

Bivariate Asymptotics for Striped Plane Partitions

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Abstract

We give a new asymptotic formula for a refined enumeration of plane partitions. Specifically: color the parts $\pi_{i,j}$ of a plane partition π according to the equivalence class of $i - j \pmod{2}$, and keep track of the sums of the 0-colored and 1-colored parts separately. We find, for large plane partitions, that the difference between these two sums is asymptotically Gaussian (and we compute the mean and standard deviation of the distribution). Our approach is to modify a multivariate technique of Haselgrove and Temperley.

1 Introduction

A *plane partition* π is an array of nonnegative integers π_{ij} , each placed at one of the lattice points (i, j) in the first quadrant. These integers, called the *parts* of π , must be weakly decreasing in both the x and the y directions. Finally, the sum of the parts is denoted $|\pi|$ and is called the *size* of π ; we require that $|\pi|$ be finite.

MacMahon [15] was the first to give the following generating function for plane partitions:

$$(1.1) \quad \prod_{n=1}^{\infty} (1 - q^n)^{-n},$$

where the variable q marks $|\pi|$. Since MacMahon, there have been many proofs of this result and many variations of it (including an in-depth study of the various symmetry classes of plane partitions); see [2] for a survey.

The classical problem of plane partition enumeration has found a surprisingly modern application in a branch of algebraic geometry called *Donaldson–Thomas (DT) Theory* which is closely related to

topological string theory in physics. DT theory attaches integer-valued deformation invariants, called “Donaldson–Thomas” invariants, to a space X ; these invariants represent “virtual counts” of sub-schemes of X [16]. If X happens to be equipped with a torus action, then the DT invariants can also be interpreted (up to a sign) as counts of torus-fixed sub-schemes of X , by means of Atiyah–Bott equivariant localization [1, 16]. In the simplest example, Equation (1.1) happens to encode all of the DT invariants of \mathbb{C}^3 , after performing the change of variables $q \mapsto -q$ [16].

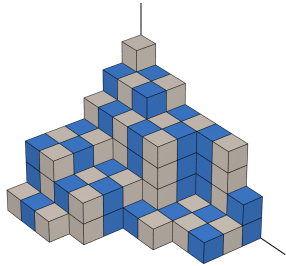
Recent developments in the DT theory of orbifolds [3, 4] have prompted the study of a certain type of refinement of MacMahon’s generating function. The simplest case (which we study here) is the following two-variable generating function:

$$(1.2) \quad \prod_{n=1}^{\infty} (1 - x^n y^n)^{-2n} (1 - x^n y^{n+1})^{-n} (1 - x^n y^{n-1})^{-n}.$$

This product specializes to MacMahon’s generating function when we set $y = x$. It is also an enumeration of plane partitions: x marks the sum of the numbers located at points (x, y) in the first quadrant such that $x - y$ is even (see Figure 1). We call $(x - y) \pmod{2}$ the *color* of (x, y) ; if we color the parts of a plane partition accordingly, the plane partition gets diagonal stripes (hence the term “striped plane partitions”). It is of geometric interest because it encodes the DT invariants of the orbifold $[\mathbb{C}^3/\mathbb{Z}_2]$; study of this generating function has led to interesting results on the behaviour of Donaldson–Thomas invariants of singular spaces and their resolutions [3, 4].

There have been several multivariate refinements of (1.1), including the following “trace” generating

Figure 1: A striped plane partition. The light boxes are marked by x ; the dark ones are marked by y .



function (see Theorem 7.2.1 of [18]):

$$(1.3) \quad \prod_{n=1}^{\infty} (1 - aq^n)^{-n}$$

in which q marks $|\pi|$ and a marks the size of the diagonal of π , $\sum \pi_{ii}$. Later, Gansner [8] refined this result to a beautiful generating function, in infinitely many variables, in which the sum of the parts in the k th diagonal slice $\sum \pi_{i,i+k}$ is marked by q_k , for $k \in \mathbb{Z}$; his result specializes to (1.2) when $q_{2k} = x$ and $q_{2k+1} = y$ for all k .

Wright [21] gave leading-term asymptotics for (1.1), using a modification of the well-known circle method of Hardy and Ramanujan [9].

The state of the art in single-variable asymptotics for plane partitions seems to be the recent preprint of Eynard [7] which, remarkably, describes a random matrix model which is precisely equivalent to many random plane partition problems. Eynard has used the powerful asymptotic technology developed for matrix models to obtain complete asymptotic expansions for many enumerative problems in plane partitions – including many for which the generating functions do not have the nice form of the ones above. It remains to be seen whether the multivariate partition functions that we study can be treated with his framework.

In this paper, we study the asymptotics of (1.2), after making the change of variables $a = y, q = xy$. This yields the following generating function:

$$(1.4) \quad \begin{aligned} G(a, q) &= \sum_m \sum_n p(m, n) a^m q^n \\ &= \prod_{r=1}^{\infty} (1 - aq^r)^{-r} \prod_{r=1}^{\infty} (1 - a^{-1}q^r)^{-r} \\ &\quad \cdot \prod_{r=1}^{\infty} (1 - q^r)^{-2r}. \end{aligned}$$

The following theorem is the main result of this paper.

THEOREM 1.1. *For n and m tending to infinity and such that $m = o(n^{1/3} \ln n)$, we have that $p(m, n)$ is asymptotic to*

$$\frac{\zeta^{1/2}(3)}{8\pi n \sqrt{\ln n}} \exp\left(3\zeta^{1/3}(3)n^{2/3}\right) \exp\left(-\frac{3\zeta^{2/3}(3)}{n^{2/3} \ln n} m^2\right).$$

We remark that this theorem implies that $p(m, n)$ satisfies a normal distribution as m and n get large. For given n , m has mean zero and standard deviation $\frac{\sqrt{6}\zeta^{1/3}(3)}{n^{1/3}\sqrt{\ln n}}$. This distribution has a direct, combinatorial interpretation: m represents the difference between the number of boxes of the two different colours.

We also remark that the asymptotics of the trace generating function (1.3) are computed in the paper by Kamenov and Mutafchiev [13]. They give distributional results while we focus on density results.

2 Preliminaries

Consider the generating function given in (1.4) of the variables a and q with coefficients $p(m, n)$. We study the asymptotic behaviour of $p(m, n)$. Observe that $p(-m, n) = p(m, n)$. Our approach is based on Haselgrove and Temperley [10]. This can be viewed as a two variable saddle point method. Near the saddle point the generating function asymptotically can be separated as the product of two terms involving independent variables.

Next we review the saddle point method in a single variable. We follow Chapter 8, Section 36 of Copson [5].

If $w(z)$ and $\phi(z)$ are analytic functions, regular in a certain domain of the complex plane, it is often possible to find the asymptotic expansion of the integral

$$\int e^{nw(z)} \phi(z) dz$$

as $n \rightarrow \infty$ by a suitable deformation of the path of integration. If z_0 is a point at which $w'(z_0) = 0$, then z_0 is called a *saddle point*. As with a function of a real variable z_0 , it is an extreme point of $|w(z)|$. The simplest case is when $w'(z)$ has a simple zero at z_0 . In this case, the level curves (those with $|w(z)|$ constant) separate the part of the complex plane near z_0 into valleys below and hills above the level curves. There is one direction along which $|w(z)|$ decreases as we move along the level curve away from z_0 , and a direction perpendicular to this direction along which $|w(z)|$ increases. When the integrand is a generating function associated with partitions of integers the integrand increases as z increases along the positive

real axis. In this case we usually take the contour to be a circle $z = re^{i\theta}$, r a function of n only, $r > 0$ chosen so that $z = r$ is a saddle point. Then $w(z) = w(r) + \frac{w''(r)}{2}(i\theta)^2 + \dots$ since $w'(r) = 0$. We usually have $w''(r) < 0$ and the contour integral is asymptotic to

$$\phi(r)e^{nw(r)} \left[\frac{-\pi}{2nw''(r)} \right]^{1/2}.$$

3 Asymptotics

In this section we prove the main result of this paper.

PROOF (OF THEOREM 1.1). From the two variable version of Cauchy's residue theorem, and using Equation (1.4), we have

$$p(m, n) = \frac{1}{(2\pi i)^2} \int_{|q|=e^{-\xi}} \int_{|a|=1} \frac{G(a, q)}{a^{m+1}q^{n+1}} da dq.$$

We write $a = e^{-v}$ and $q = e^{-w}$, and then rewrite $v = \alpha w$. This integral becomes

$$p(m, n) = \frac{1}{(2\pi i)^2} \int_{-i\pi}^{i\pi} \int_{-i\pi w^{-1}}^{i\pi w^{-1}} G(e^{-\alpha w}, e^{-w}) e^{(m\alpha+n)w} w d\alpha dw.$$

We treat w and α as independent variables. Now let $w = \xi + i\theta$ and the integral becomes

$$(3.5) \quad p(m, n) = \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} \int_{-i\pi w^{-1}}^{i\pi w^{-1}} G(e^{-\alpha w}, e^{-w}) e^{(m\alpha+n)w} w d\alpha d\theta.$$

We let

$$g(\alpha, w) = \prod_{r=1}^{\infty} \left(1 - e^{-(r+\alpha)w} \right)^{-r}.$$

Then $G(e^{-\alpha w}, e^{-w}) = g(\alpha, w)g(-\alpha, w)g^2(0, w)$ and we have that

$$\ln g(\alpha, w) = - \sum_r r \ln \left(1 - e^{-(r+\alpha)w} \right).$$

To estimate the integral for $p(m, n)$ we shall use the saddle point method. We choose ξ so that when we expand $\ln G(e^{-\alpha w}, e^{-w}) + (m\alpha+n)w$ in powers of θ the coefficient of θ is zero; in other words we choose ξ so that the partial derivative with respect to θ equals 0. Since there is symmetry between α and $-\alpha$ the coefficient of α in this expansion is automatically equal to 0, that is, the partial derivative with respect to α is zero, as required by the saddle point method in two variables.

Since $w = \xi + i\theta$, and considering the Taylor series

expansion of $\ln g(\alpha, \xi + i\theta)$ in powers of θ , we have

$$\begin{aligned} & \ln g(\alpha, \xi + i\theta) \\ &= - \sum_r r \ln \left(1 - e^{-\xi(r+\alpha)} \right) - i\theta \sum_r \frac{r(r+\alpha)}{e^{\xi(r+\alpha)} - 1} \\ & \quad + \frac{(i\theta)^2}{2} \sum_r \frac{r(r+\alpha)^2}{(e^{\xi(r+\alpha)} - 1)^2} e^{\xi(r+\alpha)} \\ & \quad - \frac{(i\theta)^3}{3!} \sum_r \frac{r(r+\alpha)^3}{(e^{\xi(r+\alpha)} - 1)^3} e^{2\xi(r+\alpha)} + \dots \end{aligned}$$

We expand the terms in this expression in powers of α to get the two variable Taylor series of $\ln g(\alpha, \xi + i\theta)$ in powers of α and θ . Consider first

$$\begin{aligned} & - \sum_r r \ln \left(1 - e^{-\xi(r+\alpha)} \right) \\ &= - \sum_r r \ln \left(1 - e^{-\xi r} \right) - \alpha \sum_r \frac{\xi r}{e^{\xi r} - 1} \\ & \quad + \frac{\alpha^2}{2} \sum_r \frac{\xi^2 r e^{\xi r}}{(e^{\xi r} - 1)^2} - \dots \end{aligned}$$

We also get

$$\begin{aligned} - \sum_r \frac{r(r+\alpha)}{e^{\xi(r+\alpha)} - 1} &= - \sum_r \frac{r^2}{e^{\xi r} - 1} \\ & \quad - \alpha \sum_r \frac{r((\xi r - 1)e^{\xi r} - 1)}{(e^{\xi r} - 1)^2} \\ & \quad - \frac{\alpha^2}{2} \sum_r \frac{\xi r(\xi r - 2)e^{\xi r}}{(e^{\xi r} - 1)^2} - \dots, \end{aligned}$$

as well as,

$$\begin{aligned} \sum_r \frac{r(r+\alpha)^2}{(e^{\xi(r+\alpha)} - 1)^2} e^{\xi(r+\alpha)} &= \sum_r \frac{r^3 e^{\xi r}}{(e^{\xi r} - 1)^2} + \dots; \\ - \sum_r \frac{r(r+\alpha)^3}{(e^{\xi(r+\alpha)} - 1)^3} e^{2\xi(r+\alpha)} &= - \sum_r \frac{r^4 e^{2\xi r}}{(e^{\xi r} - 1)^3} - \dots. \end{aligned}$$

We show later that the terms in total combined degree 3 or more are of lower order, so for our estimates we only require powers of α and θ up to total degree 2. For this we use the Mellin transform

$$e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) \tau^{-s} ds.$$

Now consider the coefficient of θ in $\ln g(\alpha, w) + \ln g(-\alpha, w) + 2 \ln g(0, w)$. From the above this coeffi-

cient is

$$\begin{aligned}
& -4i \sum_r \frac{r^2}{e^{\xi r} - 1} \\
&= -4i \sum_r \frac{r^2 e^{-r\xi}}{1 - e^{-\xi r}} = -4i \sum_{r \geq 1} \sum_{l \geq 1} r^2 e^{-\xi l r} \\
&= -4i \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) \sum_{r \geq 1} \sum_{l \geq 1} \frac{r^2}{(rl\xi)^s} ds \\
&= -4i \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) \xi^{-s} \zeta(s-2) \zeta(s) ds.
\end{aligned}$$

Since the integrand has simple poles at 3 and 1 from the ζ function and $0, -1, \dots$ from the Γ function, we obtain from the calculus of residues

$$-4i\xi^{-3}2\zeta(3) + O(\xi^{-1}).$$

Let us return to the integral equal to $p(m, n)$. To make the coefficient of θ equal to zero because of the term $e^{nw} = e^{n\xi + ni\theta}$ we must choose

$$n = 8\zeta(3)\xi^{-3} (1 + O(\xi^2))$$

or

$$\xi = \frac{2\zeta^{1/3}(3)}{n^{1/3}} \left(1 + O\left(n^{-2/3}\right)\right).$$

Thus ξ is indeed small. For $G(a, q)$ we consider

$$\begin{aligned}
& \ln G(e^{-\alpha w}, e^{-\xi - i\theta}) \\
&= \ln g(\alpha, \xi + i\theta) + \ln g(-\alpha, \xi + i\theta) \\
&\quad + 2 \ln g(0, \xi + i\theta) \\
&= -4 \sum_r r \ln(1 - e^{-\xi r}) + \alpha^2 \sum_r \frac{\xi^2 r e^{\xi r}}{(e^{\xi r} - 1)^2} \\
&\quad - 4i\theta \sum_r \frac{r^2}{e^{\xi r} - 1} + 2(i\theta)^2 \sum_r \frac{r^3 e^{\xi r}}{(e^{\xi r} - 1)^2}.
\end{aligned}$$

For the smaller order terms, due to the symmetry condition on α , we need to consider only the following terms in $\ln g(\alpha, \xi + i\theta) + \ln g(-\alpha, \xi + i\theta) + 2 \ln g(0, \xi + i\theta)$:

$$-i\theta\alpha^2 \sum_r \frac{\xi r(\xi r - 2)e^{\xi r}}{(e^{\xi r} - 1)^2} - \frac{(i\theta)^3}{3} \sum_r \frac{r^4 e^{2\xi r}}{(e^{\xi r} - 1)^3}.$$

We pick θ_0 by comparing the terms in θ^2 and θ^3 . Estimating the sums as integrals, we obtain

$$\frac{1}{\xi^4} \int_1^\infty \frac{(\xi x)^3 e^{\xi x}}{(e^{\xi x} - 1)^2} d(\xi x) \sim \frac{C_0}{\xi^4},$$

and

$$\frac{1}{\xi^5} \int_1^\infty \frac{(\xi x)^4 e^{2\xi x}}{(e^{\xi x} - 1)^3} d(\xi x) \sim \frac{C_1}{\xi^5},$$

respectively, where C_0 and C_1 are constants. We want $\theta^2/\xi^4 \rightarrow \infty$ and $\theta^3/\xi^5 \rightarrow 0$. We pick $\theta_0 = \xi^{11/6}$ and the conditions are satisfied as $\xi \rightarrow 0$. Hence the θ^3 term is $O(\xi^{1/2})$ and the θ^2 term is asymptotic to a constant times $\xi^{-1/3}$.

In a similar way, for this value of θ_0 , the term $\theta\alpha^2$ is asymptotic to a constant times $\xi^{11/6}\alpha^2$ that is of smaller order than the term α^2 .

To determine the asymptotic behavior of $p(m, n)$ we follow the method in [17]. We divide the range of integration in the integral for $p(m, n)$ up as follows:

- A : $|\theta| \leq \xi\delta\mu, |\alpha| \leq \mu$, where $\delta = \xi^{1/2}, \mu = \xi^{-1/12}$;
- B : $|\theta| \leq \xi\delta\mu, |\alpha| > \mu$;
- C : $\xi\mu\delta \leq |\theta| \leq \pi$.

We now consider the term in α^2 in $\ln g(\alpha, \xi + i\theta) + \ln g(-\alpha, \xi + i\theta) + 2 \ln g(0, \xi + i\theta)$. Let us define $K(\alpha)$ by

$$\ln K(\alpha) = \alpha^2 \sum_r \frac{\xi^2 r e^{\xi r}}{(e^{\xi r} - 1)^2}.$$

We have

$$\begin{aligned}
(3.6) \quad \ln K(\alpha) &\sim \alpha^2 \int_\xi^\infty \frac{v e^v}{(e^v - 1)^2} dv \\
&= \alpha^2 \left(\left. \frac{-v}{e^v - 1} \right|_\xi^\infty + \int_\xi^\infty \frac{1}{e^v - 1} dv \right) \\
&= \alpha^2 (1 - \ln(1 - e^{-\xi})) \\
&= \alpha^2 (1 - \ln(\xi) + O(\xi)).
\end{aligned}$$

We observe that in the range of integration $\alpha = i|\alpha|$ and as $|\alpha| \rightarrow \infty$ we have $K(\alpha) = O(|\alpha|^{-N})$ for each $N > 0$. The variables α and θ are separated in the asymptotic expansion of $G(a, q)$ near the saddle point $a = 1, \theta = \theta_0$. We change the limits of integration on α to $-i\infty$ and $i\infty$ so we can later recognize the integral as an inverse Laplace transform. To do this we break up the range of integration as in [17]. First we consider

$$\int_{-i|\mu|w}^{i|\mu|w} K(\alpha) e^{m\alpha w} d\alpha.$$

For the integration over α and θ , for $\delta = \xi^{1/2}$, we consider the cases

- (i) $|m|\xi\delta^{1/2} \leq 1$, and
- (ii) $|m|\xi\delta^{1/2} > 1$.

In case (i), $|m|\xi\delta\mu^2 \leq \delta^{1/2}\mu^2 = \xi^{1/12}$. Hence $m\alpha w = m\alpha\xi + O(m\xi\mu^2\delta) = m\alpha\xi + O(\xi^{1/12})$. We may therefore

replace w by ξ in our integral. Also $w = \xi + i\theta = \xi + o(\xi)$ so

$$(3.7) \quad \Re \left(\frac{i\mu|w|}{w} \right) = O \left(\frac{\xi\delta\mu^2}{|w|} \right) = O(\xi^{1/3}).$$

Using this and (3.6) we have

$$\begin{aligned} & \int_{-i|w|\mu/w}^{i|w|\mu/w} K(\alpha) e^{m\alpha w} d\alpha \\ &= \int_{-i|w|\mu/w}^{i|w|\mu/w} K(\alpha) e^{m\alpha\xi} \left(1 + O(\xi^{1/12}) \right) d\alpha. \end{aligned}$$

Second, we consider α going from $i\mu|w|w^{-1}$ to $i\mu$. In this case, the $\Re\alpha$ goes from $\Re(i\mu|w|/w)$ to 0. We have $\Re(m\alpha\xi) = O(m\xi\xi^{1/3}) = O(m\xi\delta^{1/2}) = o(1)$ in case (i), so by (3.6) and for all $N > 0$

$$\int_{i\mu|w|\mu/w}^{i\mu} K(\alpha) e^{m\alpha\xi} d\alpha = O(\mu^{-N}).$$

Again by (3.6) and for all $N > 0$, we have

$$\int_{i|w|\mu/w}^{i\infty} K(\alpha) e^{m\alpha\xi} d\alpha = O(\xi^N).$$

Hence in case (i) we get

$$\int_{-i\mu|w|\mu/w}^{i\mu|w|\mu/w} K(\alpha) e^{m\alpha\xi} d\alpha = \int_{-i\infty}^{i\infty} K(\alpha) e^{m\alpha\xi} d\alpha + O(\xi^{1/12}).$$

In case (ii), since $m\theta = O(\xi^{17/12})$, we have $|mw| \sim |m|\xi > \delta^{-1/2}$. Using Cauchy's integral formula for the derivative, we have

$$K'(\alpha) = \int_{\mathcal{C}} \frac{K(x)}{(x-\alpha)^2} dx,$$

where \mathcal{C} is the circle centered at α and radius $1/|\alpha|$. Then, by Equation (3.6), we have that, for all $N > 0$, $K'(\alpha) = O(|\alpha|^{-N})$. Integration by parts shows that

$$\int_{-i\mu|w|\mu/w}^{i\mu|w|\mu/w} K(\alpha) e^{m\alpha w} d\alpha = O(\xi^{1/4}).$$

As before, by Equation (3.6) and for all $N > 0$, we obtain

$$\int_{i\mu|w|\mu/w}^{i\mu} K(\alpha) e^{m\alpha\xi} = O(\mu^{-N})$$

and

$$\int_{i|w|\mu/w}^{i\infty} K(\alpha) e^{m\alpha\xi} d\alpha = O(\xi^N).$$

Thus combining case (i) and (ii) and for all $N > 0$, we have

$$\begin{aligned} & \int_{-i\mu|w|\mu/w}^{i\mu|w|\mu/w} K(\alpha) \exp(m\xi\alpha) d\alpha \\ &= \int_{-i\infty}^{i\infty} K(\alpha) \exp(m\xi\alpha) d\alpha + O(\xi^{1/12}). \end{aligned}$$

By the end of the proof we will see that the integral in the right hand side of the previous equation is the inverse Laplace transform of a normal distribution.

We now consider the integral over θ . We pick ξ so that the coefficient of θ is zero. We saw that the term of combined degree 3 or more are $O(\xi^{1/2})$. So the integral over θ is

$$\begin{aligned} & \int_{-\pi}^{\pi} \exp \left(-4 \sum_{r \geq 1} r \ln(1 - e^{-\xi r}) + n\xi \right. \\ & \quad \left. - 2\theta^2 \sum_{r \geq 1} \frac{r^3 e^{\xi r}}{(e^{\xi r} - 1)^2} \right) \cdot \left(1 + O(\xi^{1/2}) \right) d\theta. \end{aligned}$$

Since θ_0 is asymptotic to a constant times $\xi^{-1/3}$, the previous integral over the ranges $(-\pi, -\theta_0)$ and (θ_0, π) are $O(e^{-\xi^{-1/3}})$. We use the saddle point method in the variable θ for the integral

$$\begin{aligned} & \int_{-\theta_0}^{\theta_0} \exp \left(-4 \sum_{r \geq 1} r \ln(1 - e^{-\xi r}) + n\xi \right. \\ & \quad \left. - 2\theta^2 \sum_{r \geq 1} \frac{r^3 e^{\xi r}}{(e^{\xi r} - 1)^2} \right) \cdot \left(1 + O(\xi^{1/2}) \right) d\theta. \end{aligned}$$

Let $\Psi(\xi) = -4 \sum_{r \geq 1} r \ln(1 - e^{-\xi r})$. Then, we have

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} \exp \left(\Psi(\xi) + n\xi - \frac{1}{2} \theta^2 \Psi''(\xi) \right) w d\theta \\ &= \frac{\xi}{\sqrt{\pi}} \left(\frac{\Psi''(\xi)}{2} \right)^{-1/2} \exp(\Psi(\xi) + n\xi) (1 + O(\xi)). \end{aligned}$$

We conclude that the integral over the region A in the integral defining $p(m, n)$ is equal to

$$(3.8) \quad \frac{\xi}{\sqrt{\pi}} \left(\frac{\Psi''(\xi)}{2} \right)^{-1/2} \exp(\Psi(\xi) + n\xi) \cdot \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} K(\alpha) e^{m\alpha\xi} d\alpha + O(\xi^{1/12}) \right).$$

The integral over range B .

For the integration over range B we have, from (3.5),

$$\frac{|G(a, q)|}{G(1, e^{-\xi})} = \prod_r \left(\frac{1 - e^{-\xi r}}{|1 - e^{-(r+\alpha)w}|} \right)^r.$$

Recall that we have set $a = e^{-v} = e^{i\phi}$, $q = e^{-w}$ where $v = \alpha w$. We write $\beta = \arg(-(r + \alpha)w) = \arg(aq^r) = i\alpha w - r\theta$ since $\alpha w = -i\phi$; also $\Re(-rw) = -r\xi$.

In general, $\Re(\ln f(z)) = \frac{1}{2} \ln |f(z)|^2$, so

$$\begin{aligned} \left| \frac{1}{1 - e^{-(r+\alpha)w}} \right|^2 &= \frac{1}{(1 - e^{-\xi r + i\phi})(1 - e^{-\xi r - i\phi})} \\ &= \frac{1}{(1 - e^{-\xi r})^2 - 2e^{-\xi r}(\cos \beta - 1)} \end{aligned}$$

and hence

$$\begin{aligned} &\prod_r \left(\frac{1 - e^{-\xi r}}{|1 - e^{-(r+\alpha)w}|} \right)^r \\ &= \exp \left(-\frac{1}{2} \sum_r r \ln \left(1 + \frac{2e^{\xi r}}{(e^{\xi r} - 1)^2} (1 - \cos \beta) \right) \right). \end{aligned}$$

Since $\beta = i\alpha w - r\theta = \phi - r\theta$ we suppose $|r\theta| \leq 2$ so $|\beta| \leq \pi + 2 \leq 2\pi$ hence

$$1 - \cos \beta \geq K_1 \beta^2, \quad \text{for } K_1 > 0.$$

Furthermore for $|\theta| \leq \xi\mu\delta$, $w = \xi + O(\xi\mu\delta)$ so $|\phi| = |\alpha|\xi(1 + O(\mu\delta))$. Hence for $|r\theta| \leq \xi|\alpha|/2$ we have $|\beta| \geq \xi|\alpha|/2$. Let \sum' denote summation over r such that $|r\theta| \leq \xi|\alpha|/2 \leq \pi/2$, $r\xi \leq 1$. Then by the preceding inequality for $\cos \beta$ we have

$$\frac{|G(a, q)|}{G(1, e^{-\xi})} \leq \exp \left(-\frac{1}{2} \sum' r \ln \left(1 + \frac{2e^{\xi r}}{(e^{\xi r} - 1)^2} K_1 (\xi|\alpha|)^2 \right) \right)$$

since the terms neglected in the \sum' are smaller than or equal to 1 in absolute value. Also since $x \leq 1$ implies $e^{-x}/2(1 - e^{-x})^2 \geq cx^{-2}$ this implies an upper bound of

$$\exp \left(-\frac{1}{2} \sum' r \ln (1 + c_1 |\alpha|^2 r^{-2}) \right).$$

for a constant c_1 . If $r \leq \sqrt{|\alpha|}$, we have for all $N > 0$

$$\begin{aligned} &\exp \left(-\frac{1}{2} \sum' r \ln (1 + c_1 |\alpha|^2 r^{-2}) \right) \\ &\leq \exp \left(-\frac{1}{2} \sum_{r \leq \sqrt{|\alpha|}} r \ln c_1 |\alpha| \right) \\ &\leq \exp \left(-\ln c_1 |\alpha| \frac{|\alpha|}{4} \right) = O(\xi^N). \end{aligned}$$

The integral over region B is therefore negligible.

The integral over range C .

For integration over C we again start from

$$\frac{|G(a, q)|}{G(1, e^{-\xi})} = \prod_r \frac{1 - e^{-\xi r}}{|1 - e^{-(r+\alpha)w}|} = \prod_r \frac{1 - e^{-\xi r}}{|1 - e^{-\xi r + i\beta}|}$$

where $\beta = \phi - r\theta$, and we do not include the exponents r since each term in the product is smaller or equal to 1 in absolute value. We shall find that for each term in the last product with β close to 0 there are many terms with β not close to 0. Let c be any number such that $0 < c < \pi$. We first prove that if $\beta \in [c, 2\pi - c]$ then

$$(3.9) \quad \frac{1 - e^{-\xi\nu}}{|1 - e^{-\xi\nu + i\beta}|} \leq (1 - \cos \beta)^{-1/2} \frac{1 - e^{-\xi\nu}}{(1 + e^{-2\xi\nu})^{1/2}}.$$

Indeed, suppose $\beta \in [c, \pi]$. If we construct the triangle with vertices at 0, 1, $e^{-\xi\nu + i\beta}$ we obtain from the cosine law for triangles

$$\begin{aligned} |1 - e^{-\xi\nu + i\beta}| &= (1 + e^{-2\xi\nu} - 2e^{-\xi\nu} \cos \beta)^{1/2} \\ &= (1 + e^{-2\xi\nu})^{1/2} \left(1 - \frac{2e^{-\xi\nu} \cos \beta}{1 + e^{-2\xi\nu}} \right)^{1/2}. \end{aligned}$$

Since $0 < 2e^{-\xi\nu} / (1 + e^{-2\xi\nu}) \leq 1$ (since $x + x^{-1} \geq 2$) this yields that

$$|1 - e^{-\xi\nu + i\beta}| \geq (1 + e^{-2\xi\nu})^{1/2} (1 - \cos \beta)^{1/2}.$$

The case $\beta \in [\pi, 2\pi - c]$ is treated similarly.

Let $N > 0$ be an arbitrary fixed integer and ϵ be a fixed positive (small) number. For each integer r construct the sets

$$\begin{aligned} \Delta_r^j &= \left[\frac{j2\pi - \xi\epsilon/N}{r}, \frac{j2\pi + \xi\epsilon/N}{r} \right] \\ \Delta_{\phi, r}^j &= \left[\phi + \frac{j2\pi - \xi\epsilon/N}{r}, \phi + \frac{j2\pi + \xi\epsilon/N}{r} \right] \end{aligned}$$

where $1 \leq j \leq r/2$. We observe that all β with $\theta \in \Delta_r^j$ are mapped into the interval $[\phi + j2\pi - \xi\epsilon/N, \phi + j2\pi + \xi\epsilon/N]$ under the map $\phi + \theta \rightarrow \phi + \theta r$. Let l be a positive integer, let (n, m) denote the greatest common divisor of m and n . Suppose that for some j the equation

$$\frac{j}{l} = \frac{k}{r} \quad \text{or } jr = kl \quad \text{or } \phi + \frac{j}{l} = \phi + \frac{k}{r}$$

is solvable for all r with $r > l$. This implies that for each $r > l$, $l/(l, j)|r$. If $(r, l) = 1$ this is a contradiction so for each j there is an $r = r(l, j)$ such that $\phi + j/l \neq \phi + k/r$,

that is, $\theta \notin \bigcup \Delta_l^j, j = 1, 2, \dots, [l/2]$ or $\beta \notin \bigcup \Delta_{\phi, r}^j$. From (3.9) however this means

$$\frac{1 - e^{-\xi l}}{1 - e^{-\xi + i\beta}} = O\left(\xi^{1-\epsilon/N}\right)$$

as $\xi \rightarrow 0$ or $n \rightarrow \infty$. Let $R(l, j)$ be the smallest integer which is greater than each of the $r(l, j)$. We may repeat the above argument with $R(l, j)$ replacing $r(l, j)$. Doing this N times allows us to conclude that, for all $N > 0$,

$$\frac{G(a, q)}{G(1, |q|)} = O(\xi^{N-\epsilon}),$$

so the integral over region C is also negligible.

Therefore, $p(m, n)$ is asymptotic to the integral over the region A shown in Equation (3.8), and this concludes the first part of the proof.

Next we estimate this expression in terms of elementary functions. First, we have that

$$\begin{aligned} \Psi(\xi) &= -4 \sum_{r \geq 1} r \ln(1 - e^{-\xi r}) = 4 \sum_r \sum_l \frac{r}{l} e^{\xi r l} \\ &= \frac{4}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \sum_r \sum_l \frac{r}{l} \frac{\Gamma(s)}{(\xi r l)^s} ds \\ &= \frac{4}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{\xi^s} \Gamma(s) \zeta(s-1) \zeta(s+1) ds. \end{aligned}$$

There are singularities at $s = 2$, a double pole at $s = 0$, and singularities at $s = 0, -1, \dots$. The singularity at 2 gives a contribution of $4\zeta(3)\Gamma(2)/\xi^2$.

For the residue at zero we use the following estimates from Whittaker and Watson [20], pages 271 and 236, respectively:

$$\zeta(s+1) = \frac{1}{s} + \gamma + O(s), \quad \Gamma(s) = \frac{1}{s} - \gamma + O(s).$$

So, $\zeta(s+1)\Gamma(s) = 1/s^2 + O(1)$. We also have that

$$\xi^{-s} = e^{-s \ln \xi} = 1 - s \ln \xi + \frac{s^2}{2} \ln^2 \xi - \dots,$$

and so the integrand at zero is

$$\frac{\Gamma(s)\zeta(s+1)}{\xi^s} = \frac{1}{s^2} + \frac{\ln 1/\xi}{s} + O(1).$$

Hence, we have

$$\Psi(\xi) = 4 \frac{\zeta(3)}{\xi^2} + 4 \ln 1/\xi + O(\xi).$$

Next we estimate ξ in terms of n . The coefficient of θ in $\ln G(a, q)$ is $-4i \sum_r r^2 / (e^{\xi r} - 1)$ so, using that

$\zeta(-2) = 0$ and that $\zeta(-1) = -1/12$, we obtain

$$\begin{aligned} n &= 4 \sum_r \frac{r^2}{e^{\xi r} - 1} = 4 \sum_r \frac{r^2 e^{-\xi r}}{1 - e^{-\xi r}} \\ &= 4 \sum_{r \geq 1} \sum_{l \geq 1} r^2 e^{-\xi l r} = \frac{4}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{r^2 \Gamma(s)}{\xi^s l^s r^s} ds \\ &= \frac{4}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s)\zeta(s)\zeta(s-2)}{\xi^s} ds \\ &= 4 \frac{\Gamma(3)\zeta(3)}{\xi^3} + \left(-\frac{1}{12}\right) \frac{4}{\xi} + O(\xi) \\ &= \frac{8\zeta(3)}{\xi^3} - \frac{1}{3\xi} + O(\xi). \end{aligned}$$

Given the previous expression of n in terms of ξ , we can write ξ in terms of n . Indeed,

$$\xi^3 = \frac{8\zeta(3)}{n} \left(1 - \frac{\xi^2}{24\zeta(3)} + O(\xi^4)\right),$$

and so

$$\xi = \frac{2\zeta^{1/3}(3)}{n^{1/3}} \left(1 - \frac{1}{18\zeta^{1/3}(3)n^{2/3}} + O\left(\frac{1}{n^{4/3}}\right)\right).$$

Moreover,

$$\begin{aligned} \ln 1/\xi &= \ln \left(\frac{n^{1/3}}{2\zeta^{1/3}(3)} \left(1 + O\left(\frac{1}{n^{2/3}}\right)\right) \right) \\ &= \frac{\ln n}{3} - \left(\ln 2 + \frac{\ln \zeta(3)}{3} + O\left(\frac{1}{n^{2/3}}\right) \right). \end{aligned}$$

We conclude, putting all pieces together, that

$$\begin{aligned} \Psi(\xi) &= 4 \frac{\zeta(3)}{\xi^2} + 4 \ln 1/\xi + O(\xi) \\ &= \zeta^{1/3}(3)n^{2/3} + \frac{4}{3} \ln n + \left(\frac{1}{36} - \ln 2 - \frac{\ln \zeta(3)}{3} \right) \\ &\quad + O\left(\frac{1}{n^{1/3}}\right). \end{aligned}$$

Next we compute $\Psi''(\xi) = 24\zeta(3)\xi^{-4} + O(1)$. Also, $\xi^{-4} \sim 16\zeta^{-4/3}(3)n^{4/3}$. Recalling Equation (3.8) we obtain

$$\begin{aligned} &\frac{\xi}{\sqrt{\pi}} \left(\frac{\Psi''(\xi)}{2} \right)^{-1/2} \exp(\Psi(\xi) + n\xi) \\ &\sim \frac{2\zeta^{1/3}(3)}{n^{1/3}} \sqrt{\frac{2}{\pi}} \left(\frac{(16)(24)n^{4/3}}{\zeta^{1/3}(3)} \right)^{-1/2} \\ &\quad \cdot \exp\left(\zeta^{1/3}(3)n^{2/3} + 2\zeta^{1/3}(3)n^{2/3}\right) \\ &\sim \frac{\zeta^{1/2}(3)}{n} \frac{1}{4\sqrt{3\pi}} \exp\left(3\zeta^{1/3}(3)n^{2/3}\right). \end{aligned} \tag{3.10}$$

Now we focus on the factor $\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} K(\alpha) e^{m\alpha\xi} d\alpha$ of Equation (3.8). Let

$$S = \sum_r \frac{\xi^2 r e^{\xi r}}{(e^{\xi r} - 1)^2}.$$

Using Equation (3.6), we have $S \sim \ln 1/\xi \sim \ln n/3$. An inverse Laplace transform computation (see [11], for instance), and the change $\beta = \alpha\sqrt{s}$ gives

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\alpha^2 S} e^{m\xi\alpha} d\alpha &= \frac{1}{2\pi i \sqrt{S}} \int_{-i\infty}^{i\infty} e^{\beta^2} e^{\frac{m\xi\beta}{\sqrt{S}}} d\beta \\ &= \frac{1}{2\sqrt{\pi}\sqrt{S}} e^{-\left(\frac{m\xi}{\sqrt{S}}\right)^2} \\ &= \frac{\sqrt{3}}{\sqrt{\ln n}} \frac{1}{2\sqrt{\pi}} e^{-\frac{m^2\xi^2}{4S}}. \end{aligned}$$

Since $\xi^2 \sim \frac{4\zeta^{2/3}(3)}{n^{2/3}}$, we have

$$(3.11) \quad \begin{aligned} &\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\alpha^2 S} e^{m\xi\alpha} d\alpha \\ &\sim \frac{\sqrt{3}}{\sqrt{\ln n}} \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{3\zeta^{2/3}(3)m^2}{n^{2/3} \ln n}\right). \end{aligned}$$

The theorem follows from Equations (3.10) and (3.11) since we have

$$\begin{aligned} &\frac{\xi}{\sqrt{\pi}} \left(\frac{\Psi''(\xi)}{2}\right)^{-1/2} \exp(\Psi(\xi) + n\xi) \\ &\cdot \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} K(\alpha) e^{m\alpha\xi} d\alpha + O(\xi^{1/12})\right) \\ &\sim \frac{\zeta^{1/2}(3)}{n} \frac{1}{4\sqrt{3\pi}} \exp\left(3\zeta^{1/3}(3)n^{2/3}\right) \\ &\cdot \frac{\sqrt{3}}{\sqrt{\ln n}} \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{3\zeta^{2/3}(3)m^2}{n^{2/3} \ln n}\right) \\ &\sim \frac{\zeta^{1/2}(3)}{8\pi} \frac{1}{n\sqrt{\ln n}} \exp\left(3\zeta^{1/3}(3)n^{2/3}\right) \\ &\cdot \exp\left(-\frac{3\zeta^{2/3}(3)}{n^{2/3} \ln n} m^2\right). \end{aligned}$$

We observe that in order that our main term not be smaller than the error term we need that $\xi^{1/12}$ be small compared to the integral. Thus, we require that $e^{m^2/(n^{2/3} \ln n)}$ be small as compared to $\ln n$, that is, $m = o(n^{1/3} \ln n)$. \square

4 Conclusions

A related type of combinatorial structure, the *pyramid partition of length 1*, was described in [19] and enumer-

ated in [22]; the generating function is

$$(4.12) \quad \prod_{n=1}^{\infty} (1 - x^n y^n)^{-2n} (1 + x^n y^{n+1})^n (1 + x^n y^{n-1})^n.$$

Pyramid partitions also arise in Donaldson-Thomas theory, although this time it is a noncommutative variant [19]. We expect our methods to give asymptotics for these generating functions, as well.

It is likely that this particular interplay of toric algebraic geometry and algebraic combinatorics will continue to produce interesting multivariate generating functions of relevance to the geometry and physics communities [4, 6, 12, 14]. Unfortunately, (1.2) and (4.12) are definitely the simplest such examples; any other related formulae will either be significantly more complicated, or will have more than two variables. Consequently, we expect that it will be more difficult to obtain asymptotics for these generating functions.

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