Option price sensitivity to errors in stochastic dynamics modeling

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Abstract

When asset prices are modelled by stochastic dynamics, the model parameters are estimated from financial data. We study how estimation errors on model parameters impact the computed option prices, in the case where asset price and volatility follow the classical joint stochastic differential equations (SDEs) parametric model of Heston. Model parameters are estimated by an approximate maximum likelihood approach studied in [4] which presented an implementable computation of the covariance matrix for the errors in model parameters estimation. We then study and compute the sensitivity of optimal option prices to errors on the model parameters. This is achieved by numerically solving the partial differential equations (PDEs) verified by the derivatives of the option price with respect to model parameters. Combining these evaluations of derivatives with the computed covariance matrix of errors on model parameters, we obtain the errors on option price due to parametric estimation errors. We apply our method to the Standard & Poor's (S&P) 500 index options using the implied volatility index (VIX) [29] as a proxy for volatility.

1 Introduction

It is common practice in finance applications to model the price of securities and other assets using stochastic differential equations. The models are estimated using available data and are then used for forecasting or for computing the price of financial instruments.

We study the impact of errors in model estimation on the price of options where the underlying asset is assumed to verify the estimated model. We study how errors on each individual parameter affect option prices. We compute a bound on the error in the option price due to model error. Our study of parametric error sensitivity does not depend on the choice of method used for the calibration of the asset pricing model.

Extensions of the Black-Scholes models to local volatility ([8],[9]) and stochastic volatility models ([19],[25]) are often used in practice. Local volatility models allow for the instantaneous volatility to be a deterministic function of time and the asset price. In stochastic volatility models the volatility is driven by a diffusion process. We focus our study on the classical Heston stochastic volatility models, driven by two coupled SDEs.

Volatilities are not directly observed in the market, and the transition density of the coupled process is not available in closed form. We estimate the Heston model parameters using an efficient approximate maximum likelihood method introduced in [4], where variances of these parameter estimators were also computed. Here we outline the numerical computation of the derivatives of option prices with respect to each model parameter by writing and solving associated parabolic PDEs. We also show how to compute derivatives of the option price with respect to the so called "market price" of volatility risk.

We apply our methods to the case of S&P 500 index options where we use VIX as a proxy for volatility.

2 Stochastic volatility model

Consider the price process $\{X_t\}$ of a tradable asset (stock or index). Let V_t be its instantaneous volatility and $Y_t = V_t^2$. Following Heston [19] we assume that $\{X_t\}$ and $\{Y_t\}$ follow a pair of correlated diffusion processes given by,

$$dX_t = \mu X_t dt + \sqrt{Y_t} X_t d\bar{W}_1(t), \quad (2.1)$$
$$dY_t = \kappa(\theta - Y_t)dt + \gamma\sqrt{Y_t} d\bar{W}_2(t), \quad (2.2)$$

where $\{\bar{W}_1(t),\ \bar{W}_2(t)\}$ is a pair of correlated Brownian motions with $E[d\bar{W}_1(t),\ d\bar{W}_2(t)] = \rho dt$ ($|\rho| < 1$). Denote the logarithm of the stock price by $L_t = \ln(X_t)$, so that one has,

$$dL_t = (\mu - \frac{1}{2}Y_t)dt + \sqrt{Y_t} d\bar{W}_1(t). \quad (2.3)$$

The parameter μ is the mean rate of return of the asset price. For parameters $\kappa,\ \theta > 0$ Equation (2.2) is the well known mean reverting square root process ([12],[7]), originally used to model short term interest rates. The process Y_t is pulled towards the long run mean θ with speed of mean reversion κ. The parameter γ is the instantaneous relative volatility of $\{Y_t\}$. The transition density of $\{Y_t\}$ is a non central chi-square

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and its stationary distribution is a gamma distribution with mean θ and variance $\frac{\gamma^2\theta}{2\kappa}$.

We assume that $2\kappa\theta > \gamma^2$, a classical condition under which the upward drift in Equation (2.2) keeps $\{Y_t\}$ positive for all t [12] when $\{Y_0\} > 0$. Note that for $2\kappa\theta = \gamma^2$, Equation (2.2) is an Ornstein-Uhlenbeck process.

2.1 Model identification To identify parameters in the model equations (2.1)-(2.2) for best fit with a set of available jointly observed data, [1] used a Hermite approximation of the likelihood, [2] developed a weighted non parametric approach to determine the risk-neutral measure of future market states, [22] proposed a Bayesian estimation.
Here we use a very efficient parameter estimation technique introduced in [4], based on Euler discretization of the joint SDEs model; we solve first a constrained maximum likelihood estimation problem to identify the parameters in Equation (2.2). The parameter μ is then obtained by maximum likelihood based on Equation (2.1) only. Once the parameters μ, κ, θ, γ are estimated we reconstruct the underlying Brownian motions and then directly estimate their correlation coefficient ρ. This decoupling of the SDEs does not reduce the accuracy of the parameter estimates as compared to fully joint estimation, and provides a strong numerical gain, since all estimators can be explicitly computed. The natural domain for the unknown parameters is defined by the following constraints,

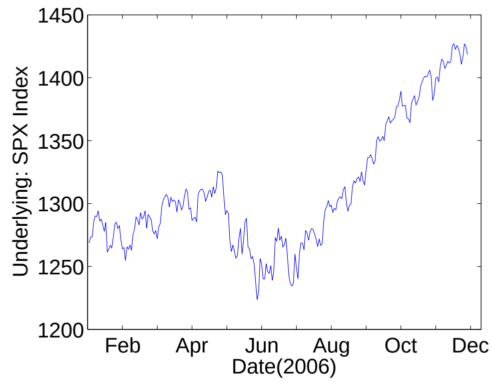
$$\kappa,\, \theta,\, \gamma > 0, \quad 0 < \theta < 1, \quad 2\kappa\theta > \gamma^2, \quad -1 \le \rho \le 1.$$

We now present our parameter identification for the modelling of observed S&P 500 index data.

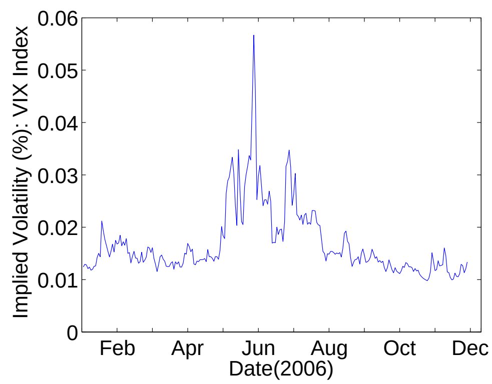
2.2 Modeling S&P 500 data by Heston SDEs We estimate the Heston stochastic volatility model for the S&P 500 index data, denoted by their ticker symbol SPX. We use the VIX index as a proxy for volatility, since the VIX values represent the implied volatility of a group of options on the S&P 500 , short dated at the money, and is officially computed by the CBOE [29].

We use 252 daily S&P 500 index data from Jan 03 to Dec 29 2006. To maintain the simultaneity of SPX and VIX we use closing data for both. Following standard practice, the time step Δ between two consecutive daily (SPX,VIX) observations is equal to 1/252. The evolution of SPX and VIX over this period is displayed in Fig. 1(a) and Fig. 1(b).

The parameter values are estimated by the algorithm outlined above [4] and displayed in Table 1, Col. 2.



(a) The SPX (S&P 500) index is the asset price value $\{X_t\}$.



(b) The VIX index represents the volatility value for options on the S&P 500 index.

Figure 1: Asset price and volatility.

2.3 Empirical variances of estimators We compute the mean bias, standard deviation, and relative errors of our parameter estimators. Here relative error is the ratio of standard deviation of the estimator to the true parameter value. To compute these quantities we simulate 1000 trajectories of length $N = 252$ for the Heston joint SDEs with the estimated parameter values as true values and then obtain the mean bias, standard deviation and relative error over the 1000 trajectories. Columns 3, 4 and 5 of Table 1 display these quantities. The Monte Carlo simulations technique we have used to evaluate empirical bias and variance of our parameter estimators could naturally be applied to any other choice of parameter estimation technique for Heston model calibration. However our choice [4] of parameter estimators based on constrained maximum likelihood of the discretized model is nearly optimal since the corresponding estimators are asymptotically efficient (see [4]).

The fairly large dispersion of the μ-estimate is due

Parameter	Estimate	Bias	Std	Relative Error
κ	16.60	1.00	2.87	16%
θ	.017	.000	.002	13%
γ	.283	-.002	.014	5%
ρ	-.544	.004	.059	10%
μ	.135	.033	.084	60%

Table 1: Estimated parameters for a Heston model fitted to the (SPX,VIX) one year dataset ($N = 252$). Column 2 = parameter estimates. Column 3 = Empirical Mean Bias, Column 4 = Empirical Standard Deviation, Column 5 = Standard Deviation / True Value.

to the fairly high fluctuations of volatility estimates when $N = 252$. Fortunately, as is known from the no arbitrage pricing theory, the risk neutral value of the option price does not depend directly on the parameter μ. Therefore we will not need to include the parameter μ in our sensitivity analysis. We will refer to

$$PAR = (\kappa, \theta, \gamma, \rho)$$

as the vector of four key model parameters for our sensitivity study. Then $PAR \in \Omega_{PAR} \subset \mathbb{R}^4$ where

$$\begin{aligned} \Omega_{PAR} \quad &= \quad \{PAR \in \mathbb{R}^4 : \kappa, \theta, \gamma > 0, \\ &\qquad \gamma^2 < 2\kappa\theta,\ |\rho| < 1\}. \end{aligned} \tag{2.4}$$

Let $P = [\kappa, \theta, \gamma, \rho]$ be the true value of the parameter vector PAR and let $\hat{P}_N = [\hat{\kappa}, \hat{\theta}, \hat{\gamma}, \hat{\rho}]$ be the maximum likelihood estimator of P (see [4]) based on N observations. Let Σ_N be the 4×4-covariance matrix of the parameter estimators $\hat{P}_N$, so that

$$\Sigma_N(i,j) = cov(\hat{P}_N(i), \hat{P}_N(j)),\ i, j = 1, 2, 3, 4.$$

These estimators are asymptotically unbiased, and (see [3]) as the number of observations N tends to ∞, the covariance matrix Σ_N has the following equivalent

$$\Sigma_N = cov(\hat{P}_N - P) \sim \frac{1}{N} L(P),$$

where the fixed 4×4-matrix $L(P)$ can be explicitly computed (see [4]). Hence the variances and covariances of our estimators tend to zero as N tends to ∞, at speeds proportional to $\frac{1}{N}$, as confirmed by our numerical results.

We give an empirical estimate of the matrix L for the model parameters of the S&P 500 data, by $L \sim N\Sigma_N$ for $N = 10,000$, where Σ_N is the empirical covariance matrix of $\hat{P}_N$ estimated from 1000 simulated trajectories of length $N = 10,000$ for the joint Heston SDEs. For $N = 10,000$, the covariances of the parameter estimators are all almost zero, and the standard deviations of the estimators $\hat{P}_N$ are given by

$$\text{stand.dev}(\hat{\kappa}) = 0.87, \quad \text{stand.dev}(\hat{\theta}) = 0.0003,$$

$$\text{stand.dev}(\hat{\gamma}) = 0.002, \quad \text{stand.dev}(\hat{\rho}) = 0.009.$$

As N increases from 252 to 10,000, standard deviations approximately do decrease by the theoretical multiplicative factor $\sqrt{\frac{10,000}{252}}$. The limiting matrix L, approximated by $N\Sigma_N$ for $N = 10000$, is given by

$$L = \begin{pmatrix} 7570 & -1.3 & .1 & -.1 \\ -1.3 & .001 & 0 & -.001 \\ .1 & 0 & .043 & -.094 \\ -.1 & -.001 & -.094 & .885 \end{pmatrix}$$

From Fig. 1 we see that when the value of the S&P 500 index falls the value of the VIX index increases. This explains the negative correlation coefficient (ρ) for this (SPX,VIX) dataset, and is called the *leverage effect* in financial literature.

For parameter fitting of the model equations (2.1)-(2.2) to given sets of daily data, one has to fix a time unit for the variable "t", which is equivalent to selecting a value Δ for the time period between two successive daily observations. Usually one sets $\Delta = 1/252$ for daily data, but we point out that for our parameter estimators, the relative error sizes, defined by the ratios of standard deviations over true values, are independent of the choice of Δ.

3 Option pricing

A European call (resp. put) option is a financial contract which gives its holder the right to buy (resp. sell) the underlying asset, at a future fixed "maturity" date T, at a predetermined "strike price" K, fixed at the time the option is created. The option holder can exercise his right to buy (or sell) at maturity only. However the option can be traded at anytime between its creation and maturity.

The European call option with maturity T is characterized by its payoff function,

$$\Phi(X_T) = (X_T - K)^+ = \max(X_T - K, 0).$$

Let $\{Z_t, 0 \le t \le T\}$ be the price of a European call option with maturity T. It is standard to assume that under stochastic volatility the price Z_t is a function of X_t and Y_t [19],

$$Z_t = f(X_t, Y_t, t).$$

The theory of no arbitrage pricing implies that the option price is computed under the so called equivalent martingale measure (EMM). The discounted price of any traded asset is a martingale under this measure. The dynamics of $\{X_t,\, Y_t\}$ under the EMM is given by [18],

$$dX_t = rX_t dt + \sqrt{Y_t} X_t dW_1(t), \tag{3.5}$$

$$dY_t = \kappa((\theta - Y_t) - \lambda_t \gamma \sqrt{Y_t})dt + \gamma\sqrt{Y_t} dW_2(t). \tag{3.6}$$

The dynamics for the log of the stock price L_t under the risk neutral measure is given by,

$$dL_t = (r - \frac{1}{2}Y_t)dt + \sqrt{Y_t} dW_1(t), \tag{3.7}$$

where $E[dW_1(t),\ dW_2(t)] = \rho dt$. The parameter r is a risk free short term rate of return assumed to be constant, for instance, short term yield of U.S. treasury bonds.

The random variable λ_t is a function of the market price of volatility risk and arises due to the non tradability of the volatility. The market is said to be incomplete in this case. For equity options [23] presents evidence of non zero market price of volatility risk.
Define

$$g(X_t, Y_t, t) = f(X_{T-t}, Y_{T-t}, T-t), \quad 0 \le t \le T.$$

Then g satisfies the following parabolic PDE ([5],[19]),

$$\frac{\partial g}{\partial t} - Lg = 0 \text{ on } (0,\infty) \times (0,\infty) \times (0,T] \tag{3.8}$$

with initial condition,

$$g(x,y,0) = (x-K)^+ \text{ on } (0,\infty)\times(0,\infty)\times\{t=0\},$$

where L is the linear operator,

$$\begin{aligned} L = \quad & rx\frac{\partial}{\partial x} + [\kappa(\theta - y) - \lambda\sqrt{y}\gamma]\frac{\partial}{\partial y} + \frac{1}{2}x^2 y \frac{\partial^2}{\partial x^2} \\ & + \frac{1}{2}\gamma^2 y \frac{\partial^2}{\partial y^2} + \rho\gamma x y \frac{\partial^2}{\partial x \partial y} - r. \end{aligned}$$

We have assumed λ_t to be constant similar to [13]. We now describe the boundary conditions satisfied by the option price [19]. The boundary conditions remain unchanged after the transformation in the time variable. When the asset price is zero there is no rational interest in buying the call option and therefore the option is worthless,

$$g(0,y,t) = 0 \text{ on } \{x=0\}\times(0,\infty)\times(0,T].$$

For large values of the asset the option price grows linearly with the asset price,

$$\lim_{x\to\infty} \frac{\partial g(x,y,t)}{\partial x} = 1, \forall y,\ t.$$

For large values of Y_t the option price tends to be constant as a function of the square of volatility,

$$\lim_{y\to\infty} \frac{\partial g(x,y,t)}{\partial y} = 0, \forall x,\ t.$$

The boundary condition at $y = 0$ is obtained by setting y to be zero in the PDE (3.8),

$$\frac{\partial g}{\partial t} - rx\frac{\partial g}{\partial x} - \kappa\theta\frac{\partial g}{\partial y} + rg = 0, \tag{3.9}$$

on $(0,\infty)\times\{y=0\}\times(0,T]$. For all practical applications we can consider the problem on the bounded domain,

$$U_T = U_B \times (0,T],$$

where

$$U_B = (0, X_{max}) \times (0, Y_{max}).$$

The option pricing problem is a two dimensional second order parabolic partial differential equation where the operator L is elliptic for each t in x and y. More precisely it is a convection-diffusion type equation with Dirichlet boundary at $x = 0$ and Neumann boundary as x and y approach X_{max} and Y_{max} respectively [17]. The boundary condition at $y = 0$ is not of a standard type. We refer to Equation (3.8) together with the initial and boundary conditions as an initial/boundary-value problem [11]. The article [10] shows that the option price given by the stochastic representation formula is a classical solution of the PDE (3.8) and satisfies the boundary condition at vanishing volatility.

3.1 Market price estimates for volatility risk

We need an estimate of λ in order to solve the pricing equation. According to the no arbitrage theory, λ should be estimated from one volatility dependent asset and used to price all other volatility dependent assets. Considering a stochastic market price of volatility risk λ_t would amount to introducing a third order model where the volatility of volatility is also stochastic. This is not the purpose of this study.[1] Among the existing literature on this subject, [13] uses and estimates a constant λ for stochastic volatility models based on observed option prices. As suggested in [19] one could estimate the unknown λ using average returns on option positions that are hedged against the risk of changes in the asset price.

In a parallel paper [30], two of the authors introduce and study a new algorithm to estimate the market price λ of volatility risk for a tradable asset X_t, derived from time dependent Black-Scholes approximations of the jointly observed dynamics of several options. We

[1] Pointed out by a referee.

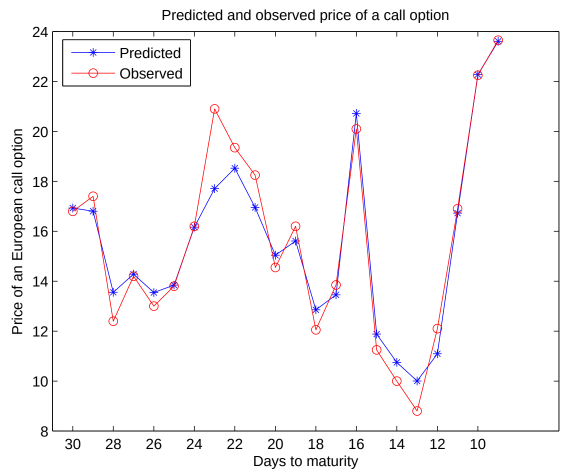


Figure 2: Predicted and observed option price of a European call option with strike price 1430 for $\lambda = 2$.

have applied this method to the (SPX,VIX) dataset to estimate the corresponding λ. The results on the sensitivity of the option price with respect to λ are presented in Section 7. We conclude that for options with strike price close to the asset price (near the money options) the relative change in option price due to small changes in λ is less than 5% (see Table 3). Hence we chose and estimated a constant λ for this study.

From a group of short maturity options with strike price close to the prevailing SPX values, we get an estimate $\lambda = 2$, validated by simultaneously comparing the mean squared error in time between predicted and actual option prices for a range of different λ values. We report concrete results here for one specific European option EOP_t with strike price 1430 and maturity date Feb 17 2007, observed daily from Jan 03 2007 to Feb 02 2007 .

We used the estimated model parameters of Table 1 for the dynamics of the asset price, and we set the value of the risk free rate of return at $r = .01$. Using the estimated $\lambda = 2$, we solved the option price PDE (3.8) to compute dynamic estimates $E\hat{O}P_t$ of the option price EOP_t, and we derived the mean value 0.66 and standard deviation 0.96 of the prediction error $E\hat{O}P_t - EOP_t$ for the option price, which had a mean value of 15.63 during this period. Fig. 2 represents the computed and market price of this European option EOP_t.

To study, in the next sections, how sensitive the option price is to the model parameters PAR and the impact of model estimation errors on the option price, we fixed the value $\lambda = 2$ just estimated from market data.

4 Modeling errors : impact on option price

In this section we compute a bound on the error in the option price generated by random errors on parameters of the joint Heston SDEs. These random errors are due to the estimation of the model parameters from observed data on asset price and volatility. The option price $g(x, y, t)$ as a solution of the PDE (3.8), depends on all the model parameters except μ. We will use both an extended and a shorthand notation for g, defined by

$$g(x, y, t; PAR, \lambda) = g(PAR) = g(x, y, t),$$

where $PAR = (PAR_i)_{i=1}^4$ is the vector,

$$PAR_1 = \kappa, \quad PAR_2 = \theta, \quad PAR_3 = \gamma, \quad PAR_4 = \rho.$$

It can be proved that $g(PAR)$ is a smooth function of the parameter vector PAR in the domain Ω_{PAR} defined above by Equation (2.4).

Let P be the true but unknown value of the parameter vector PAR of the model equations (2.1)-(2.2). We expect P to be in a neighborhood of the parameter estimator $Q = \hat{P}_N$, where N is the number of daily observations of the pair (asset price, volatility) used to compute $Q = \hat{P}_N$. For all $i = 1, 2, 3, 4$ define the partial derivatives of the option price $g(P)$ at P by

$$D_{i,P} = D_i(g)(x, y, t; PAR, \lambda) = \frac{\partial g}{\partial PAR_i}(x, y, t; PAR, \lambda)$$

and denote by D_P the corresponding gradient of $g(P)$, viewed as a column vector in $\mathbb{R}^4$. A first order Taylor expansion of $g(PAR)$ at $PAR = P$ yields

$$g(Q) \simeq g(P) + \sum_{i=1}^{4} D_{i,P}.(Q_i - P_i). \tag{4.10}$$

As noted in Section 2.3, as $N \to \infty$, the covariance matrix $Cov(Q - P) = \Sigma_N$ is equivalent to $(1/N)L(P)$, where the deterministic symmetric non negative matrix $L(P)$ does not depend on N but only on the true parameter values P, and can be estimated empirically by intensive simulations for each given P.

The option price error induced by errors in parameter estimation has a mean quadratic error which depends on P and is defined by

$$\delta(P)^2 = E[(g(Q) - g(P))^2].$$

Since our parameter estimators are asymptotically unbiased, squaring both sides of Equation (4.10) and taking expectation with respect to the distribution of $Q = \hat{P}_N$ yields

$$\delta(P)^2 \simeq \frac{1}{N} D_P^* L(P) D_P. \tag{4.11}$$

The true P is unknown, but given the estimated Q, we expect that with high probability, the unknown P will belong to a "confidence neighborhood" of Q, denoted $B(Q) \subset \Omega_{PAR} \subset \mathbb{R}^4$. When P varies within $B(Q)$, the correct option price $g(P)$ and the option price $g(Q)$ computed from the estimated Heston model differ by a random error of size $|g(Q) - g(P)|$, which has a deterministic L_2-size $\delta(P)$ computed by Equation (4.11). To evaluate the sensitivity of the computed option price to errors in the estimated parameter values Q, we need to compute the maximum $\varepsilon(Q)$ of $\delta(P)$ for arbitrary $P \in B(Q)$.

A typical "best" choice of $B(Q)$ would be the ellipsoid centered at Q and defined by the covariance matrix Σ_N. For large N, Σ_N is equivalent to $(1/N)L(P)$, and the off diagonal entries of the covariance matrix $(1/N)L(P)$ are negligible. Hence, in practical applications, we define the confidence neighborhood $B(Q)$ by

$$B(Q) = B_1 \times B_2 \times B_3 \times B_4$$

with

$$B_i = [Q_i - \frac{1}{\sqrt{N}} L_{i,i}(P), Q_i + \frac{1}{\sqrt{N}} L_{i,i}(P)],$$

where $\frac{1}{\sqrt{N}} L_{i,i}(P)$ is the standard deviation of Q_i. To improve the level of confidence for the neighborhood $B(Q)$ one may clearly double the length of the intervals B_i.

We then define the *sensitivity* of the option price $g(PAR)$ with respect to each parameter PAR_i as the product $|D_{i,P}|\sqrt{var(PAR_i)}$ of the absolute value of the partial derivative $D_{i,P}$ of $g(PAR)$ at P with the standard deviation of PAR_i. With these notations, the 4 option price sensitivities can be expressed as,

$$Sen_\kappa = |D_{1,P}|\sigma_\kappa, \quad Sen_\theta = |D_{2,P}|\sigma_\theta,$$

$$Sen_\gamma = |D_{3,P}|\sigma_\gamma, \quad Sen_\rho = |D_{4,P}|\sigma_\rho,$$

where $\sigma_\kappa, \sigma_\theta, \sigma_\gamma, \sigma_\rho$ are the respective standard deviations of the estimators $\hat{\kappa}, \hat{\theta}, \hat{\gamma}, \hat{\rho}$. We now compute the partial derivatives of the option price with respect to the model parameters.

5 Derivatives of option price : PDEs

Before computing sensitivities of the option price to parameter errors, we review the question of existence, uniqueness, and differentiability (with respect to parameters) for the solution of the parabolic PDE and boundary value problem defining the option price.

The extensive literature on parabolic PDEs does not seem to cover explicitly the specific initial/boundary value problem that we have here; see for instance the semi-group method [28] or the weak sense solutions approach [26]. So we sketch here a proof for the differentiability of the option price with respect to its parameters. The option price is a function of the logarithm lsp of the stock price, denoted by

$$h(lsp, y, t) = f(exp(lsp), y, t) \text{ where}$$

$$t \in (0, T), l \in (-\infty, \infty), y \in (0, \infty).$$

Let $Z_t = (L_t, Y_t)$. We will also use the shorthand notations $z = (lsp, y)$ and $h(lsp, y, t) = h(z, t)$. A probabilistic expression for the option price at time t from the no arbitrage pricing theory is given by [18],

$$h(z, t) = e^{-r(T-t)} E(\phi(Z_T)|Z_t = z), \tag{5.12}$$

where $z = (lsp, y)$. Recall the definition of the payoff function in terms of lsp,

$$\phi(z) = \phi(lsp, y) = \max(exp(lsp) - K, 0).$$

The conditional expectation of the random variable $\phi(Z_T)$ given $Z_t = z$ in Equation (5.12) is computed under the risk neutral measure. Let

$$u(t) = exp(-r(T-t)).$$

Then,

$$h(z, t) = u(t) E(\phi(Z_T)|Z_t = z) \text{ and}$$

$$h(z, t) = u(t) \int_\Omega q(t, z; T, Z)\phi(Z) dZ, \tag{5.13}$$

where $q(t, z; T, Z)$ is the conditional density of the random variable Z_T given $Z_t = z$. The set Ω is the spatial domain,

$$\Omega = \mathbb{R} \times \mathbb{R}^+.$$

The transition density function $q(t, z; T, Z)$ is known to satisfy, as a function of $T > t$ and Z, the Kolmogorov forward parabolic PDE [15],

$$(\frac{\partial}{\partial T} - \Delta) q = 0 \text{ on } (t, \infty) \times \mathbb{R} \times \mathbb{R}^+, \tag{5.14}$$

with initial condition

$$\lim_{T \to t} q(t, z; T, Z) = \delta_z.$$

Here Δ is the second order differential operator

$$\Delta = - < b, \nabla > + \frac{1}{2} tr(H\sigma\sigma^*),$$

where the column vectors b and σ are respectively the drift coefficients and the diffusion coefficients of

Equation (3.7) and Equation (3.6), ∇ denotes the gradient operator, H is the Hessian operator in square matrix form, and "tr" denotes the trace of a matrix. In Equation (5.14), we have omitted the dependence of q on T and Z.

It is well known that the solution q of the forward equation is a smooth function of the parameters in Equation (5.14)([14],[20]). To compute the derivative of the option price with respect to the model parameters we need to differentiate with respect to the parameters under the integral sign on the right hand side of Equation (5.13). Since the function ϕ does not depend explicitly on the parameters, one only needs to verify that the derivatives of q with respect to the parameters are integrable in the domain Ω. This requires deriving good upper bounds on the derivatives at (lsp, y, t) when y tends to zero, which can be done by applications of the maximum principle. We have verified numerically that the evaluations of the derivatives obtained by solving the formal PDEs generated by differentiating the PDE for $g(PAR)$ with respect to each parameter do converge to the derivative of the option price with respect to that parameter.

We now study the sensitivity of the option price with respect to λ.

Define the vector $\mathbf{p} \in \mathbb{R}^5$ by $\mathbf{p}_i = PAR_i$ for $i = 1, 2, \ldots, 4$ and $\mathbf{p}_5 = \lambda$. For all $i = 1, 2, \ldots, 5$ define,

$$D_i(g)(x, y, t; \mathbf{p}) = \frac{\partial g}{\partial p_i}(x, y, t; \mathbf{p}).$$

Differentiating Equation (3.8) with respect to each of the parameters p_i the *sensitivity PDEs* take the following form,

$$(5.15) \qquad (\frac{\partial}{\partial t} - L)D_i = G_i(x, y, t; \mathbf{p}) \text{ on } U_T,$$

with initial condition

$$(5.16) \qquad D_i(x, y, 0; \mathbf{p}) = 0 \text{ on } U_B \times \{t = 0\}.$$

We see that the sensitivity equations are two dimensional non-homogeneous parabolic partial differential equations where the right hand side is given by,

$$G_1(x, y, t; \mathbf{p}) = (\theta - y)\frac{\partial g}{\partial y},$$

$$G_2(x, y, t; \mathbf{p}) = \kappa\frac{\partial g}{\partial y},$$

$$G_3(x, y, t; \mathbf{p}) = \gamma y\frac{\partial^2 g}{\partial y^2} + \rho xy\frac{\partial^2 g}{\partial x\partial y} - \lambda\sqrt{y}\frac{\partial g}{\partial y},$$

$$G_4(t, x, y; \mathbf{p}) = \gamma xy\frac{\partial^2 g}{\partial x\partial y},$$

$$G_5(t, x, y; \mathbf{p}) = -\gamma\sqrt{y}\frac{\partial g}{\partial y}.$$

The boundary conditions for the sensitivity PDEs are,

$$D_i = 0, \text{ on } \{x = 0\} \times (0, Y_{max}) \times (0, T], \text{ with}$$

$$\lim_{x \to X_{max}} \frac{\partial D_i}{\partial x} = 0, \forall y,\ t, \text{ and}$$

$$\lim_{y \to Y_{max}} \frac{\partial D_i}{\partial y} = 0, \forall x,\ t$$

for all $i = 1, 2, \ldots, 5$. For the boundary $y = 0$ we differentiate Equation (3.9) with respect to each of the parameters to obtain,

$$(\frac{\partial}{\partial t} - rx\frac{\partial}{\partial x} - \kappa\theta\frac{\partial}{\partial y} + r)D_i = F_i(x, y, t; \mathbf{p}),$$

where

$$F_1 = \theta\frac{\partial g}{\partial y}, \quad F_2 = \kappa\frac{\partial g}{\partial y}$$

and $F_i = 0$ for $i = 3, 4, 5$. We do not have a sensitivity equation corresponding to the data r because r is a known deterministic constant for our purpose and which we do not estimate from the asset price data.

6 Derivatives of option price : Numerics

We use standard schemes for discretization of the partial differential equations for which stability of option price solutions has been shown. We apply a uniform space-time finite difference grid on the computational domain U_T. Let the number of grid steps be m, n and s in the x, y and t directions respectively. The grid steps in each direction are denoted

$$\Delta x = \frac{X_{max}}{m}; \quad \Delta y = \frac{Y_{max}}{n}; \quad \Delta t = \frac{T}{s}.$$

We use superscripts to denote the time variable and subscripts to denote the spatial variable at the grid point values,

$$g_{ij}^k = g(x_i, y_j, t_k) = g(i\Delta x, j\Delta y, k\Delta t),$$

where

$$i = 0, 1, 2, \ldots, m;\ j = 0, 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots, s.$$

We observe that for American options the pricing equation (3.8) is replaced by an inequality but the differential operator L remains unchanged. We therefore apply the space discretization scheme used in [21] for American options to our European option pricing. We use a second order accurate finite difference scheme for the space derivatives. We use the classical central difference scheme for the first order derivatives and the usual three point scheme for the second order derivatives. At the boundary $j = 0$ an upwind discretization scheme is used [16] for the derivative in the y direction in Equation (3.9) which reads,

$$\delta_y g_{i0}^k = \frac{-3g_{i0}^k + 4g_{i1}^k - g_{i2}^k}{2\Delta y}.$$

The mixed derivatives are discretized using the seven point stencil described in [21]. The properties of the resulting discretization matrix $\mathbf{A}$ are studied in [21]. The strictly diagonally dominant matrix with positive diagonal elements and non-positive off diagonal elements is known to have good stability properties. In general the matrix $\mathbf{A}$ is not an M-matrix but as remarked in [21], with sufficiently small time steps, the resulting matrix is diagonally dominant. The space discretization leads to a semi-discrete equation,

$$\frac{d\mathbf{g}}{dt} + \mathbf{A}\mathbf{g} = \mathbf{b},$$

where $\mathbf{A}$ is an $mn \times mn$ matrix and $\mathbf{b}$ is a column vector of length mn. The vector $\mathbf{g}$ of length mn is the option price at the grid points. The vector $\mathbf{b}$ consists of terms due to the Neumann boundary condition in the x direction and does not depend on t.

For time discretization we model similar to [24], the so called *Backward Difference Formula*, BDF2 scheme. This is an implicit scheme with second order accuracy. The stability of time discretization schemes is considered in [21] and [24] among others. At time $k\Delta t$ the BDF2 scheme reads,

$$\frac{3\mathbf{g}^{k+1} - 4\mathbf{g}^k + \mathbf{g}^{k-1}}{2\Delta t} + \mathbf{A}\mathbf{g}^{k+1} = \mathbf{b}$$

for $k = 1, 2, \ldots, l-1$. This scheme is L-stable which is a stronger property than unconditional stability. Due to this property the BDF2 scheme does not cause oscillations in the solution. The other favorable properties of this scheme are considered in [24]. At each iterate of the BDF2 scheme we require the value of the last two iterates. As is typical, we obtain the first iterate using an Implicit Euler scheme. That is, given $\mathbf{g}^0$ (the initial value), we obtain $\mathbf{g}^1$ using an implicit Euler scheme,

$$\frac{\mathbf{g}^1 - \mathbf{g}^0}{\Delta t} + \mathbf{A}\mathbf{g}^1 = \mathbf{b}.$$

We have verified that this choice of space-time discretization of the initial/boundary-value problem gives us stable solutions for the option price at moderate grid sizes. At each time step we solve the following system of linear equation,

$$(\mathbf{I} + \frac{2}{3}\Delta t\mathbf{A})\mathbf{g}^{k+1} = \frac{4}{3}\mathbf{g}^k - \frac{1}{3}\mathbf{g}^{k-1} + \Delta t\mathbf{b}, \tag{6.17}$$

where $\mathbf{I}$ is an mn by mn identity matrix. We solve this system of equations using a classical LU-decomposition[2].

After we obtain the option price at the discrete grid points, we solve the sensitivity equations. The right hand side of the sensitivity equations is approximated using the central difference scheme for the first derivatives, the 3-point stencil for the second derivatives and the 7 point stencil for the mixed derivatives. We solve the sensitivity equations on the same grid that we used for the option price and use the same space-time discretization to discretize the sensitivity equations. We verify empirically that the scheme for the solution of the sensitivity equations converges.

7 Option price sensitivity to modeling errors

We now compute the sensitivity of S&P 500 options pricing to errors in parameter estimations, where the VIX index is considered as a proxy for the volatility value. In Section 2 above, we have computed the maximum likelihood estimate Q of the unknown Heston model parameter vector P, based on $N = 252$ daily observed values of SPX and VIX. The covariance matrix of $Q - P$ was also computed in Section 2. We can then compute the associated concrete sensitivity values for the pricing of options based on SPX, for $N = 252$ observations.

7.1 Computational Domain Numerically we solve for the option price, viewed at each time t as a function $h(lsp, y, t)$ of t, of the log of the stock price ($lsp = \ln(x)$), and of the squared volatility y of the stock price x. This change of variable gives a transformed initial/boundary value problem that we solve on the following computational domain for $[lsp, y, t]$, where

$$[lsp, y, t] \in (7, 8) \times (0, 1) \times (0, 0.25).$$

This domain was chosen after verifying carefully the robustness of the solution when the domain boundary $lsp = 7$ is reduced to $lsp = 5$. Since the time increment between two consecutive daily observations is set to 1/252 (by convention), the maturity date $T = .25$ indicates that we study an option with three months to maturity. We set the risk free rate of return at value $r = .01$ and we have estimated the market price of volatility risk by $\lambda = 2$. We use the following grid size for our computation,

$$\Delta lsp = .016, \quad \Delta y = .006, \quad \Delta t = .004,$$

[2]Decompose the matrix $\mathbf{I} + \frac{2}{3}\Delta t\mathbf{A}$ as the product of a lower and an upper triangular matrix.

Grid size ($m \times n \times s$)	Δl	Δy	Δt	CPU-time (secs)
226800	1/60	1/60	1/252	95
308700	1/70	1/70	1/252	240

Table 2: CPU-time for option price & derivatives.

where Δlsp is the grid size for the space variable lsp computed as the "lsp" interval length (equal to 1 here) divided by the number of grid meshes in the lsp direction. Table 2 displays the computing time (CPU-time) in seconds required to solve the 6 PDEs determining the option price and its 5 sensitivities. The LU-decomposition is performed only once for each solution. The key computational cost is the LU-decomposition and backward substitution for a matrix of size $mn \times mn$ where m and n are the grid sizes in the lsp and the y direction. The computing times increase more rapidly as we increase the spatial grid size mn than when we increase the grid size in time. For coarser grids the computing times are much lower but the solution is not accurate. We performed extensive numerical simulations to verify that the solutions converge and that the boundary conditions are satisfied.

7.2 The ranges of (SPX,VIX) data The Heston model parameters in Section 2 were estimated using 252 daily (SPX,VIX) data for the year 2006. To present our sensitivity results within a realistic range, we consider 63 SPX and VIX index daily values for the first quarter of 2007, and use these 63 values as the successive values of asset price X_t and volatility (V_t), with the time "t" varying between 0 and $T = 0.25$ by time steps equal to 1/252. In this period, the median SPX index value is 1426, and the median VIX index value is 11%.

We have computed precise sensitivity values for an option with strike price 1430 and three months to maturity ($t = 63/252$). The SPX index values during this period are between 1370 and 1460. The corresponding VIX values are between 10 % and 20 %. We display the dependence of option price sensitivities on asset price and volatility within these two realistic ranges. When we display the dependence of sensitivities on asset price, we keep the volatility fixed at 11% . To display sensitivities dependence on volatility, we keep the asset price fixed at 1408.

7.3 Option price errors due to empirical models We compute the estimation error on the option price for the estimated model $Q = (\hat{\kappa}, \hat{\theta}, \hat{\gamma}, \hat{\rho})$ given in Table 1. As described in Section 4 we compute the impact of parameter estimation errors on the option price through

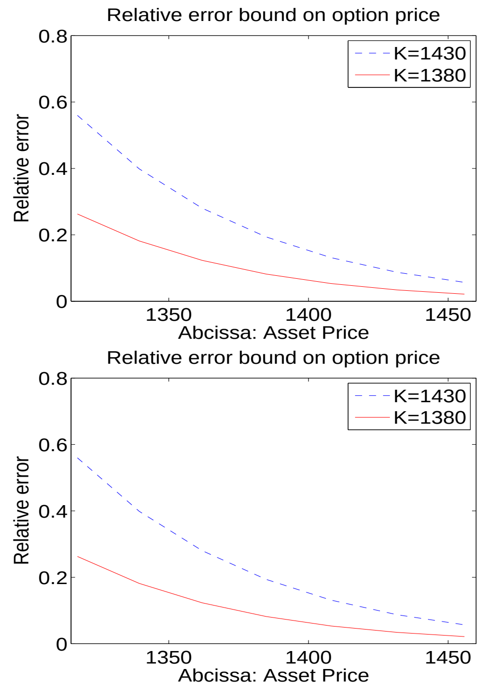


Figure 3: Relative error bound in option pricing.

the formula,

$$\varepsilon(Q) = \max_{P \in B(Q)} \delta(P),$$

were the L^2-error $\delta(P)$ is computed by Equation (4.11), and where the neighborhood $B(Q)$ is the product of the 4 intervals $B_i = [Q_i - \hat{\sigma}_i, Q_i + \hat{\sigma}_i]$ where

$$\hat{\sigma}_1 = \hat{\sigma}_\kappa, \quad \hat{\sigma}_2 = \hat{\sigma}_\theta, \quad \hat{\sigma}_3 = \hat{\sigma}_\gamma, \quad \hat{\sigma}_4 = \hat{\sigma}_\rho$$

are the estimates for standard deviation of κ, θ, γ and ρ displayed in Table 1.

We discretize the set $B(Q)$ using three grid points for each parameter, which gives us a grid of $3^4 = 81$ vectors. At each one of these 81 grid vectors P we compute the 4 partial derivatives of the option price with respect to the 4 key model parameters and the standard deviations of the 4 key parameter estimators, in order to compute the L^2-error $\delta(P)$. We then take the maximum of these 81 values of $\delta(P)$ to obtain the option pricing error $\varepsilon(Q)$. In Fig. 3 we illustrate the impact of parameter estimation errors on the pricing of two options written on the SPX index. The *relative pricing error* $\varepsilon(Q)/g(Q)$, where the bound $\varepsilon(Q)$ on the option pricing error is divided by the option price $g(Q)$, is plotted as a function of first the asset price and then volatility.

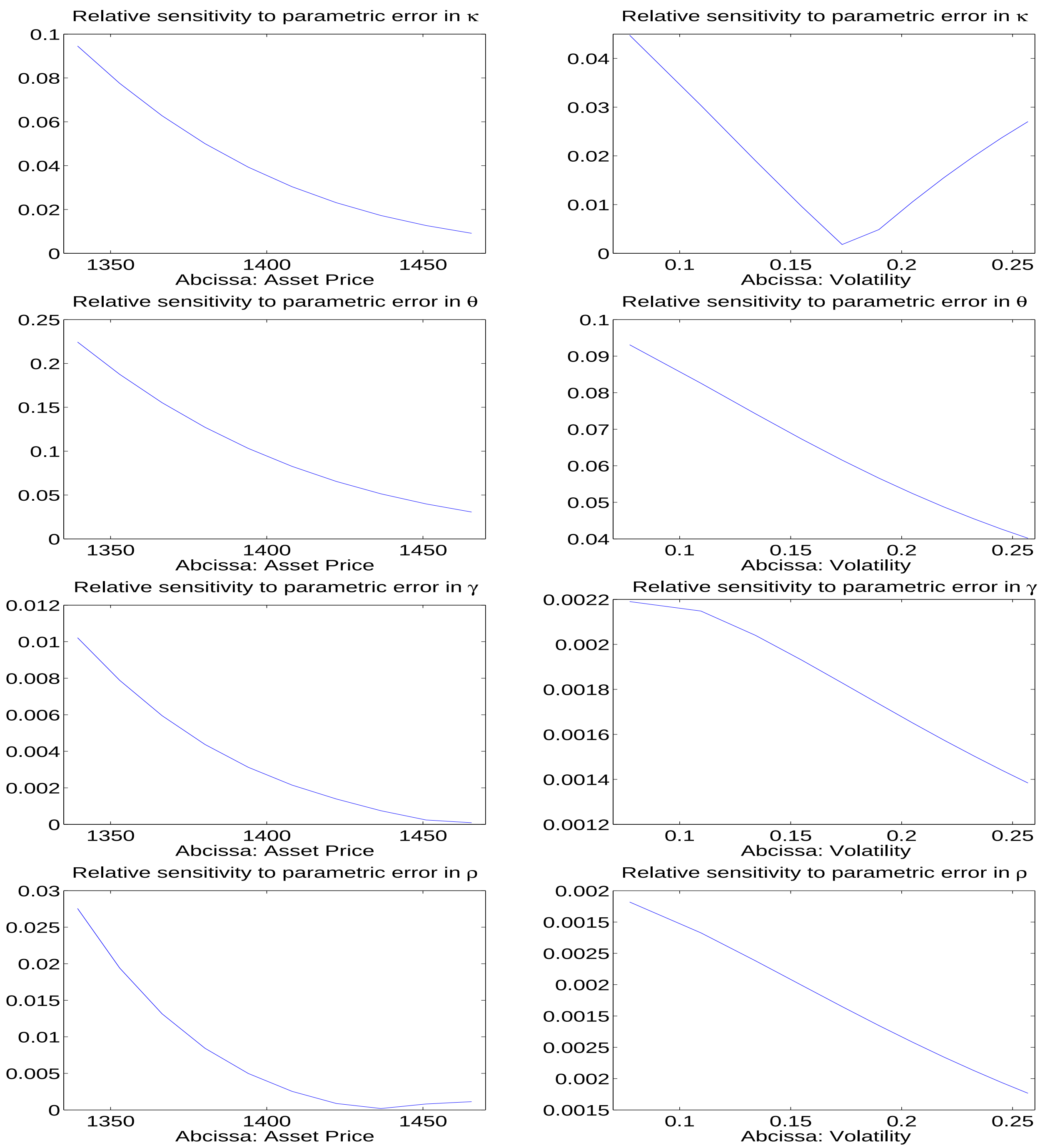


Figure 4: Relative sensitivities of option pricing to estimation errors in κ, θ, γ, ρ as a function of asset price(left) and as a function of volatility(right).

For the first SPX option, with strike price 1430 and 3 months to maturity, the option pricing error bounds $\varepsilon(Q)$ are between 1.5 and 4, for asset prices ranging from 1320 to 1460, and volatility fixed at 11%. For the second SPX option, which has strike price 1380 and 3 months to maturity, the option pricing error bounds are between 1.8 and 3, for the same asset price range and volatility value.

7.4 Option pricing accuracy for S&P 500 data

The sensitivity of the option price to each model parameter was defined in Section 4. In Fig. 4, for each model parameter, we show how the corresponding option pricing sensitivity depends on asset price (for volatility fixed at 11%) and on volatility (for asset price fixed at 1408). For each model parameter PAR_i, we plot the relative sensitivity of option pricing to estimation errors on PAR_i, (for an option with strike price 1430 and 3 months to maturity). Relative sensitivity to PAR_i is computed at the estimated parameter values Q by $|D_{i,Q}|\sqrt{\Sigma_N(i,i)}/g(Q)$, where $D_{i,Q}$ is the partial derivative of the option price $g(Q)$, and $\Sigma_N(i,i)$ is the variance of the estimator Q_i. The model parameters vector PAR is here fixed at the value $PAR = Q$ estimated from the 2006 (SPX,VIX) dataset.

For each parameter PAR_i, we also compute the empirical mean sensitivity $MSEN_{PAR_i}$ of the option price with respect to PAR_i. This mean sensitivity computation is based on the empirical distribution of the asset price over the life of the option. The mean sensitivity of the option price studied here are given by $MSEN_\kappa = 0.42$, $MSEN_\theta = 1.57$, $MSEN_\gamma = 0.02$, $MSEN_{\rho=.04}$. The mean option price during this period is 30.37 and the median option price is 27.60. The option price is most sensitive to errors on the parameter θ.

7.5 Price of volatility risk: impact on options

We compute the derivative $D_5(g(Q)) = \partial_\lambda g(Q)$ of the option price with respect to the market price of risk, to evaluate the option price sensitivity to errors on this unknown market price λ. Table 3 presents $|\partial_\lambda g(Q)|$ at realistic levels of asset price and volatility values. These derivatives are computed at the estimated model Q for an option with strike price equal to 1430 and 3 months to maturity.

Asset price	Volatility	Derivative	Option price
1422	11%	1.55	29.90
1422	19%	1.64	35.74
1357	11%	.96	5.65
1357	19%	1.07	9.29
1480	11%	1.30	67.43
1480	19%	1.45	72.35

Table 3: Absolute value of partial derivative with respect to λ, for the price of an SPX option, at $\lambda = 2$.

8 Conclusion

We have defined a generic mathematical approach, and implemented corrresponding numerical algorithms to compute option price sensitivities to estimation errors on each model parameter of the joint SDEs Heston model for the underlying asset price and volatility.

Our approach combines efficient estimation of the underlying model parameters and of their variances and covariances, with the numerical solution of 6 parabolic equations on $\mathbb{R}^2$, satisfied by the option price and by its partial derivatives with respect to the model parameters of the underlying joint SDEs model.

We also had to develop and apply a consistent method to estimate the market price of the volatility risk on concrete option data.

We have applied our approach to model the 2006 S&P 500 daily data by a Heston pair of coupled SDEs, and to study the option price sensitivities of several European call options based on this index. The results are coherent and quite satisfactory from a pragmatic point of view.

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