

Improved bounds and new techniques for Davenport–Schinzel sequences and their generalizations

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Abstract

We present several new results regarding $\lambda_s(n)$, the maximum length of a Davenport–Schinzel sequence of order s on n distinct symbols.

First, we prove that

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{(1/t!) \alpha(n)^t + O(\alpha(n)^{t-1})}, & s \geq 4 \text{ even;} \\ n \cdot 2^{(1/t!) \alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)}, & s \geq 3 \text{ odd;} \end{cases}$$

where $t = \lfloor (s-2)/2 \rfloor$, and $\alpha(n)$ denotes the inverse Ackermann function. The previous upper bounds, by Agarwal, Sharir, and Shor (1989), had a leading coefficient of 1 instead of $1/t!$ in the exponent. The bounds for even s are now tight up to lower-order terms in the exponent. These new bounds result from a small improvement on the technique of Agarwal et al.

More importantly, we also present a new technique for deriving upper bounds for $\lambda_s(n)$. This new technique is based on some recurrences very similar to those used by the author, together with Alon, Kaplan, Sharir, and Smorodinsky (SODA 2008), for the problem of stabbing interval chains with j -tuples. With this new technique we: (1) re-derive the upper bound of $\lambda_3(n) \leq 2n\alpha(n) + O(n\sqrt{\alpha(n)})$ (first shown by Klazar, 1999); (2) re-derive our own new upper bounds for general s ; and (3) obtain improved upper bounds for the generalized Davenport–Schinzel sequences considered by Adamec, Klazar, and Valtr (1992).

Regarding lower bounds, we show that $\lambda_3(n) \geq 2n\alpha(n) - O(n)$ (the previous lower bound (Sharir and Agarwal, 1995) had a coefficient of $\frac{1}{2}$), so the coefficient 2 is tight. We also present a simpler variant of the construction of Agarwal, Sharir, and Shor that achieves the known lower bounds of $\lambda_s(n) \geq n \cdot 2^{(1/t!) \alpha(n)^t - O(\alpha(n)^{t-1})}$ for $s \geq 4$ even.

1 Introduction

Given a sequence S , denote by $|S|$ the length of S , and by $\|S\|$ the number of distinct symbols in S . If u is another sequence, we write $u \subset S$ if S contains a subsequence u' (not necessarily contiguous) which is isomorphic to u (i.e., u' can be made equal to u by a one-to-one renaming of its symbols). In this case we say that S contains u or that u is contained in S . Otherwise, we write $u \not\subset S$ and we say that S is u -free. For example, $S = abcdcb$ contains $u = abab$, but it is v -free for $v = abba$.

A sequence S is called r -sparse if S contains no pair of equal symbols at distance less than r . In other words, S is r -sparse if every interval in S of length at most r contains only distinct symbols.

A Davenport–Schinzel sequence of order s , for $s \geq 1$, is a sequence that is 2-sparse (i.e., contains no adjacent repeated symbols) and is u -free for $u = ababab\dots$ of length $s+2$. Let $\lambda_s(n)$ denote the maximum length of a Davenport–Schinzel sequence of order s on n distinct symbols ($\lambda_s(n)$ is finite for all s and n). We always take s to be fixed, and consider $\lambda_s(n)$ as a function of n .

These sequences are named after Harold Davenport and Andrzej Schinzel, who first studied them in 1965 [5]. The main motivation for Davenport–Schinzel sequences is the complexity of the lower envelope of a set of curves in the plane. However, Davenport–Schinzel sequences have a large number of applications in computational and combinatorial geometry; the book [14] by Sharir and Agarwal is entirely devoted to this topic. Given the prominent role these sequences play in computational geometry, it is of great interest to derive tight asymptotic bounds for $\lambda_s(n)$. This goal is quite challenging, given the complicated form of the known bounds (see below). There has been little progress in the problem for nearly 20 years.

The bounds $\lambda_1(n) = n$ (no aba) and $\lambda_2(n) = 2n-1$ (no $abab$) are quite easy to obtain. But for $s \geq 3$ the problem becomes much more complicated—it turns out that $\lambda_s(n)$ is slightly superlinear in n .

Hart and Sharir showed in 1986 [6, 14] that $\lambda_3(n) =$

*Supported by ISF Grant 155/05 and by the Hermann Minkowski–MINERVA Center for Geometry at Tel Aviv University.

$\Theta(n\alpha(n))$, where $\alpha(n)$ denotes the inverse Ackermann function. The tightest known bounds for $\lambda_3(n)$ are

$$(1.1) \quad \frac{1}{2}n\alpha(n) - O(n) \leq \lambda_3(n) \leq 2n\alpha(n) + O(n\sqrt{\alpha(n)}).$$

(Sharir and Agarwal [14], and Klazar [8], respectively.) Klazar [9] asks whether $\lim_{n \rightarrow \infty} \lambda_3(n)/(n\alpha(n))$ exists.

The current upper and lower bounds for $\lambda_s(n)$ for general s were established by Agarwal, Sharir, and Shor in 1989 [2, 14], and are as follows. Let $t = \lfloor (s-2)/2 \rfloor$. Then,

$$(1.2) \quad \lambda_s(n) \leq \begin{cases} n \cdot 2^{\alpha(n)^t + O(\alpha(n)^{t-1})}, & s \geq 4 \text{ even;} \\ n \cdot 2^{\alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)}, & s \geq 3 \text{ odd;} \end{cases}$$

$$\lambda_s(n) \geq n \cdot 2^{(1/t)\alpha(n)^t - O(\alpha(n)^{t-1})}, \quad s \geq 4 \text{ even.}$$

For odd $s \geq 5$ the asymptotically best lower bounds known are obtained by $\lambda_s(n) \geq \lambda_{s-1}(n)$.

In 2008 the author, together with Alon, Kaplan, Sharir, and Smorodinsky, conjectured that:

CONJECTURE 1.1. ([4]) *The true bounds for $\lambda_s(n)$ are*

$$\lambda_s(n) = \begin{cases} n \cdot 2^{(1/t)\alpha(n)^t \pm O(\alpha(n)^{t-1})}, & s \geq 4 \text{ even;} \\ n \cdot 2^{(1/t)\alpha(n)^t \log_2 \alpha(n) \pm O(\alpha(n)^t)}, & s \geq 3 \text{ odd;} \end{cases}$$

where $t = \lfloor (s-2)/2 \rfloor$.

This conjecture is based on some surprisingly similar tight bounds that they obtained for an unrelated problem called *stabbing interval chains with j -tuples*.

1.1 Generalized Davenport–Schinzel sequences. Adamec, Klazar, and Valtr [1] considered a generalization of Davenport–Schinzel sequences, in which the forbidden pattern is not limited to $abab\dots$, but can be an arbitrary sequence.

Let u (the *forbidden pattern*) be a sequence with $\|u\| = r$ distinct symbols and length $|u| = s$. Then we denote by $\text{Ex}_u(n)$ the maximum length of an r -sparse, u -free sequence on n distinct symbols. The standard Davenport–Schinzel sequences are obtained by taking $r = 2$ and $u = abab\dots$ of length $s + 2$.

The requirement of r -sparsity is necessary, since an $(r-1)$ -sparse, u -free sequence can be arbitrarily long. The requirement of r -sparsity, however, ensures that $\text{Ex}_u(n)$ is finite.

Generalized Davenport–Schinzel sequences have found several applications in discrete mathematics. Valtr [16] used generalized Davenport–Schinzel sequences to obtain bounds for some Turán-type problems for geometric graphs. Alon and Friedgut [3] used

them to derive an almost-tight upper bound for the so-called Stanley–Wilf conjecture (the conjecture was later proved by Marcus and Tardos [11] by a different technique). For more information see the surveys by Klazar [9] and by Valtr [16]. More recently, Pettie [12] used generalized Davenport–Schinzel sequences to improve Sundar’s [15] near-linear upper bound for the *deque conjecture* for splay trees.

1.2 Formation-free sequences. Klazar in 1992 [7] developed a general technique for bounding $\text{Ex}_u(n)$ in terms of only $r = \|u\|$ and $s = |u|$. His technique is based on considering what we call *formation-free sequences* (our name). Given integers r and s , an (r, s) -*formation* is a sequence of s permutations on r symbols. For example, $abcd \ dcab \ dcab \ cdab \ dabc$ is a $(4, 5)$ -formation. An (r, s) -*formation-free sequence* is a sequence which is r -sparse and does not contain any (r, s) -formation as a subsequence.

Denote by $F_{r,s}(n)$ the length of the longest possible (r, s) -formation-free sequence on n distinct symbols. Let u be a sequence with $\|u\| = r$ and $|u| = s$. Since u is trivially contained in every (r, s) -formation, it follows that $\text{Ex}_u(n) \leq F_{r,s}(n)$.

Klazar made a slight improvement to this observation, by noting that if $r \geq 2$, then u is contained in every $(r, s-1)$ -formation, and thus,

$$(1.3) \quad \text{Ex}_u(n) \leq F_{r,s-1}(n) \quad \text{for } r \geq 2.$$

(The case $r = 1$ is not interesting in any case.) Klazar proved the bound

$$(1.4) \quad F_{r,s}(n) \leq n \cdot 2^{O(\alpha(n)^{s-3})},$$

where the O notation hides constants that depend on r and s . Together with (1.3), this implies that

$$\text{Ex}_u(n) \leq n \cdot 2^{O(\alpha(n)^{s-4})}.$$

1.3 Our results. In this paper we present several new results.

First, we make a small improvement on the argument of Agarwal et al. [2, 14] and prove:

THEOREM 1.2. *Let $s \geq 3$ be fixed, and let $t = \lfloor (s-2)/2 \rfloor$. Then*

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{(1/t)\alpha(n)^t + O(\alpha(n)^{t-1})}, & s \text{ even;} \\ n \cdot 2^{(1/t)\alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)}, & s \text{ odd.} \end{cases}$$

Thus, the upper bounds for $\lambda_s(n)$ are now in line with Conjecture 1.1, and for s even they are also tight up to lower-order terms in the exponent.

More importantly, we also present a new technique for deriving upper bounds for $\lambda_s(n)$. Our new technique is based on some recurrences very similar to those used by Alon et al. [4], for the problem of stabbing interval chains with j -tuples.

With our new technique we re-derive Klazar’s upper bound (1.1) for $\lambda_3(n)$, as well as our new bounds in Theorem 1.2 for $\lambda_s(n)$, $s \geq 4$. We also apply our technique to formation-free sequences, proving that:

THEOREM 1.3. *For $s \geq 4$ we have*

$$F_{r,s}(n) \leq \begin{cases} n \cdot 2^{(1/t!) \alpha(n)^t + O(\alpha(n)^{t-1})}, & s \text{ odd}; \\ n \cdot 2^{(1/t!) \alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)}, & s \text{ even}; \end{cases}$$

where $t = \lfloor (s-3)/2 \rfloor$. (The O notation hides factors dependent on r and s .)

As an aside, we improve on Klazar’s bound (1.3):

LEMMA 1.4. *Let u be a sequence with $\|u\| = r$, $|u| = s$. Then, $\text{Ex}_u(n) \leq F_{r,s-r+1}(n)$.*

This, together with Theorem 1.3, yields:¹

THEOREM 1.5. *Let u be a sequence with $\|u\| = r$, $|u| = s$, and $s \geq r+3$. Let $t = \lfloor (s-r-2)/2 \rfloor$. Then,*

$$\text{Ex}_u(n) \leq \begin{cases} n \cdot 2^{(1/t!) \alpha(n)^t + O(\alpha(n)^{t-1})}, & s-r \text{ even}; \\ n \cdot 2^{(1/t!) \alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)}, & s-r \text{ odd}. \end{cases}$$

Note that Theorem 1.5 is a generalization of Theorem 1.2: Taking $r=2$ and $u = abab\dots$ of length $s+2$ yields the theorem once again.

Regarding lower bounds, we prove:

THEOREM 1.6. $\lambda_3(n) \geq 2n\alpha(n) - O(n)$.

COROLLARY 1.7. $\lim_{n \rightarrow \infty} \lambda_3(n)/(n\alpha(n)) = 2$.

Finally, we present a simpler variant of the construction of Agarwal, Sharir, and Shor [2, 14], which achieves the lower bounds (1.2) for $s \geq 4$ even.

1.4 The Ackermann function and its inverse.

The *Ackermann hierarchy* is a sequence of functions $A_k(n)$, for $k \geq 1$ and $n \geq 0$, where $A_1(n) = 2n$, and for $k \geq 2$ we let $A_k(n) = A_{k-1}^{(n)}(1)$. (Here $f^{(n)}$ denotes the n -fold composition of f .) The definition of $A_k(n)$ for $k \geq 2$ can also be written recursively by $A_k(0) = 1$, and $A_k(n) = A_{k-1}(A_k(n-1))$ for $n \geq 1$. We have $A_2(n) = 2^n$, and $A_3(n) = 2^{2^{\dots^2}}$ is a “tower” of n twos.

¹Klazar himself [7] speculated that it should be possible to achieve roughly $\text{Ex}_u(n) \leq n \cdot 2^{O(\alpha(n)^{s/2})}$.

We have $A_k(1) = 2$ and $A_k(2) = 4$, but $A_k(3)$ already grows very rapidly with k . We define the *Ackermann function* as $A(n) = A_n(3)$. Thus, $A(n) = 6, 8, 16, 65536, \dots$ for $n = 1, 2, 3, \dots$ ²

We then define the slow-growing inverses of these rapidly-growing functions as $\alpha_k(x) = \min\{n \mid A_k(n) \geq x\}$ and $\alpha(x) = \min\{n \mid A(n) \geq x\}$ for all real $x \geq 0$.

Alternatively, and equivalently, we can define these inverse functions directly: We define the *inverse Ackermann hierarchy* by letting $\alpha_1(x) = \lceil x/2 \rceil$ and, for $k \geq 2$, defining $\alpha_k(x)$ recursively by $\alpha_k(x) = 0$ for $x \leq 1$, and $\alpha_k(x) = 1 + \alpha_k(\alpha_{k-1}(x))$ for $x > 1$. In other words, for each $k \geq 2$, $\alpha_k(x)$ denotes the number of times we must apply α_{k-1} , starting from x , until we reach a value not larger than 1. Thus, $\alpha_2(x) = \lceil \log_2 x \rceil$, and $\alpha_3(x) = \log^* x$. Finally, we define the *inverse Ackermann function* by $\alpha(x) = \min\{k \mid \alpha_k(x) \leq 3\}$.

1.5 Organization of this paper. Because of space constraints we omit most proofs, as well as some of the results themselves. For the omitted material refer to the full version of this paper.

Section 2 shows how Theorem 1.2 reduces to bounding a function denoted $\psi_s(m, n)$. In Section 3 we briefly sketch our improvement on the technique of Agarwal et al. [2, 14] for bounding $\psi_s(m, n)$. In Section 4 we present an alternative technique, which yields the same improved bounds for $\psi_s(m, n)$.

Section 5 addresses formation-free sequences. We first prove Lemma 1.4, and then we extend our new technique to formation-free sequences, proving Theorem 1.3.

Finally, Section 6 presents our construction for $\lambda_3(n)$ that proves Theorem 1.6. The simplified construction of Davenport–Schinzel sequences of even order $s \geq 4$ is omitted from this extended abstract.

2 Upper bounds for Davenport–Schinzel sequences

The upper bounds for $\lambda_s(n)$ are obtained by considering a function with an additional parameter m :

DEFINITION 2.1. *Let $\psi_s(m, n)$ be the maximum length of a Davenport–Schinzel sequence of order s on n distinct symbols that can be partitioned into m or fewer contiguous blocks, where each block contains only distinct symbols.*

²The Ackermann function is usually defined by “diagonalizing” the hierarchy, letting $A(n) = A_n(n)$. This does not make any asymptotic difference, and we prefer the above definition because, first, diagonalization is unnecessary, and second, the corresponding “direct” definition of $\alpha(x)$ comes out simpler. For other references where $\alpha(x)$ is defined similarly, see Pettie [12] and Seidel [13, slide 85].

The relation between $\lambda_s(n)$ and $\psi_s(m, n)$ is as follows:

LEMMA 2.2. ([2, 14]) *Given $s \geq 3$, let $\varphi_{s-2}(n)$ be a nondecreasing function in n such that $\lambda_{s-2}(n) \leq n\varphi_{s-2}(n)$ for all n . Then,*

$$\lambda_s(n) \leq \varphi_{s-2}(n)(\psi_s(2n, n) + 2n).$$

For $s = 3$ we this gives $\lambda_3(n) \leq \psi_3(2n, n) + 2n$ (by taking $\varphi_1(n) = 1$, since $\lambda_1(n) = n$). Actually for $s = 3$ we have $\lambda_3(n) = \psi_3(2n, n)$ (Hart and Sharir [6, 14]).

The main issue, then, is to bound $\psi_s(m, n)$. We present two different techniques for bounding $\psi_s(m, n)$. The first one is a minor modification of the technique of Agarwal et al. [2, 14]. The second one is our new technique. Both techniques yield the following bounds:

LEMMA 2.3. *For $s = 3$ we have*

$$\psi_3(m, n) = O(km\alpha_k(m) + kn) \quad \text{for all } k.$$

In general, for every fixed $s \geq 3$ we have

$$\psi_s(m, n) \leq C_{s,k}(m\alpha_k(m)^{s-2} + n) \quad \text{for all } k,$$

for some constants $C_{s,k}$ of the form

$$C_{s,k} = \begin{cases} 2^{(1/t!)k^t \pm O(k^{t-1})}, & s \text{ even;} \\ 2^{(1/t!)k^t \log_2 k \pm O(k^t)}, & s \text{ odd;} \end{cases}$$

where $t = \lfloor (s-2)/2 \rfloor$.

(Equivalent bounds for $\psi_3(m, n)$ and $\psi_4(m, n)$ were previously derived by Hart and Sharir [6, 14], and Agarwal, Sharir, and Shor [2, 14], respectively. For $s \geq 5$ these are improvements over [2, 14], which for $s \geq 6$ yield improved bounds for $\lambda_s(n)$.)

Proof of Theorem 1.2. Take $k = \alpha(m)$ in Lemma 2.3 (recalling that $\alpha_{\alpha(m)}(m) \leq 3$ by definition), and substitute into Lemma 2.2. For $s = 3, 4$ we get $\lambda_3(n) = O(n\alpha(n))$, $\lambda_4(n) = O(n \cdot 2^{\alpha(n)})$ (by taking $\varphi_1(n) = 1$, $\varphi_2(n) = 2$). For $s \geq 5$ we bound $\varphi_{s-2}(n)$ by induction on s and we get the desired bounds (the factor $\varphi_{s-2}(n)$ only affects lower-order terms in the exponent). \square

3 Bounding $\psi_s(m, n)$

The bounds for $\psi_s(m, n)$ given in Lemma 2.3 result from the following complicated-looking recurrence relation. This is a small modification of the recurrence in [2, 14] (and more complicated).

RECURRENCE 3.1. *Let $m, n \geq 1$ and $b \leq m$ be integers, and let*

$$m = m_1 + m_2 + \cdots + m_b$$

be a partition of m into b nonnegative integers. Then, there exists a partition of n into nonnegative integers

$$n = n_1 + n_2 + \cdots + n_b + n^*,$$

and there exist nonnegative integers $n_1^, n_2^*, \dots, n_b^* \leq n^*$ satisfying*

$$n_1^* + n_2^* + \cdots + n_b^* \leq \psi_s(b, n^*) + b,$$

such that

$$\psi_s(m, n) \leq 2\psi_{s-1}(m, n^*) + 4m + \sum_{i=1}^b (\psi_{s-2}(m_i, n_i^*) + \psi_s(m_i, n_i)).$$

See the full version of this paper for more details.

4 A new technique for bounding $\psi_s(m, n)$

We now present an alternative technique for bounding $\psi_s(m, n)$. Our new technique is based on a variant of Davenport–Schinzel sequences, in which we turn the problem around, in a sense. We call our variant sequences *almost-DS sequences*.

An *almost-DS sequence of order s with multiplicity k and m blocks* (or an $\text{ADS}_k^s(m)$ -sequence, for short) is a sequence that satisfies the following properties:

- It is a concatenation of m blocks, each block containing only distinct symbols.
- Each symbol appears at least k times (in different blocks, so we must have $m \geq k$ for there to be any symbols at all).
- The sequence contains no alternation $abab\dots$ of length $s + 2$.

Note that we do allow repetitions at the interface between adjacent blocks (this simplifies matters). This is why these are *almost* Davenport–Schinzel sequences.

We now pose a different problem: We ask for *maximizing the number of distinct symbols*. Let $\Pi_k^s(m)$ denote the maximum number of distinct symbols in an $\text{ADS}_k^s(m)$ -sequence. (Note that $\Pi_k^s(m) = 0$ for $m < k$.)

The connection between $\psi_s(m, n)$ and $\Pi_k^s(m)$ is based on the following lemma:

LEMMA 4.1. *For all s, n, m , and k we have $\psi_s(m, n) \leq k(\Pi_k^s(m) + n)$.*

Proof. Let S be a maximum-length Davenport–Schinzel sequence of order s on n distinct symbols that is partitionable into m blocks, each of distinct symbols. Thus, $|S| = \psi_s(m, n)$. Let $k \geq 1$ be a parameter.

We transform S into another sequence S' in which every symbol appears exactly k times as follows:³ For each symbol a , group the occurrences of a in S from left to right into “clusters” of size k , deleting the last remaining $\leq k - 1$ occurrences of a . Make the occurrences of a in different clusters different, by replacing each a in the i -th cluster by a new symbol a_i .

We deleted at most kn symbols from S , so $|S'| \geq |S| - kn$. On the other hand, S' is clearly an $\text{ADS}_k^s(m)$ -sequence (the symbol deletions might have created repetitions at the interface between blocks, but these are permitted in almost-DS sequences; on the other hand, the symbol replacements do not introduce any forbidden alternations). Thus, S' contains at most $\Pi_k^s(m)$ distinct symbols. Since each symbol appears exactly k times, we have $|S'| \leq k \cdot \Pi_k^s(m)$. The claim follows. \square

Thus, our problem reduces to bounding $\Pi_k^s(m)$.

LEMMA 4.2. *For all $s \geq 1$, $m \geq s$ we have $\Pi_s^s(m) = \infty$.*

LEMMA 4.3. *We have $\Pi_2^1(m) = m - 1$.*

LEMMA 4.4. *For all $s \geq 2$ we have $\Pi_{s+1}^s(m) \leq \binom{m-2}{s-1} = O(m^{s-1})$.*

We now bound $\Pi_k^s(m)$ by deriving recurrences and solving them, in a manner almost entirely analogous to [4]. The following recurrence and corollary are analogous to Lemma 3.2 in [4]:

RECURRENCE 4.5. *For every $s \geq 3$ and every k and m we have*

$$\Pi_{2k-1}^s(2m) \leq 2\Pi_{2k-1}^s(m) + 2\Pi_k^{s-1}(m).$$

Proof sketch. Given an $\text{ADS}_{2k-1}^s(2m)$ -sequence S , partition the $2m$ blocks of S into a “left half” and a “right half” of m blocks each. The term $2\Pi_{2k-1}^s(m)$ accounts for those symbols that appear only in the left half or only in the right half. And the symbols that appear in both halves must appear at least k times in one of the halves. The term $2\Pi_k^{s-1}(m)$ accounts for them. \square

COROLLARY 4.6. *For every fixed $s \geq 2$, if we let $k = 2^{s-1} + 1$, then $\Pi_k^s(m) = O(m(\log m)^{s-2})$.*

Proof. Apply Recurrence 4.5 using induction on s , using Lemma 4.4 as base case for $s = 2$. \square

The following recurrence and corollary for $\Pi_k^3(m)$ are analogous to Recurrence 3.3 and Lemma 3.5 in [4]:

³A similar argument has been used by Sundar [15, Lemma 9] for a different problem.

RECURRENCE 4.7. *Let t be an integer parameter, with $t \leq \sqrt{m}$. Then,*

$$\Pi_k^3(m) \leq \left(1 + \frac{m}{t}\right) \Pi_k^3(t) + \Pi_{k-2}^3\left(1 + \frac{m}{t}\right) + 3m.$$

Proof. Take a sequence S that maximizes $\Pi_k^3(m)$. Let $b = \lceil m/t \rceil \leq 1 + m/t$. Partition the m blocks of S from left to right into b layers L_1, \dots, L_b of at most t blocks each.

We classify the symbols of S into different types. A symbol is *local* for layer L_i if it only appears in L_i . Taking just the symbols local to L_i produces an $\text{ADS}_k^3(t)$ -sequence. Therefore, the number of local symbols is at most $\Pi_k^3(t)$ per layer, or at most $b\Pi_k^3(t) \leq \left(1 + \frac{m}{t}\right) \Pi_k^3(t)$ altogether.

Symbols which appear in at least two layers are called *global symbols*.

Call a global symbol *left-concentrated* for layer L_i if it makes its first appearance in L_i , and it appears at least three times in L_i . Given a layer L_i , take just the left-concentrated symbols for L_i , and just their occurrences within L_i . The resulting sequence S'_i cannot contain an alternation $abab$, or else S would contain $ababa$. Therefore, S'_i is an $\text{ADS}_3^2(t)$ -sequence, so by Lemma 4.4 it has at most $t - 2$ different symbols. Thus, there are at most $b(t - 2) \leq \left(1 + \frac{m}{t}\right)(t - 2) \leq m$ left-concentrated symbols altogether (since $t \leq \sqrt{m}$).

Similarly, there are at most m *right-concentrated* symbols.

Next, call a symbol *middle-concentrated* for layer L_i if it appears at least twice in L_i , and it also appears before L_i and after L_i .

Given L_i , take just the middle-concentrated symbols for L_i , and just their occurrences within L_i . The resulting sequence S''_i cannot contain an alternation aba , so S''_i is an $\text{ADS}_2^1(t)$ -sequence, and so by Lemma 4.3 it contains at most $t - 1$ different symbols. Therefore, there are at most $b(t - 1) \leq m$ middle-concentrated symbols. (Note that we might have counted the same middle-concentrated symbol more than once.)

Finally, take all the global symbols we have not accounted for so far—the *scattered symbols*. Each of these symbols appears in at least $k - 2$ different layers. Build a subsequence of S by taking just the scattered symbols, and for each scattered symbol, just one occurrence per layer. Each layer becomes a block, and no new forbidden alternation can arise. Hence, we get an $\text{ADS}_{k-2}^3(b)$ -sequence, which can have at most $\Pi_{k-2}^3\left(1 + \frac{m}{t}\right)$ different symbols. \square

COROLLARY 4.8. *There exists an absolute constant c such that, for every $k \geq 2$, we have*

$$\Pi_{2k+1}^3(m) \leq cm\alpha_k(m) \quad \text{for all } m.$$

The bound for $\psi_3(m, n)$ in Lemma 2.3 now follows from Corollary 4.8 and Lemma 4.1.

4.1 Obtaining Klazar's improved upper bound for $\lambda_3(n)$. Klazar's tighter upper bound (1.1) for $\lambda_3(n)$ follows by using the following relation between $\lambda_3(n)$ and $\psi_3(m, n)$, instead of Lemma 2.2:

LEMMA 4.9. (KLAZAR [8]) *We have $\lambda_3(n) \leq \psi_3(1 + 2n/\ell, n) + 3n\ell$, where $\ell \leq n$ is a free parameter.*

COROLLARY 4.10. $\lambda_3(n) \leq 2n\alpha(n) + O(n\sqrt{\alpha(n)})$.

Proof. Taking $k = 2\alpha(m) + 1$ in Lemma 4.1, and bounding $\Pi_{2\alpha(m)+1}^3(m)$ by Corollary 4.8, we get

$$\begin{aligned} \psi_3(m, n) &\leq (2\alpha(m) + 1)(cm\alpha_{\alpha(m)}(m) + n) \\ &= 2n\alpha(m) + n + O(m\alpha(m)). \end{aligned}$$

We now apply Lemma 4.9 with $\ell = \sqrt{\alpha(n)}$. \square

4.2 Bounding $\Pi_k^s(m)$ for general s . The following recurrence and corollary for $\Pi_k^s(m)$ are analogous to Recurrence 3.6 and Lemma 3.8 in [4]:

RECURRENCE 4.11. *Let $s \geq 3$ be fixed. Let k_1, k_2, k_3 be integers, and put $k = k_2k_3 + 2k_1 - 3k_2 - k_3 + 2$. Then,*

$$\begin{aligned} \Pi_k^s(m) &\leq \left(1 + \frac{m}{t}\right) (\Pi_k^s(t) + 2\Pi_{k_1}^{s-1}(t) + \Pi_{k_2}^{s-2}(t)) \\ &\quad + \Pi_{k_3}^s\left(1 + \frac{m}{t}\right), \end{aligned}$$

where t is a free parameter.

Proof. Take a sequence S that maximizes $\Pi_k^s(m)$. Again partition the m blocks of S into $b = \lceil m/t \rceil \leq 1 + m/t$ layers L_1, \dots, L_b , with at most t blocks per layer.

We again classify the symbols of S into *local* (if the symbol appears in only one layer), or *global*. As before, there are at most $(1 + \frac{m}{t}) \Pi_k^s(t)$ local symbols.

And we again classify the global symbols into *left-concentrated*, *right-concentrated*, *middle-concentrated*, and *scattered*.

A global symbol is *left-concentrated* for layer L_i if its first k_1 occurrences fall in L_i . The number of left-concentrated symbols is at most $(1 + \frac{m}{t}) \Pi_{k_1}^{s-1}(t)$ altogether. *Right-concentrated* symbols are defined and handled analogously.

A global symbol is *middle-concentrated* for layer L_i if it appears at least k_2 times in L_i , and it also appears before L_i and after L_i . There are at most $(1 + \frac{m}{t}) \Pi_{k_2}^{s-2}(t)$ such symbols altogether.

Finally, a global symbol is *scattered* if it appears in at least k_3 different layers. Taking just these symbols,

and for each symbol, just one occurrence per layer, we obtain an $\text{ADS}_{k_3}^s(b)$ -sequence. Thus, there are at most $\Pi_{k_3}^s(b) \leq \Pi_{k_3}^s(1 + \frac{m}{t})$ scattered symbols.

All that remains is to show that we did not miss any global symbol. Suppose a global symbol is neither left-, middle-, nor right-concentrated, nor scattered. Then the symbol appears at most $2(k_1 - 1) + (k_3 - 3)(k_2 - 1) = k - 1$ times in S , a contradiction. \square

COROLLARY 4.12. *Let $R_s(d)$ be given for $s \geq 1$, $d \geq 2$ by $R_1(d) = 2$, $R_2(d) = 3$, and for $s \geq 3$ by*

$$\begin{aligned} R_s(2) &= 2^{s-1} + 1, \\ R_s(d) &= R_s(d-1)R_{s-2}(d) + 2R_{s-1}(d) - 3R_{s-2}(d) \\ &\quad - R_s(d-1) + 2, \quad \text{for } d \geq 3. \end{aligned}$$

Then, for every $s \geq 2$ and $d \geq 2$, if $k \geq R_s(d)$ then

$$\Pi_k^s(m) \leq cm\alpha_d(m)^{s-2} \quad \text{for all } m.$$

Here $c = c(s)$ is a constant that depends only on s .

We have $R_3(d) = 2d + 1$, $R_4(d) = 5 \cdot 2^d - 4d - 3$, and in general, letting $t = \lfloor (s-2)/2 \rfloor$,

$$(4.5) \quad R_s(d) = \begin{cases} 2^{(1/t)d^t + O(d^{t-1})}, & s \text{ even;} \\ 2^{(1/t)d^t \log_2 t + O(d^t)}, & s \text{ odd.} \end{cases}$$

Lemma 2.3 now follows from Lemma 4.1 with $k = R_s(d)$ and Corollary 4.12.

5 Bounding formation-free sequences

We start by proving Lemma 1.4.

LEMMA 1.4. *Let u be a sequence with $\|u\| = r$, $|u| = s$. Then, $\text{Ex}_u(n) \leq F_{r, s-r+1}(n)$.*

Proof. Suppose $u = u_1u_2 \dots u_s$, where $1 \leq u_i \leq r$ for each i . We can assume that the symbols in u make their first appearances in the order $1, 2, \dots, r$.

Let $s' = s - r + 1$, and let $\ell = \ell_1\ell_2 \dots \ell_{s'}$ be an arbitrary (r, s') -formation, where each ℓ_j is a permutation of $\{1, \dots, r\}$. We want to show that $u \subset \ell$.

Define a partition $u = B_1B_2 \dots B_{s'}$ of u into s' blocks as follows: First let each symbol of u constitute its own block of length 1. Then, for each $2 \leq j \leq r$, merge the block that contains the first occurrence of j in u with the block containing the immediately preceding symbol. The number of blocks goes down from s to s' .

Here is an example of a sequence thus partitioned:

$$(5.6) \quad u = [1][1][12][134][2][4][1][25][5].$$

Clearly, each block B_j is an increasing sequence.

Now we are going to define a permutation σ on $\{1, \dots, r\}$ such that, for each block B_j with $1 \leq j \leq s'$, its image $\sigma(B_j)$ is a subsequence of ℓ_j . We do this by examining the blocks from right to left, and by defining σ in the order $\sigma(r), \sigma(r-1), \dots, \sigma(1)$. Note that blocks of length 1 can be safely ignored.

Suppose we have already dealt with blocks $B_{s'}, B_{s'-1}, \dots, B_{j+1}$, and that now is the turn of block B_j , where $|B_j| > 1$. Let k be the last symbol in B_j . The symbols preceding k in B_j are $k-1, k-2, \dots$, up to the second symbol of B_j . All these symbols make their first appearance in u in B_j . Call these the “new” symbols of B_j .

Suppose we have already assigned values to $\sigma(k+1), \dots, \sigma(r)$ in such a way that, no matter how we assign $\sigma(1), \dots, \sigma(k)$, the images $\sigma(B_{j+1}), \dots, \sigma(B_{s'})$ will always be subsequences of $\ell_{j+1}, \dots, \ell_{s'}$, respectively.

Now consider the symbols of ℓ_j . Call a symbol of ℓ_j “free” if it has not yet been assigned as image $\sigma(i)$ to any symbol i , for $k+1 \leq i \leq r$.

We scan ℓ_j from right to left, considering only its free symbols, and we assign in a greedy fashion these free symbols as images $\sigma(k), \sigma(k-1), \dots$ to $k, k-1, \dots$ (the “new” symbols of B_j).

After we are done with these assignments, the only symbol of B_j which has not been assigned an image is the first symbol of B_j —call it b_j . But no matter how we define $\sigma(b_j)$ later on, we will always have that $\sigma(B_j)$ is a subsequence of ℓ_j (because of our greedy approach).

At the end, the assignment $\sigma(1)$ of 1 will be forced.

For example, with u as in (5.6), suppose that

$$\ell = \ell_1 \ell_2 32514 35421 \ell_5 \ell_6 \ell_7 35142 \ell_9$$

(where $\ell_1, \ell_2, \ell_5, \ell_6, \ell_7, \ell_9$ do not matter). Then, our algorithm will assign $\sigma(5) = 2, \sigma(4) = 1, \sigma(3) = 4, \sigma(2) = 5$, and finally $\sigma(1) = 3$. Then the sequence

$$\sigma(u) = [3][3][35][341][5][1][3][52][2]$$

is a subsequence of ℓ , as desired. \square

REMARK 5.1. *Lemma 1.4 is not the last word in finding sequences in formations. For example, consider the sequence $u = abcabca$. Lemma 1.4 states that u is contained in every $(3, 5)$ -formation, but in fact u is contained in every $(3, 4)$ -formation: Let $\ell = \ell_1 \ell_2 \ell_3 \ell_4$ be a $(3, 4)$ -formation. Suppose $\ell_1 = abc$. Then, if u itself is not a subsequence of ℓ , then ℓ_2 must have b before a , ℓ_3 must have c before b , and ℓ_4 must have a before c . But then ℓ contains the subsequence $cbacbac$.*

Now we set out to bound $F_{r,s}(n)$.

LEMMA 5.2. (KLAZAR [7]) *We have $F_{r,2}(n) \leq rn$ and $F_{r,3}(n) \leq 2rn$. Furthermore, for every s we have $F_{r,s}(n) \leq sn^r$ for all $n \geq r$.*

LEMMA 5.3. (KLAZAR [7]) *Let $S = S_1 S_2 \dots S_m$ be a sequence which is a concatenation of m blocks, each block containing only distinct symbols. Then S can be made r -sparse by deleting at most $(r-1)(m-1)$ symbols.*

Next, we make a definition analogous to Definition 2.1:

DEFINITION 5.4. *Let $\psi'_{r,s}(m, n)$ be the length of the longest r -sparse, (r, s) -formation-free sequence on n distinct symbols that can be partitioned into m or fewer blocks, each block containing only distinct symbols.*

REMARK 5.5. *The reader need not be intimidated (more than necessary) by the double subscript r, s in $\psi'_{r,s}(m, n)$. We are never going to use induction on r , only on s .*

The following lemma is analogous to Lemma 2.2, and relates $F_{r,s}(n)$ to $\psi'_{r,s}(m, n)$.

LEMMA 5.6. *Given fixed integers r and s , let $\varphi_{r,s-2}(n)$ be a nondecreasing function of n such that $F_{r,s-2}(n) \leq n\varphi_{r,s-2}(n)$ for all n . Then,*

$$F_{r,s}(n) \leq 2n + \varphi_{r,s-2}(n)(2(r-1)n + \psi'_{r,s}(2n, n)).$$

(This constitutes a minor improvement over Klazar [7], since Klazar related $F_{r,s}(n)$ to $\varphi_{r,s-1}(n)$.)

We now apply our “almost-DS” technique to formation-free sequences. For this, we introduce and analyze “almost-formation-free” sequences.

5.1 Almost-formation-free sequences. If S is a sequence, we say that S is an $\text{AFF}_{r,s,k}(m)$ sequence if S contains no (r, s) -formation, can be partitioned into m of fewer blocks of distinct symbols, and each symbol appears at least k times (in k different blocks).

Note that we do not require r -sparsity; this is the reason for calling S “almost” formation-free.

Let $\Pi'_{r,s,k}(m)$ denote the maximum possible number of distinct symbols in an $\text{AFF}_{r,s,k}(m)$ sequence.

We first show the connection between AFF sequences and $\psi'_{r,s}(m, n)$.

LEMMA 5.7. *For all $s \geq 2$ and all k we have $\psi'_{r,s}(m, n) \leq k(\Pi'_{r,s,k}(m) + n)$.*

Now we derive upper bounds for $\Pi'_{r,s,k}(m)$.

LEMMA 5.8. *For every $r \geq 2$ we have $\Pi'_{r,2,2}(m) = (r-1)(m-1)$.*

LEMMA 5.9. *For every fixed $r \geq 2$ and $s \geq 3$ we have $\Pi'_{r,s,s}(m) \leq (r-1)\binom{m-2}{s-2} = O(m^{s-2})$.*

RECURRENCE 5.10. *We have*

$$\Pi'_{r,s,2k-1}(2m) \leq 2\Pi'_{r,s,2k-1}(m) + 2\Pi'_{r,s-1,k}(m).$$

COROLLARY 5.11. *For fixed $r \geq 2$ and $s \geq 3$, if we let $k = 2^{s-2} + 1$, then $\Pi'_{r,s,k}(m) = O(m(\log m)^{s-3})$ (where the implicit constant might depend on r and s).*

RECURRENCE 5.12. *Let $r \geq 2$ and $s \geq 3$ be fixed. Let k_1, k_2, k_3 , and k be integers satisfying $k = k_2k_3 + 2k_1 - 3k_2 - k_3 + 2$. Then,*

$$\begin{aligned} \Pi'_{r,s,k}(m) \leq & \left(1 + \frac{m}{t}\right) \left(\Pi'_{r,s,k}(t) + 2\Pi'_{r,s-1,k_1}(t)\right) \\ & + \Pi'_{r,s-2,k_2}(t) + \Pi'_{r,s,k_3}\left(1 + \frac{m}{t}\right), \end{aligned}$$

where t is a free parameter.

COROLLARY 5.13. *Let $R_s(d)$ be the sequences defined in Corollary 4.12. Then, for every $s \geq 3$ and $d \geq 2$, if $k \geq R_{s-1}(d)$ then*

$$\Pi'_{r,s,k}(m) \leq cm\alpha_d(m)^{s-3} \quad \text{for all } m.$$

Here, $c = c(r, s)$ is a constant that depends only on r and s .

From Corollary 5.13 and Lemma 5.7 we obtain:

COROLLARY 5.14. *Let $s \geq 4$. Then, for all r, m , and n we have*

$$\psi'_{r,s}(m, n) \leq C_{r,s,d}(m\alpha_d(m)^{s-3} + n) \quad \text{for all } d,$$

for some constants $C_{r,s,d}$ of the form

$$C_{r,s,d} = \begin{cases} 2^{(1/t)d^t \pm O(d^{t-1})}, & s \text{ odd}; \\ 2^{(1/t)d^t \log_2 d \pm O(d^t)}, & s \text{ even}; \end{cases}$$

where $t = \lfloor (s-3)/2 \rfloor$.

We can finally prove our upper bounds for $F_{r,s}(n)$.

Proof of Theorem 1.3. Take $d = \alpha(m)$ in Corollary 5.14, then substitute into Lemma 5.6, bounding $\varphi_{r,s-2}(n)$ by induction on s . Use the base cases $F_{r,2}(n), F_{r,3}(n) = O(n)$ (by Lemma 5.2). (As before, $\varphi_{r,s-2}(n)$ contributes only to lower-order terms in the exponent.) \square

6 The lower bound construction for $s = 3$

We now prove Theorem 1.6 by constructing, for every n , a Davenport–Schinzel sequence of order 3 on n distinct symbols with length at least $2n\alpha(n) - O(n)$.

For this purpose, we first define a two-dimensional array of sequences $Z_d(m)$, for $d, m \geq 1$, with the following properties:

- Each symbol in $Z_d(m)$ appears exactly $2d+1$ times.
- $Z_d(m)$ contains no forbidden alternation $ababa$. (We do not preclude the presence of adjacent repeated symbols in $Z_d(m)$.)
- $Z_d(m)$ is partitioned into *blocks*, where each block contains only distinct symbols. Some of the blocks in $Z_d(m)$ are *special blocks*. Each symbol in $Z_d(m)$ makes its first and last occurrences in special blocks. Furthermore, the special blocks are entirely composed of first and last occurrences of symbols (there might be *both* first and last occurrences in the same special block). Moreover, each special block in $Z_d(m)$ has length exactly m .
- For $d \geq 2$, each special block is surrounded by regular blocks on both sides, and *no* regular block is surrounded by special blocks on both sides. For the former property, we place empty regular blocks at the beginning and end of $Z_d(m)$, for $d \geq 2$.

In what follows, we enclose regular blocks by $()$'s, and special blocks by $[]$'s.

The base cases of the construction are as follows: For $d = 1$, we let

$$Z_1(m) = [12 \dots m](m \dots 21)[12 \dots m].$$

For $m = 1$ and $d \geq 2$ we let

$$Z_d(1) = ()1(1) \dots (1)[1](),$$

with $2d+1$ ones. Note the empty regular blocks at the beginning and end of $Z_d(1)$.

Let $S_d(m)$ be the number of special blocks in $Z_d(m)$.

The recursive construction. For $d, m \geq 2$, we construct $Z_d(m)$ recursively as follows. Let $Z' = Z_d(m-1)$. Let $f = S_d(m-1)$ be the number of special blocks in Z' , and let $Z^* = Z_{d-1}(f)$. Thus, the special blocks in Z^* have length f . Let $g = S_{d-1}(f)$ be the number of special blocks in Z^* .

Create g copies of Z' , each copy using “fresh” symbols which do not occur in Z^* nor in any preceding copy of Z' . Thus, we have one copy of Z' for each special block in Z^* . And each special block in Z^* has as many symbols as there are special blocks in the corresponding copy of Z' .

Let C_i be the i -th special block in Z^* , and let Z'_i be the i -th copy of Z' . Let a be the ℓ -th symbol in C_i , and let D_ℓ be the ℓ -th special block in Z'_i . We duplicate a into aa , and we insert the aa into Z'_i as follows:

If the a in C_i is the first a in Z^* , then the first of the two a 's falls at the end of D_ℓ and the second a falls at the beginning of the block after D_ℓ . And if the a in

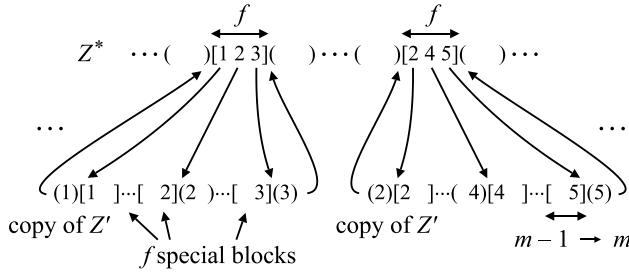


Figure 1: Construction of $Z_d(m)$ from Z^* and many copies of Z' .

C_i is the last a in Z^* , then the first of the two a 's falls at the end of the block before D_ℓ and the second a falls at the beginning of D_ℓ . (Recall that D_ℓ is surrounded by regular blocks in Z'_i .)

Since no regular block in Z'_i is surrounded by special blocks on both sides, it follows that no block in Z'_i receives more than one symbol from Z^* . Thus, even after the insertions, no block in Z'_i has repeated symbols.

After these insertions, at the place in Z^* where the block C_i used to be there is now a hole. We insert Z'_i (with its extra symbols) into this hole. After doing this for all special blocks C_i in Z^* , we obtain the desired sequence $Z_d(m)$. See Figure 1.

It is easy to check that every symbol in $Z_d(m)$ has multiplicity $2d + 1$: The symbols of the copies of Z' already had multiplicity $2d + 1$, and the symbols of Z^* had their multiplicity increased from $2d - 1$ to $2d + 1$.

It is also clear that each symbol makes its first and last occurrences in special blocks, that the special blocks in $Z_d(m)$ contain only first and last occurrences, and that their length increased from $m - 1$ to m . Furthermore, every special block is surrounded by regular blocks on both sides, and no regular block is surrounded by special blocks on both sides. And $Z_d(m)$ contains empty regular blocks at the beginning and at the end.

No *ababa*. Let us now verify that $Z_d(m)$ contains no alternation *ababa* of length 5. Assume by induction that this is true for the component sequences Z' and Z^* .

Suppose for a contradiction that $Z_d(m)$ contains an alternation *ababa*. The symbols a and b cannot come from the same copy of Z' , by induction, and they cannot come from different copies of Z' , since they would not alternate at all.

Further, a and b cannot both come from Z^* : By the induction assumption, Z^* contains no forbidden alternation. And the duplications of symbols $a \rightarrow aa$ cannot create a forbidden alternation, since the two a 's end up being adjacent in $Z_d(m)$.

Next, suppose that a comes from a copy of Z' and b

comes from Z^* . Then this copy of Z' received two non-adjacent b 's. But this is impossible by construction: Our copy of Z' received symbols from a single special block of Z^* , which contained at most one b . This b was duplicated into two *adjacent* copies bb .

Finally, suppose that a comes from Z^* and b comes from a copy of Z' . Then this copy of Z' received an a that is neither the first nor the last a in Z^* . This is also a contradiction.

REMARK 6.1. *The above construction shares some similarities with an earlier construction by Komjáth [10].*

Analysis. The sequences $Z_d(m)$ are not necessarily Davenport–Schinzel sequences, since they might contain adjacent repeated symbols at the interface between blocks. But these repetitions can be easily eliminated by deleting at most one symbol per block of $Z_d(m)$. Let $Z'_d(m)$ be the resulting repetition-free (and so Davenport–Schinzel) sequences.

It can be shown that the average block length in $Z_d(m)$ is at least $m/2$. Therefore,

$$(6.7) \quad |Z'_d(m)| \geq (1 - 2/m)|Z_d(m)|.$$

Now diagonalize by taking the sequences $Z'_d = Z'_d(d)$ for $d \geq 1$. Let $Z_d^* = Z_d(d)$, and let $N_d^* = \|Z_d^*\|$. It can be shown that $N_d^* \leq A(d + c)$ for some constant c —or in other words, $d \geq \alpha(N_d^*) - c$. Thus, by (6.7), and since each symbol appears $2d + 1$ times in Z_d^* ,

$$\begin{aligned} |Z'_d| &\geq (1 - 2/d)|Z_d^*| = (1 - 2/d)(2d + 1)N_d^* \\ &\geq 2N_d^* \cdot \alpha(N_d^*) - O(N_d^*). \end{aligned}$$

We have thus proven that $\lambda_3(n) \geq 2n\alpha(n) - O(n)$ for n of the form $n = N_d^*$. We need to interpolate to intermediate values of n . Given an arbitrary integer n , let $d = d(n)$ be the unique integer such that

$$N_d^* < N_{d+1}^* \leq n < N_{d+2}^*.$$

Let $t = \lfloor n/N_d^* \rfloor$, and let $Z''(n)$ be a concatenation of t copies of Z'_d , with disjoint sets of symbols. It can be shown that $\lambda_3(n) \geq |Z''(n)| \geq 2n\alpha(n) - O(n)$, and we are done.

REMARK 6.2. *The coefficient 2 in our bound comes from the fact that each symbol appears roughly $2d$ times in $Z_d(m)$. In previous constructions [17, 10, 14] each symbol appears only $d \pm O(1)$ times in the equivalent sequence. Sharir and Agarwal [14] lost an additional factor of 2 in the interpolation step; we avoided this loss by letting $Z''(n)$ consist of many copies of Z'_d , instead of using Z'_{d+1} (which would have been a more obvious choice).*

7 Conclusion and open problems

Unfortunately, the bounds for $\lambda_s(n)$ for odd $s \geq 5$ are still not completely tight. We believe the new upper bounds are the true bounds, simply by analogy to the interval-chain bounds. But the construction that gives the lower bounds does not seem to work when s is odd.

Are there other problems that, like interval chains and almost-DS sequences, satisfy recurrences like Recurrence 4.7 and Recurrence 4.11? If so, it would be interesting to find more examples of such problems.

The reason we can unambiguously talk about the coefficient that multiplies $\alpha(n)$ (e.g., in Theorems 1.2 and 1.6), despite the fact that there are several different versions of $\alpha(n)$ in the literature, is that all these versions differ from one another by at most an *additive* constant. Thus, the coefficient multiplying $\alpha(n)$ is not affected. On the other hand, one cannot talk about the leading coefficient in $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$, for example, unless a standard definition of $\alpha(n)$ is agreed upon.

Can our construction of Section 6 be realized as the lower envelope of segments in the plane? If so, it would yield a factor-of-2 improvement for this problem as well.

Acknowledgements. The “inverse” problem of almost-DS sequences was raised by my advisor, Micha Sharir, during a discussion with Haim Kaplan and me. Recurrence 4.7 for $\Pi_k^3(m)$ is also due to Micha, as well as the proof of Lemma 1.4. Haim found how an upper bound for $\Pi_k^3(m)$ yields an upper bound for $\lambda_3(n)$; his argument has been generalized in Lemma 4.1. I also owe thanks to Micha for many intensive discussions, and for reading carefully several drafts of this paper.

Finally, I wish to thank Martin Klazar for some useful email correspondence, and to the anonymous referees for their helpful suggestions.

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