Tunnel - Aquifer State Function and Flow Integrals

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Abstract
An integral representation of the state function of a semi-infinite aquifer that is drained by a circular tunnel is used to deduce an exact analytic solution. A set of single and double integrals, whose integrands are products of modified Bessel functions of the 2nd kind and trigonometric functions, is generated. Another set of integrals, the flow integrals, containing the gradient of the state function, is generated when determining the flow induced by the tunnel. Both sets of integrals are computed exactly and expressed in closed forms. Although the analytic solution is not a compact equation it can be truncated retaining its first term which gives an indication on the relation that binds tunnel and soil parameters, and it can be used for a first evaluation of the water inflow and pressure for engineering practice.

1 Introduction.

The state of an aquifer in a fractured rock mass is obtained from the hydraulic part of the evolution equations of the contact consolidation [1,2]. In fractured rock mass where the fractures rarefy and close with depth, the hydraulic conductivity decreases exponentially [3] and the aquifer is non-homogeneous. In this case, the evolution equations merge into a linear governing equation that acts on a state function that is related to the hydraulic potential [4]. An integral form of the state function is obtained subsequently and is then transformed into an integral equation that is solved analytically considering a semi-infinite aquifer drained by a circular tunnel. The solution is a series which requires evaluating double integrals involving modified Bessel functions. These integrals are calculated exactly and obtained in closed form in Lemma 2.1.

Due to environmental constraints and safety problems during tunnel construction and exploitation, the amount of water that flows through the tunnel edge or into the aquifer through the water table needs to be calculated and monitored. The flow integrals are integrals over the water table and the tunnel edge of the normal flow. Their difference is the volume rate or the instantaneous water gained or lost by the aquifer. The flow integrals operate on a combination of the state function and its gradient and are calculated in closed analytic forms using Lemma 3.1.

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Analytic and closed form solutions are important in engineering especially with the increasing use of numerical commercial codes. It helps in evaluating the correctness of these codes and to find bugs in their source files and numerical schemes. Also in everyday practice a simple equation that gives a quick first approximation with a straight insight on the important parameters is preferred to black boxes, which may be used subsequently in case more details are needed.

The governing equation of a homogeneous aquifer in the steady state is the Laplace equation. Analytic solutions of the Laplace equation considering a circular tunnel may be found in [4,5] and are partly reviewed in [6]. The governing equation in the transient state or the steady state of a non-homogeneous aquifer is the modified Helmholtz equation [7]. Analytic solutions for the Helmholtz modified equation considering a draining circular tunnel may be found in [4,7]. They are based on the lemmas 2.1 and 3.1 that will be proven.

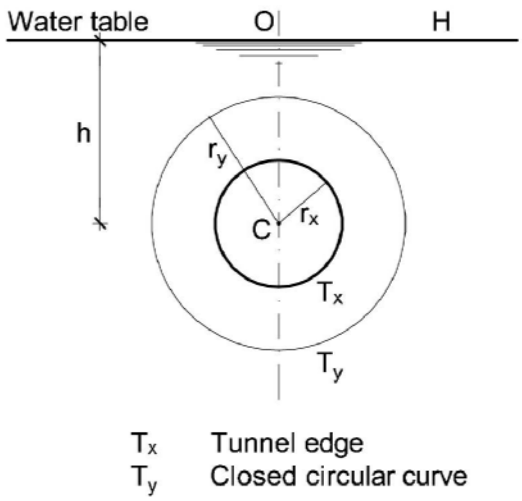

Figure 1. The semi-infinite aquifer is bounded by H the water table and T_x the tunnel edge with center C.

2 State function.

Figure 1 shows a semi-infinite aquifer bounded by a horizontal water table H and a draining circular tunnel T_x. The state function Ω is defined in the aquifer domain and satisfies Dirchelet conditions on the boundaries H and T_x. It is often preferred to define an internal condition instead of a condition on the tunnel edge. This will be considered in this paper and a function ω will be assigned to the state function on T_y, which is a closed curve that encircles the tunnel (Figure 1). If the internal condition is not needed it is always possible to approach the tunnel edge undertaking the limit T_y-->T_x. So, the state function is supposed to be zero on the water table (Ω=0 on H) and equal to ω on T_y (Ω=ω on T_y).

An integral form of the state functions that satisfies the modified Helmholtz equation ($\nabla^2\Omega = \mathbf{b}^2\Omega$) and is zero on H, is

$$\Omega(\mathbf{y}) = \int_{T_x} ds(\mathbf{x})\varphi(\mathbf{x})[K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|) - K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|)]\,, \quad (1)$$

in which the single layer φ is defined on the tunnel edge T_x with center C(0,-h) and radius r_x smaller than the depth h; K_0 is the modified Bessel function of the second kind of order zero; **b** is a constant vector field which modulus is b; and the star stands for the mirror image with respect to the water table.

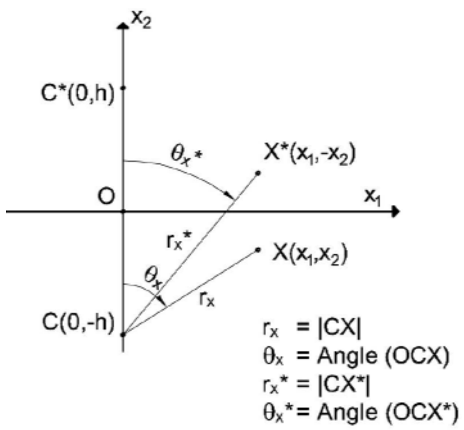

*Figure 2. Notation: X*and C* are respectively the mirror image of X and C with respect to H (x_2=0). Two systems of coordinates are used; a Cartesian one with origin O on H and a polar one with pole C.*

The single layer is an unknown function that is deduced resolving an integral equation that is obtained when ω(**y**) is assigned to the state function Ω on T_y. Before doing that, the single layer is written as a Fourier series with cosine and sine coefficients φ^c_n and φ^s_n and is

$$\varphi(\mathbf{x}) = \varphi_0^c + \sum_{m=1}^{\infty}(\varphi_m^c \cos m\theta_x + \varphi_m^s \sin m\theta_x)\,. \quad (2)$$

Inserting, Equation (2) in Equation (1) leads to

$$\Omega = 2\pi r_x \sum_{m=0}^{\infty} I_m(br_x)[\varphi_m(\theta_y)K_m(br_y) - \varphi_m(\theta_{y*})K_m(br_{y*})] \quad (3)$$

with

$$\varphi_0(\theta) = \varphi_0^c \quad \varphi_m(\theta) = \varphi_m^c \cos(m\theta) + \varphi_m^s \sin(m\theta) \quad , \quad (4)$$

in which I_m and K_m are the modified Bessel functions of the first and second kind of order m, r_x is the length of CX; θ_x is the angle which is made up of CX and the vertical axis (see Figure 2 and Appendix A for notations); and r_y, θ_y, r_{x*}, θ_{x*} , r_{y*}, θ_{y*} are defined equivalently for CX*, CY and CY*. The proof of Equation (3) is obtained after having computed the following integrals

$$J_1 = \int_{T_x} ds(\mathbf{x})\cos(m\theta_x)K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|)\,, \quad (5)$$

$$J_2 = \int_{T_x} ds(\mathbf{x})\cos(m\theta_x)K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|)\,, \quad (6)$$

$$J_3 = \int_{T_x} ds(\mathbf{x})\sin(m\theta_x)K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|) \quad (7)$$

and

$$J_4 = \int_{T_x} ds(\mathbf{x})\sin(m\theta_x)K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|)\,. \quad (8)$$

On the circle T_y with center C(0,-h) and radius r_y (greater than r_x), ω(**y**) is written as a Fourier series with cosine coefficients ω_n^c and sine coefficients ω_n^s. Inserting the Fourier series for ω in (1), when restricting Ω on T_y, leads to two independent systems of equations. The two systems are obtained by multiplying Equation (3) by cos(nθ_y) and sin(nθ_y) respectively and integrating over T_y, and are

$$\sum_{m=0}^{\infty} C_{nm}\varphi_m^c = \pi r_y \omega_n^c \Delta_{nn} \quad n \geq 0 \quad (9)$$

and

$$\sum_{m=1}^{\infty} S_{nm}\varphi_m^s = \pi r_y \omega_n^s \quad n \geq 1\,, \quad (10)$$

in which the matrix coefficients C_{mn} and S_{mn} are

$$C_{nm} = 2\pi^2 r_x r_y I_m(br_x)[K_m(br_y)\Delta_{mn} - I_n(br_y)(K_{n+m}(2bh) + K_{n-m}(2bh))] \quad (11)$$

and

$$S_{nm} = 2\pi^2 r_x r_y I_m(br_x)[K_m(br_y)\Delta_{mn} - I_n(br_y)(K_{n+m}(2bh) - K_{n-m}(2bh))] \quad (12)$$

and in which Δ_{mn} equals 2 when m=n=0, 1 when m=n>0 and 0 when m is different from n. The odd and even parts of φ, which are represented by the sine and cosine coefficients, are determined separately. It is deduced, from Equations (9) and (10), that parities do not mix and, from Equations (11) and (12), that for identical parities the modes are not orthogonal. When r_y is equal to r_x the coefficient C_{nm} and S_{nm} remain unchanged when exchanging m and n, which leads to symmetric matrixes. The proofs for Equations (9) to (12) are straightforward once the following integrals have been evaluated

$$J_5 = \iint_{T_x T_y} ds(\mathbf{x}) ds(\mathbf{y}) \cos(m\theta_x)\cos(n\theta_y) K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|), \quad (13)$$

$$J_6 = \iint_{T_x T_y} ds(\mathbf{x}) ds(\mathbf{y}) \cos(m\theta_x)\cos(n\theta_y) K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|), \quad (14)$$

$$J_7 = \iint_{T_x T_y} ds(\mathbf{x}) ds(\mathbf{y}) \sin(m\theta_x)\sin(n\theta_x) K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|), \quad (15)$$

$$J_8 = \iint_{T_x T_y} ds(\mathbf{x}) ds(\mathbf{y}) \sin(m\theta_x)\sin(n\theta_y) K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|), \quad (16)$$

$$J_9 = \iint_{T_x T_y} ds(\mathbf{x}) ds(\mathbf{y}) \cos(m\theta_x)\sin(n\theta_y) K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|) \quad (17)$$

and

$$J_{10} = \iint_{T_x T_y} ds(\mathbf{x}) ds(\mathbf{y}) \cos(m\theta_x)\sin(n\theta_y) K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|). \quad (18)$$

LEMMA 2.1. *Integrals J_1 to J_{10} are*

$$J_1 = 2\pi r_x \cos(m\theta_y) I_m(br_x) K_m(br_y), \quad (19)$$

$$J_2 = 2\pi r_x \cos(m\theta_{y*}) I_m(br_x) K_m(br_{y*}), \quad (20)$$

$$J_3 = 2\pi r_x \sin(m\theta_y) I_m(br_x) K_m(br_y), \quad (21)$$

$$J_4 = 2\pi r_x \sin(m\theta_{y*}) I_m(br_x) K_m(br_{y*}), \quad (22)$$

$$J_5 = 2\pi^2 r_x r_y I_m(br_x) K_m(br_y)\Delta_{mn}, \quad (23)$$

$$J_6 = 2\pi^2 r_x r_y I_m(br_x) I_n(br_y)[K_{n+m}(2bh) + K_{n-m}(2bh)], \quad (24)$$

$$J_7 = 2\pi^2 r_x r_y I_m(br_x) K_m(br_y)\Delta_{mn} \quad m \geq 1, \quad (25)$$

$$J_8 = 2\pi^2 r_x r_y I_m(br_x) I_n(br_y)[K_{n+m}(2bh) - K_{n-m}(2bh)], \quad (26)$$

$$J_9 = 0 \quad (27)$$

and

$$J_{10} = 0. \quad (28)$$

Proof of Lemma 2.1.
The Neumann addition theorem and Graf additions formulae will be used at many different stages in proving Lemma 2.1. They are expanded in Appendix B.

Proofs for J_1 and J_3. The Neumann addition theorem will be applied to the triangle XYC; X is on T_x and Y is any point in between T_x and H. Since in XYC the opposite angle to XY is θ_x-θ_y and CY is longer than CX, The Neumann addition theorem, in the present notation and after having scaled XYC sides by b, leads to

$$K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|) = I_0(br_x)K_0(br_y) + 2\sum_{k=1}^{\infty} I_k(br_x)K_k(br_y)\cos(k(\theta_x-\theta_y)). \quad (29)$$

Multiplying Equation (29) by $\cos(m\theta_x)$ and integrating over T_x gives, on the left hand side, J_1 as it is defined in Equation (5). On the right hand side Equation (19) is obtained using (a) the trigonometric identity $\cos(k(\theta_x-\theta_y)) = \cos(k\theta_x)\cos(k\theta_y) + \sin(k\theta_x)\sin(k\theta_y)$, (b) $ds(\mathbf{x}) = r_x d\theta_x$ and (c) the orthogonal integration properties of the product of trigonometric functions. Multiplying Equation (29) again by $\sin(m\theta_x)$ and integrating over T_x gives, on the left hand side, J_3 in Equation (7). On the right hand side Equation (21) is obtained as before by using the trigonometric angle sum and difference identities, and using the properties of the integration of the product of trigonometric functions.

Proofs for J_2 and J_4. The Neumann relation is rewritten considering the triangle CXY*. The opposite angle to XY* is θ_x-θ_{y*} and CY* is longer than CX. Hence,

$$K_0(b\,|\,\mathbf{x}-\mathbf{y}^*\,|) = I_0(br_x)K_0(br_{y*}) + 2\sum_{k=1}^{\infty} I_k(br_x)K_k(br_{y*})\cos(k(\theta_x-\theta_{y*})) \quad . (30)$$

Multiplying Equation (30) by $\cos(m\theta_x)$ and $\sin(m\theta_x)$ and integrating over T_x gives, on the left hand, J_2 in Equations (6) and J_4 in Equation (8) respectively after having exchanged **x*-y** with **x-y***, which is possible since they have equal lengths. On the right hand side Equations (20) and (24) are obtained, using again the trigonometric angle sum and difference identities and the properties of the integration of the product of trigonometric functions.

Proofs for J_5, J_7 and J_9. Multiplying J_1 in Equation (5) by $\cos(n\theta_y)$ and $\sin(n\theta_y)$ and integrating over the circle T_y leads, on the right hand side, to J_5 in Equation (13) and J_9 in Equation (17). On the left hand side, J_5 in Equation (23) and J_9 in Equation (27) are then obtained using the integration result for J_1 in Equation (19) and the properties of the integration of the product of trigonometric functions. J_7 in Equation (15) is obtained multiplying J_3 in Equation (7) by $\sin(n\theta_y)$ and integrating over the circle T_y. J_7 in Equation (25) is then obtained using the integration result for J_3 in Equation (21) and the properties of the integration of the products of trigonometric functions.

Proofs for J_6 and J_{10}. Multiplying J_2 in Equation (6) by $\cos(n\theta_y)$ and integrating over T_y gives J_6 in Equation (14). Using Equation (20) for J_2, J_6 becomes

$$J_6 = 2\pi r_x I_m(br_x)\int_{T_y} ds(\mathbf{x})\cos(n\theta_y)\cos(m\theta_{y*})K_m(br_{y*})\,. \quad (31)$$

The Graf cosine addition formula will be applied to the triangle YCC*. The angle C*CY is θ_y and is opposite to C*Y, whose length is equal to CY* and therefore is r_{y*}. Angle CC*Y is equal to the angle C*CY* and therefore is θ_{y*}. Angle CC*Y is opposite to CY whose length is r_y. The length of CC* is 2h and is greater than CY. Graf cosine addition formula in the present notation after having scaled the side of the triangle by b is

$$K_m(br_{y*})\cos(m\theta_{y*}) = \sum_{k=-\infty}^{\infty} I_k(br_y)K_{k+m}(2bh)\cos(k\theta_y)\,. \quad (32)$$

Multiplying Equation (32) by $2\pi r_x\cos(n\theta_y)I_m(br_x)$ and integrating over the circle T_y lead to (31) on the left hand side and to Equation (24) on the right hand side when using the properties of the integration of the product of trigonometric functions and the basic property of the modified Bessel functions (I_n=I_{-n} and K_n=K_{-n}).

Multiplying J_2 in Equation (6) by $\sin(n\theta_y)$ and integrating over T_y gives J_{10} in Equation (18). Using Equation (20) for J_2, J_{10} becomes

$$J_{10} = 2\pi r_x I_m(br_x)\int_{T_y} ds(\mathbf{x})\sin(n\theta_y)\cos(m\theta_{y*})K_m(br_{y*})\,. \quad (33)$$

Multiplying Equation (32) by $2\pi r_x\sin(n\theta_y)I_m(br_x)$, integrating over the circle T_y, using Equation (33) and the properties of the integration of the product of trigonometric functions give Equation (28).

Proof for J_8. Multiplying J_4 in Equation (8) by $\sin(n\theta_y)$ and integrating over T_y gives J_8 in Equation (16). Using Equation (22) for J_4, J_8 becomes

$$J_8 = 2\pi r_x I_m(br_x)\int_{T_y} ds(\mathbf{x})\sin(n\theta_y)\sin(m\theta_{y*})K_m(br_{y*})\,. \quad (34)$$

The Graf sine addition formula will be applied to the triangle YCC*, leading to

$$K_m(br_{y*})\sin(m\theta_{y*}) = \sum_{k=-\infty}^{\infty} I_k(br_y)K_{k+m}(2bh)\sin(k\theta_y)\,. \quad (35)$$

Multiplying Equation (35) by $2\pi r_x\sin(n\theta_y)I_m(br_x)$ and integrating over the circle T_y, lead to Equation (34) on the left hand side and to Equation (26) on the right hand side when using the properties of the integration of the product of trigonometric functions and the basic property of the modified Bessel functions.

3. Flow integrals.

The flow integrals are necessary to evaluate the amount of water that flows normally to the water table H or to a closed curve T_y that surrounds the tunnel, per unit time. The normal is denoted by **n** and points upward on H and toward the center on the circular curve T_y. The flow integrals have a different form depending on the flow conditions and the material behavior of the aquifer. Three of the flow integrals that cover a wide range of applications, including steady flow in a non-homogeneous aquifer and the transient flow in a homogeneous aquifer, are

$$F_1 = \int_{T_y} ds(\mathbf{y}) e^{\mathbf{b}.\mathbf{y}} (\nabla\Omega - \mathbf{b}\Omega).\mathbf{n} , \tag{36}$$

$$F_2 = \int_{T_y} ds(\mathbf{y})\mathbf{n}.\nabla\Omega , \tag{37}$$

and

$$F_3 = \int_{H} ds(\mathbf{y})\mathbf{n}.\nabla\Omega . \tag{38}$$

In F_1, **b** is a constant vector field whose modulus is b; the angle it makes with the vertical Cartesian coordinate axis is denoted by θ_b. The flow integrals F_1 to F_3 may be evaluated exactly considering the sine and cosine coefficients of the single layer as

$$\begin{aligned} F_1 &= 4\pi^2 r_x e^{-bh\cos\theta_b} [\varphi_0^c I_0(br_x) \\ &\quad + \sum_{n=1}^{\infty} (\varphi_n^c \cos(n\theta_b) + \varphi_n^s \sin(n\theta_b)) I_n(br_x)], \end{aligned} \tag{39}$$

$$\begin{aligned} F_2 &= 4\pi^2 r_x r_y b[\varphi_0^c I_0(br_x) K_1(br_y) \\ &\quad + I_1(br_y) \sum_{n=0}^{\infty} \varphi_n^c I_n(br_x) K_n(2bh)] \end{aligned} \tag{40}$$

and

$$F_3 = -4\pi^2 r_x e^{-bh} [\varphi_0^c I_0(br_x) + \sum_{n=1}^{\infty} \varphi_n^c I_n(br_x)] . \tag{41}$$

The proofs for Equations (39) to (41) are straightforward after having proven the following lemma.

LEMMA 3.1. *The flow integrals F_1, F_2 and F_3 may be transformed into integrals on the tunnel edge with an integrand that considers the single layer instead of the state function as*

$$F_1 = 2\pi \int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) e^{\mathbf{b}.\mathbf{x}} , \tag{42}$$

$$F_2 = 2\pi b r_y \int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) [I_0(br_x) K_1(br_y) + I_1(br_y) K_0(br_{x^*})] \tag{43}$$

and

$$F_3 = -2\pi \int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) e^{bx_2} . \tag{44}$$

Proof of Lemma 3.1.

Proof for F_1. K_0 is a fundamental 2D solution of the modified Helmholtz equation, so

$$\nabla^2 K_0(b|\mathbf{x}-\mathbf{y}|) - \mathbf{b}^2 K_0(b|\mathbf{x}-\mathbf{y}|) = -2\pi\delta(\mathbf{x}-\mathbf{y}) . \tag{45}$$

This latter leads to

$$\nabla . e^{\mathbf{b}.\mathbf{y}} (\nabla K_0(b|\mathbf{x}-\mathbf{y}| - \mathbf{b} K_0(b|\mathbf{x}-\mathbf{y}|) = -2\pi e^{\mathbf{b}.\mathbf{y}} \delta(\mathbf{x}-\mathbf{y}) . \tag{46}$$

Integrating Equation (46) on the disc with boundary T_y, applying Gauss theorem and remembering that the exterior normal to the disc is **–n** leads to

$$\int_{T_y} ds(\mathbf{y}) e^{\mathbf{b}.\mathbf{y}} \mathbf{n}.(\nabla K_0(b|\mathbf{x}-\mathbf{y}| - \mathbf{b} K_0(b|\mathbf{x}-\mathbf{y}|) = 2\pi e^{\mathbf{b}.\mathbf{x}} . \tag{47}$$

When changing **x** into **x***, Equation (46) remains valid but the integral in Equation (47) is zero since **x*** is outside of the disc and its boundary T_y. The proof that F_1 is given by Equation (42) is completed after having inserted the integral representation of the state function in the definition of F_1 (Equation (36)), and considering Equation (47) and the comment that follows it.

Proof for F_2. Inserting the integral representation of the state function in the definition of F_2 gives

$$F_2 = \int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) \int_{T_y} ds(\mathbf{y}) \mathbf{n}.\nabla (K_0(b|\mathbf{x}-\mathbf{y}| - K_0(b|\mathbf{x}^*-\mathbf{y}|) . \tag{48}$$

Rewriting the gradient in Equation (48) using polar coordinates gives

$$F_2 = -\int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) r_y \frac{\partial}{\partial r_y} \int_{-\pi}^{\pi} d\theta_y (K_0(b|\mathbf{x}-\mathbf{y}| - K_0(b|\mathbf{x}^*-\mathbf{y}|) . \tag{49}$$

The integrals on the extreme right of (49) are deduced from J_1 and J_2 with m=0 leading to

$$F_2 = -2\pi r_y \int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) \frac{\partial}{\partial r_y} [I_0(br_x) K_0(br_y) - I_0(br_y) K_0(br_{x^*})] . \tag{50}$$

The proof that F_2 is also given by Equation (43) is completed remembering that the derivative of K_0 is $-K_1$ and of I_0 is I_1.

Proof for F_3. Inserting the integral form of the state function in the definition of F_3 gives

$$F_3 = \int_{T_x} ds(\mathbf{x}) \varphi(\mathbf{x}) \int_{H} ds(\mathbf{y}) \mathbf{n}.\nabla (K_0(b|\mathbf{x}-\mathbf{y}| - K_0(b|\mathbf{x}^*-\mathbf{y}|) . \tag{51}$$

On H the normal **n** is (0,1) and the following relations are obtained

$$\mathbf{n}.\nabla K_0(b\,|\,\mathbf{x}-\mathbf{y}\,|) = -\frac{\partial}{\partial x_2} K_0(b\sqrt{(x_1^2 - y_1^2) + x_2^2}) \quad (52)$$

and

$$\mathbf{n}.\nabla K_0(b\,|\,\mathbf{x}^*-\mathbf{y}\,|) = \frac{\partial}{\partial x_2} K_0(b\sqrt{(x_1^2 - y_1^2) + x_2^2})\,. \quad (53)$$

The gradient on the left hand sides of equations (52) and (53) is driven by **y** after y_2 is changed to zero. Inserting Equations (52) and (53) in Equation (51) leads to

$$F_3 = -2\int_{T_x} ds(\mathbf{x})\varphi(\mathbf{x}) \frac{\partial}{\partial x_2} \int_{-\infty}^{\infty} dx_1 K_0(b\sqrt{(x_1^2 - y_1^2) + x_2^2})\,. \quad (54)$$

The integral on the extreme right of Equation (44) is given in [8] (section 6.677, 5th integral) and is

$$\int_{-\infty}^{\infty} dx_1 K_0(b\sqrt{x_1^2 + x_2^2}) = \frac{\pi}{b} e^{-b|x_2|} \quad . \quad (55)$$

The proof that F_3 is also given by Equation (44) is completed after having inserted Equation (55) in (54) and remembering that x_2 is negative on T_x.

4. Ultimate form.

Lemma 2 is used to prove that inserting the Fourier series of the single layer in F_1, F_2 and F_3 lead to Equations (39), (40) and (41) respectively.

Proof for F_1. Inserting in Equation (42) the Fourier series of the single layer leads to Equation (39) using the following relation

$$\mathbf{b}.\mathbf{x} = -hb\cos\theta_b + br_x\cos(\theta_x - \theta_b) \quad (56)$$

and using the following integral that is given in [8] (section 3.937, 2nd integral)

$$\int_{-\pi}^{\pi} d\theta\, e^{b\cos\theta} \cos n\theta = 2\pi I_n(b)\,. \quad (57)$$

Proof for F_2. When inserting the Fourier series of the single layer in Equation (43) there are two integrations to undertake. The first is obtained easily. The second is

$$\int_{T_x} ds(\mathbf{x})[\varphi_0^c + \sum_{n=1}^{\infty}(\varphi_n^c \cos n\theta_x + \varphi_n^s \sin n\theta_x)]K_0(br_{x^*}) = 2\pi r_x \sum_{n=0}^{\infty} \varphi_n^c I_n(br_x) K_n(2bh)\,. \quad (58)$$

The proof for Equation (58) requires the application of the Neumann theorem to the triangle CC*X*. The opposite angle to CC* is CX*C* which is equal to angle C*XC and is θ_x. CC* is 2h and longer than C*X* which is equal to r_x. In the present notation the Neumann relation is

$$K_0(br_{x^*}) = I_0(br_x)K_0(2bh) + 2\sum_{m=1}^{\infty} I_m(br_x)K_m(2bh)\cos(m\theta_x)\,. \quad (59)$$

Inserting Equation (59) on the left hand side of Equation (58) and using the properties of the integration of the product of trigonometric functions complete the proof that F_2 is given by Equation (40).

Proof for F_3. The proof that F_3 is given by Equation (41) is obtained after having inserted the Fourier series for the single layer in Equation (44) and using Equation (57).

5. Application.

In practice the flow is symmetric with respect to the tunnel axis, the aquifer properties change with depth and the state function is restricted on the tunnel edge. The analytical solution is then greatly simplified. First, the single layer has zero sine coefficients. Secondly, the angle of the vector field **b** is zero. Finally, the radius r_y and r_x are equals and will be written identically without the index as r. Considering a truncated single layer limited to its first constant term, the following approximations for the state function and the flow integrals are deduced

$$\Omega = \omega_0^c \frac{K_0(br_y) - K_0(br_{y^*})}{K_0(br) - I_0(br)K_0(2bh)}\,, \quad (60)$$

$$F_1 = 2\pi e^{-bh}\omega_0^c \frac{1}{K_0(br) - I_0(br)K_0(2bh)}\,, \quad (61)$$

$$F_2 = 2\pi r b\omega_0^c \frac{K_1(br) + I_1(br)K_0(2bh)}{K_0(br) - I_0(br)K_0(2bh)} \quad (62)$$

and

$$F_3 = -F_1\,. \quad (63)$$

In Equation (60) r_y is not the radius of T_y but the modulus of CY; Y is any point in the aquifer. The limit case when b approaches zero gives the analytical approximations for a homogeneous aquifer. Using the first expansion term of the modified Bessel functions and of the exponential in Equations (60) to (63) give the following approximated equations for the state function and the flow integrals

$$\Omega = \frac{\omega_0^c}{\ln\frac{2h}{r}} \ln\frac{r_{y*}}{r_y}, \tag{64}$$

$$F_1 = 2\pi \frac{\omega_0^c}{\ln\frac{2h}{r}} \tag{65}$$

and

$$F_2 = F_1 = -F_3 \ . \tag{66}$$

Equation (64) may be identified with the Muskat potential of a point source and its image in a homogeneous aquifer after considering a mean pressure change on the tunnel edge given by ω_0^c. Equation (65) is the Muskat-Goodman water inflow equation, provided that it is divided by the hydraulic conductivity and that the same mean pressure change ω_0^c on the tunnel edge is considered. Equation (66) is the obvious fact that in a homogeneous aquifer in the steady state F_1 and F_2 are equal and that the water flowing across the water table F_1 is the water that flows out into the tunnel F_3.

6. Conclusion.

Many sets of integrals containing Bessel functions are generated when, in the state function and the flow integrals of a non-homogeneous aquifer drained by a circular tunnel, the single layer is written as a Fourier series. All of the generated sets of integrals were calculated exactly and are now available for analytical calculations. It has been observed that parities do not mix but the modes of identical parities do.

Even though the analytical solution is found in a closed form, it needs to be improved by finding new series for which the modes do not mix or equivalently remain orthogonal. Actually, the analytical solution may serve in engineering practice. First, it is helpful in controlling the output of numerical codes and verifying if they are able to calculate the flow and the state function of a non-homogeneous aquifer correctly. Secondly, truncating the solution to its first order term provides approximated practical formulae which estimate the water potential and the amount of water that is flowing in and out of the aquifer.

The approximated formulae that are obtained, truncating the single layer are extensions to non-homogeneous aquifer of the Muskat potential and the Muskat-Goodman water inflow, which are commonly used for homogeneous aquifer in everyday engineering practice.

7. References and Bibliography.

[1] El Tani, M., 2007. Hydrostatic paradox of saturated media. Geotechnique, 57, 9, 773–777.

[2] Kolymbas, D. & Verruijt A., 2008. Discussion on: Hydrostatic paradox of saturated media. Geotechnique, 58, 10, 835–837.

[3] Snow, T. 1968. Hydraulic character of fractured metamorphic rocks of the Front Range and implications to the rocky mountain arsenal well, Quarterly Colorado School Mines, 63, 167-199.

[4] El Tani, M., 1999. Water inflow into tunnels. Challenges for the 21st Century, Proceedings of the World Tunnel Congress 99, Oslo, pp. 61–70, Balkema.

[5] El Tani, M., 2003. Circular tunnel in a semi-infinite aquifer. Tunnelling and underground space technology, 18, 1, 49–55.

[6] Park K H, Owatsiriwonga A. & Leeb J-G, 2008. Analytical solution for steady-state groundwater inflow into a drained circular tunnel in a semi-infinite aquifer: A revisit. Tunnelling and underground space technolgy, 23, 206-209.

[7] El Tani, M., 2010. Helmholtz evolution of a semi-infinite aquifer drained by a circular tunnel. Tunnelling and underground space technology, 25, 1, 54–62.

[8] Gradshteyn, I.S. & Ryzhik I.M. 1996. Table of Integrals, Series and Products, 5th edition, Edited by Alan Jeffrey. Academic Press.

[9] Watson, G.N. 1944. A treatise on the theory of Bessel functions. Cambridge university press.

Appendix A: Notation

Latin

b	Intensity of non-homogeneity (b=0 homogeneous; b>0 non-homogeneous)
b	Non-homogeneity vector (b is its modulus)
C	Tunnel center and pole of the polar coordinates
C_{mn}	Matrix coefficient
H	Water table
h	Distance of the tunnel center to H
I_n	Modified Bessel function of the 1st kind of order n
K_n	Modified Bessel function of the 2nd kind of order n
J_1 to J_{10}	Simple and double integrals of the product of modified Bessel functions and trigonometric functions

n Exterior normal
O Origin of the Cartesian coordinates
F_1, F_2, F_3 Flow integrals
r radius of the tunnel when $T_x = T_y$
r_x radius of the tunnel T_x
r_y radius of T_y
r_x, r_{x*} Lengths of CX, CX*
r_y, r_{y*} Lengths of CY, CY*
S_{mn} Matrix coefficient
T_y Circle of radius r_y and center C
T_x Tunnel edge with radius r_x and center C
x_1, x_2 Horizontal and vertical Cartesian coordinates

Greek symbols
Δ_{mn} Modified Kronecker ($\Delta_{00} = 2$, $\Delta_{mm} = 1$ for m>0, $\Delta_{mn} = 0$ for m different to n)
δ Dirac delta distribution
θ_x, θ_{x*} Angles OCX, OCX*
θ_y, θ_{y*} Angles OCY, OCY*
Ω State function
φ Single layer
φ^c_n , φ^s_n Cosine and sine Fourier coefficients of φ
ω Assigned function to Ω on T_y
ω^c_n , ω^s_n Cosine and sine Fourier coefficients of ω
∇ Gradient operator

Appendix B: *Neumann addition theorem and Graf additions formulae*

For a triangle with sides a, b and c and respective opposite angles A, B and C, the Graf cosine addition formula, when b>c or b=c, is (See Watson [9])

$$K_m(a)\cos(mC) = \sum_{n=-\infty}^{\infty} K_{n+m}(b) I_n(c)\cos(nA) \quad b \geq c\,. \quad (67)$$

The Graf sine addition formula is obtained from The Graf cosine addition formula exchanging the cosine with the sine.

The Neumann addition formula is a particular case of Graf cosine addition considering m=0

$$K_0(a) = K_0(b) I_0(c) + 2\sum_{n=1}^{\infty} K_n(b) I_n(c)\cos(nA) \quad b \geq c\,. \quad (68)$$

In the literature the Neumann addition formula is also called the Neumann addition theorem.