$, there exists an integer N such that whenever $n > N$, we have, for all j, $y_{n,j}(1-\epsilon) < x_{n,j} < y_{n,j}(1+\epsilon)$, which yields

$$\sum_{j=1}^{pn-1}y_{n,j}(1-\epsilon) < \sum_{j=1}^{pn-1}x_{n,j} < \sum_{j=1}^{pn-1}y_{n,j}(1+\epsilon).$$

That is,

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{pn-1}x_{n,j}}{\sum_{j=1}^{pn-1}y_{n,j}} = \lim_{n\to\infty}\frac{\sum_{j=1}^{pn-1}x_{n,j}/n}{\sum_{j=1}^{pn-1}y_{n,j}/n} = 1.$$

Replacing the $x_{n,j}$ and the $y_{n,j}$ back into the above formulas, we prove the claim.

We can generalize even more to allow the jumps to be shifted slightly from linear:

Corollary 3. *Let $1 \le s_1 < \cdots < s_k$ and $a_1 \le \cdots \le a_l < p$ be postive integers and q an integer such that $0 < |q| < p$. Furthermore let $b_1, b_2, \cdots, b_l$ be any arbitrary integers. Then*

$$\begin{aligned}
&\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\cdots,s_k,a_1n+b_1,\cdots,a_ln+b_l}\right)^{\frac{1}{n}} \\
&\quad = \lim_{n\to\infty} T\left(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}\right)^{\frac{1}{n}}
\end{aligned}$$

Proof. Applying the formula $\cos\frac{2\pi(a_in+b_i)}{pn+q} = \cos\frac{2\pi a_in}{pn+q}\cos\frac{2\pi b_i}{pn+q} - \sin\frac{2\pi a_in}{pn+q}\sin\frac{2\pi b_i}{pn+q}$ and (almost) copying the proof of the previous Corollary (reusing the $x_{n,j}$, $y_{n,j}$ defined there) we have

$$\lim_{n\to\infty} \frac{\ln\left[T\left(C_{pn+q}^{s_1,\cdots,s_k,a_1n+b_1,\cdots,a_ln+b_l}\right)\right]/n}{\ln\left[T\left(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln}\right)\right]/n}$$

$$= \lim_{n\to\infty} \frac{\sum_{j=1}^{pn+q-1} x_{n,j}/n + O(\frac{1}{n})}{\sum_{j=1}^{pn-1} y_{n,j}/n}$$

$$= \lim_{n\to\infty} \frac{\sum_{j=1}^{pn-1} x_{n,j}/n + O(\frac{1}{n})}{\sum_{j=1}^{pn-1} y_{n,j}/n}$$

$$= 1.$$

The proof is completed.

Example. Consider the spanning trees of $C_{2n}^{1,n}$ and $C_{2n+1}^{1,n}$. Note that

$$T(C_{2n}^{1,n}) = \frac{n}{2}[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n]^2 \tag{2.5}$$

(Theorem 4 in [18]). This gives

$$\lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = (\sqrt{2}+1)^2 = 2\sqrt{2}+3 = 5.82843....$$

Alternatively, Theorem 3 tells us

$$\lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = 4^2 \exp\left(\int_0^1 \ln(\sin^2 \pi x + 1) + \ln(\sin^2 \pi x) dx\right). \tag{2.6}$$

Mathematica gives $\int_0^1 \ln\,(\sin^2 \pi x + 1)dx = .376453\ldots$, and $\int_0^1 \ln\,(\sin^2 \pi x)dx = -1.38629\ldots$. Plugging these values into (2.6) gives

$$\lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = 4^2 \exp(.376453 - 1.38629) = 5.82843...$$

which is exactly what we had before. Returning to (2.5), Corollary 2 tells us that

$$\lim_{n\to\infty} T(C_{2n+1}^{1,n})^{\frac{1}{n}} = \lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = (\sqrt{2}+1)^2$$

We therefore see that $\lim_{m\to\infty} T\left(C_m^{1,\lfloor\frac{m}{2}\rfloor}\right)^{\frac{1}{m}}$ exists and has value

$$\begin{aligned}\lim_{m\to\infty} T\left(C_m^{1,\lfloor\frac{m}{2}\rfloor}\right)^{\frac{1}{m}} &= \lim_{n\to\infty} T(C_{2n+1}^{1,n})^{\frac{1}{2n}} \\ &= \lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{2n+1}} \\ &= \sqrt{2}+1 = 2.4142\ldots\end{aligned}$$

We can now prove our main theorem:

THEOREM 4. *Let $1 \le s_1 < \cdots < s_k$, $1 \le d_1 \le \cdots \le d_l$, be positive integers and and $c_1 \le \cdots \le c_l$ be arbitrary non-zero integers. Set[2] $p = lcm(a_1, a_2, \ldots, a_l)$. Then*

$$\lim_{m\to\infty} T\left(C_m^{s_1,s_2,\ldots,s_k,\lfloor\frac{m}{d_1}\rfloor+e_1,\ldots\lfloor\frac{m}{d_l}\rfloor+e_l}\right)^{\frac{1}{m}}$$

$$= \left(4^k \exp\left(\sum_{t=1}^{p}\int_0^1 \ln \left(\sum_{i=1}^{k} \sin^2 \pi s_i x + \frac{1}{2}(l - \sum_{i=1}^{l} \cos\frac{2t\pi}{d_i})\right) dx\right)\right)^{1/p}$$

Proof. Set $a_i = \frac{p}{d_i}$ and for $0 \le q < p$ also set $b_{q,i} = \left\lfloor\frac{q}{d_i}\right\rfloor + e_i$, so,

$$\text{if } m = pn+q \text{ then } \left\lfloor\frac{m}{d_i}\right\rfloor + e_i = a_i n + b_{q,i}. \tag{2.7}$$

Now, combining Theorem 3, Corollary 2 and Corollary 3 yields, for fixed q,

$$\begin{aligned}&\lim_{n\to\infty} T(C_{pn+q}^{s_1,\cdots,s_k,a_1n+b_{q,1},\cdots,a_ln+b_{q,l}})^{\frac{1}{n}} \\ =\ &\lim_{n\to\infty} T(C_{pn}^{s_1,\cdots,s_k,a_1n,\cdots,a_ln})^{\frac{1}{n}} \\ =\ &4^k \exp\left(\sum_{t=1}^{p}\int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi s_i x + \frac{1}{2}(l - \sum_{i=1}^{l} \cos\frac{2t\pi a_i}{p})\right) dx\right) \\ =\ &4^k \exp\left(\sum_{t=1}^{p}\int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi s_i x + \frac{1}{2}(l - \sum_{i=1}^{l} \cos\frac{2t\pi}{d_i})\right) dx\right)\end{aligned}$$

This implies

$$\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\cdots,s_k,a_1n+b_{q,1},\cdots,a_ln+b_{q,l}}\right)^{\frac{1}{pn+q}}$$

$$= \left(4^k \exp\left(\sum_{t=1}^{p}\int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi s_i x + \frac{1}{2}(l - \sum_{i=1}^{l} \cos\frac{2t\pi}{d_i})\right) dx\right)\right.$$

Since this is true for *every* q we have proved the theorem.

3 The Asymptotics of $T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}}$ when $n, s_1, \cdots, s_k \to \infty$

In the previous section we saw that, if $s_1, s_2, \cdots, s_k$ are fixed then $T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}}$ converges to a constant dependent upon the s_i. In this section we briefly discuss how this constant changes as the jumps s_i themselves go to infinity.

THEOREM 5. *Let $s_1, s_2, \cdots, s_k$ be abitrary positive integers. Then*

$$\lim_{s_1,s_2,\cdots,s_k\to\infty}\lim_{n\to\infty} T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}}$$

$$= 4^k \exp(\int_0^1\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2 \pi x_i\right) dx_1\cdots dx_k).$$

[2] where *lcm* denotes *least common multiple.*

Proof. From Lemma 2, we have, if $s_1, s_2, \cdots, s_k$ are fixed, that

$$\lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} = \ln 4 + \int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi s_i x\right) dx.$$

Letting $s_1 x = x_1$, yields

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} \\
= \;& \ln 4 + \int_0^{s_1} \ln\left(\sin^2 \pi x_1 + \sum_{i=2}^{k} \sin^2 \pi s_i \frac{x_1}{s_1}\right) dx_1 \frac{1}{s_1} \\
= \;& \ln 4 + \\
&\sum_{j=1}^{s_1-1} \left(\int_j^{j+1} \ln\left(\sin^2 \pi x_1 + \sum_{i=2}^{k} \sin^2 \pi s_i \frac{x_1}{s_1}\right) dx_1\right) \frac{1}{s_1}.
\end{aligned}$$

Note that in the integral, for all s_1, j and x_1, we have that $\frac{j}{s_1} \le \frac{x_1}{s_1} \le \frac{j+1}{s_1}$. Therefore, letting $s_1 \to \infty$ and fixing all $s_2, s_3, \cdots, s_k$, we have

$$\begin{aligned}
&\lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} \\
= \;& \ln 4 + \\
&\lim_{s_1\to\infty} \sum_{j=1}^{s_1-1} \left(\int_j^{j+1} \ln\left(\sin^2 \pi x_1 + \sum_{i=2}^{k} \sin^2 \pi s_i \frac{x_1}{s_1}\right) dx_1\right) \frac{1}{s_1} \\
= \;& \ln 4 + \int_0^1 \int_0^1 \ln\left(\sin^2 \pi x_1 + \sum_{i=2}^{k} \sin^2 \pi s_i y\right) dx_1 dy.
\end{aligned}$$

Now letting $s_2 y = x_2$, then similar to the above discussion,

$$\begin{aligned}
&\lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} \\
= \;& \ln 4 + \int_0^{s_2} \int_0^1 \ln\Big(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \\
&\qquad \sum_{i=3}^{k} \sin^2 \pi s_i \frac{x_2}{s_2}\Big) dx_1 dx_2 \times \frac{1}{s_2} \\
= \;& \ln 4 + \sum_{j=1}^{s_2-1} \Big(\int_j^{j+1} \int_0^1 \ln\Big(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \\
&\qquad \sum_{i=3}^{k} \sin^2 \pi s_i \frac{x_2}{s_2}\Big) dx_1 dx_2 \frac{1}{s_2}.
\end{aligned}$$

Note, again, that in the integral, for all s_2, j and x_2, we have that $\frac{j}{s_2} \le \frac{x_2}{s_2} \le \frac{j+1}{s_2}$. Therefore, letting $s_2 \to \infty$ and fixing all the $s_3, s_4, \cdots, s_k$, we have

$$\begin{aligned}
&\lim_{s_2\to\infty} \lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} \\
= \;& \ln 4 + \int_0^1 \int_0^1 \int_0^1 \ln\Big(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \\
&\qquad \sum_{i=3}^{k} \sin^2 \pi s_i y\Big) dx_1 dx_2 dy.
\end{aligned}$$

Continuing the same approach, we come to

$$\begin{aligned}
&\lim_{s_k\to\infty} \lim_{s_{k-1}\to\infty} \cdots \lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} \\
= \;& \ln 4 + \int_0^1 \cdots \int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi x_i\right) dx_1 dx_2 \cdots dx_k.
\end{aligned}$$

Now making use of the symmetricity of $T(C_n^{s_1,s_2,\cdots,s_k})$ with respect to $s_1, s_2, \cdots, s_k$, we finally have

$$\begin{aligned}
&\lim_{s_1,s_2,\cdots,s_k\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\cdots,s_k}))}{n} \\
= \;& \ln 4 + \int_0^1 \cdots \int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi x_i\right) dx_1 dx_2 \cdots dx_k.
\end{aligned}$$

This proves the theorem.

We note that, interestingly, this quantity is exactly the asymptotic limit of the number of spanning trees in k-dimensional square tori as derived by Garcia, Noy and Tejel in [7].[3]

Let us consider a very special case of the above theorem:

[5] tells us that grids $G(m,n)$ and tori $TS(m,n)$ share the same asymptotic number of spanning trees. With this result plus the additional numerical result obtained in [8], we know that

$$\begin{aligned}
&\lim_{m,n\to\infty} T(TS(m,n))^{\frac{1}{mn}} \\
&= \lim_{m,n\to\infty} T(G(m,n))^{\frac{1}{mn}} \\
&= 4\exp\left(\int_0^1 \int_0^1 \ln(\sin^2 \pi x + \sin^2 \pi y) dx dy\right) \\
&= 3.20991230....
\end{aligned}$$

As noted in [9], when we draw the circulant graph $C_{mn}^{1,n}$ on the grid $G(n,m)$ (mapping node k in $C_{mn}^{1,n}$ to the unique node (i,j) in $G(n,m)$ where $k = ni + j$) then $C_{mn}^{1,n}$ is actually identical to the torus $TS(n,m)$ except for side edges all of whose left endpoints are shifted up by one. Thus, it seems possible that the asymptotics of the circulant would be similar to that of the torus. In fact, Theorem 5 says that this is the case:

[3] Let T_n^k be the number of spanning trees in the k-dimensional square torus with n-vertices, i.e., each dimension has span $n^{1/k}$. [7] showed that $\lim_{n\to\infty} \left(T_n^k\right)^{1/n}$ is exactly the quantity given in Theorem 5.

COROLLARY 4.

$$\lim_{n\to\infty}\lim_{m\to\infty} T(C^{1,n}_{mn})^{\frac{1}{mn}} = \lim_{m,n\to\infty} T(TS(m,n))^{\frac{1}{mn}} = \lim_{m,n\to\infty} T(G(m,n))^{\frac{1}{mn}}$$

Theorem 5 shows that the limit is fixed if we first take the limit over n and then take the limit over the s_i (in any order). An interesting remaining open question would be to show, as in the case of k-dimensional grids and tori, that the *order* in which the limit is taken doesn't matter at all, i.e., that we can mix the n in with the s_i arbitrarily and still get

$$\lim_{n,s_1,s_2,\cdots,s_k\to\infty}\lim_{n\to\infty} T(C_n^{s_1,s_2,\cdots,s_k})^{\frac{1}{n}} = 4^k \exp(\int_0^1\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi x_i\right) dx_1\cdots dx_k).$$

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