The cover time of random geometric graphs

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Abstract

We study the cover time of random geometric graphs. Let $I(d) = [0,1]^d$ denote the unit torus in d dimensions. Let $D(x,r)$ denote the ball (disc) of radius r. Let Υ_d be the volume of the unit ball $D(0,1)$ in d dimensions. A random geometric graph $G = G(d,r,n)$ in d dimensions is defined as follows: Sample n points V independently and uniformly at random from $I(d)$. For each point x draw a ball $D(x,r)$ of radius r about x. The vertex set $V(G) = V$ and the edge set $E(G) = \{\{v,w\} : w \neq v, w \in D(v,r)\}$. Let $G(d,r,n)$, $d \geq 3$ be a random geometric graph. Let $c > 1$ be constant, and let $r = (c \log n/(\Upsilon_d n))^{1/d}$. Then **whp**

$$C_G \sim c \log\left(\frac{c}{c-1}\right) n \log n.$$

1 Introduction

Let $G = (V,E)$ be a connected graph with $|V| = n$ vertices, and $|E| = m$ edges. For $v \in V$ let C_v be the expected time taken for a simple random walk W on G starting at v, to visit every vertex of G. The *vertex cover time* C_G of G is defined as $C_G = \max_{v \in V} C_v$. The (vertex) cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [2] that $C_G \leq 2m(n-1)$. It was shown by Feige [12], [13], that for any connected graph G, the cover time satisfies $(1-o(1))n \log n \leq C_G \leq (1+o(1))\frac{4}{27}n^3$. As an example of a graph achieving the lower bound, the complete graph K_n has cover time determined by the Coupon Collector problem. The *lollipop* graph consisting of a path of length $n/3$ joined to a clique of size $2n/3$ gives the asymptotic upper bound for the cover time.

A few words on notation. Results on random graphs are always asymptotic in n, the size of the vertex set. The notation $A_n \sim B_n$ means that $\lim_{n\to\infty} A_n/B_n = 1$, and **whp** (with high probability) means with probability tending to 1 as $n \to \infty$. Poly-log factors are suppressed in $\tilde{O}, \tilde{\Omega}$.

In a series of papers, we have studied the cover time of various models of a random graph, see [7], [8], [5], [6] and [9]. In this paper we study random geoemtric graphs.

Let I denote the unit interval $[0,1]$ and let $I(d) = [0,1]^d$ denote the unit torus in d dimensions. We use the torus for convenience, to avoid boundary effects. Let $D(x,r)$ denote the ball (disc) of radius r, and thus

$$D(x,r) =$$
$$\{y \in I(d) : \sum_{i=1}^{d} \min\left\{|x_i - y_i|^2, |x_i - (1+y_i)|^2\right\} \leq r^2\}$$

Let Υ_d be the volume of the unit ball $D(0,1)$ in d dimensions. Thus

$$\Upsilon_d = (\pi^{d/2})/\Gamma(d/2+1) = \begin{cases} \frac{\pi^k}{k!} & d = 2k, \text{ even} \\ \frac{2^d k! \pi^k}{d!} & d = 2k+1, \text{ odd} \end{cases}$$

A random geometric graph $G = G(d,r,n)$ in d dimensions is defined as follows: Sample n points V independently and uniformly at random from $I(d)$. For each point x draw a ball $D(x,r)$ of radius r about x. The vertex set $V(G) = V$ and the edge set $E(G) = \{\{v,w\} : w \neq v, w \in D(v,r)\}$

Geometric graphs are widely used as models of adhoc wireless networks [14], [15], [19] in which each transmitter has transmission radius r and can only communicate with other transmitters within that radius. In the simplest model of a random geometric graph, the n points representing transmitters, are distributed uniformly at random (uar) in the unit square. Any other point v within the circle radius r centered at a transmitter u, is joined to u by an edge. If $r \geq \sqrt{c \log n/(\pi n)}$, $c > 1$, the graph formed in this way is connected **whp** [14, 17].

Avin and Ercal [3] considered the cover time of geometric graphs in the case $d = 2$:

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THEOREM 1.1. *If* $G = G(2,r,n)$ *and* $r^2 > \frac{8\log n}{n}$ *then* **whp**

$$C_G = \Theta(n\log n).$$

They indicate that their result can be generalized to $d \geq 3$.

In this paper we consider $d \geq 3$ and replace $\Theta(n\log n)$ by an asymptotically correct constant:

THEOREM 1.2. *Let* $G(d,r,n)$, $d \geq 3$ *be a random geometric graph. Let* $c > 1$ *be constant, and let* $r = \left(\frac{c\log n}{\Upsilon_d n}\right)^{1/d}$. *Then* **whp**

$$C_G \sim c\log\left(\frac{c}{c-1}\right)n\log n. \tag{1.1}$$

As c increases, the RHS of (1.1) is asymptotic to $n\log n$. It will be clear that we can allow $c \to \infty$ in our analysis and obtain this estimate rigorously. We find it convenient however just to deal with the case of c constant.

Structure of the paper To prove Theorem 1.2, we establish bounds on the cover time using the method of [5]. In Section 3 we state a general lemma, the first visit time lemma, on which our results are based. In the next few sections we estimate various quantities needed for this lemma. Then in Section 6, we establish upper and lower bounds on the cover time of G.

2 Some properties of G

The first thing to note is that G is connected **whp**. See for example [14] and [17].

We next give easy upper and lower bounds on vertex degrees.

LEMMA 2.1. *For* $c > 1$ *there exists a constant* $c_0 > 0$ *such that* **whp**

$$c_0\log n \leq d(v) \leq \Delta_0 = (c+10)\log n \qquad \textit{for all } v \in V.$$

Proof omitted. □

It is a simple matter to show using the Chebychev inequality that the number of edges m of G satisfies

$$m \sim \frac{1}{2}cn\log n. \tag{2.2}$$

Let

$$I_c = [c_0\log n, \Delta_0].$$

We have shown that **whp** all degrees lie in I_c and we now estimate how many vertices there are of degree i, $i \in I_c$.

LEMMA 2.2. *Let*

$$\overline{D}(k) = n\binom{n-1}{k}p^k(1-p)^{n-1-k} \leq \frac{2}{n^{c-1}}\left(\frac{ne}{k}\right)^k.$$

denote the expected number of vertices v *with* $d(v) = k$. *Let* $D(k)$ *denote the actual number and*

$$\begin{aligned} K_0 &= \{k \in I_c : \overline{D}(k) \leq (\log n)^{-2}\}. \\ K_1 &= \{k \in I_c : (\log n)^{-2} \leq \overline{D}(k) \leq (\log n)^2\}. \\ K_2 &= I_c \setminus (K_0 \cup K_1). \end{aligned}$$

Then, **whp**

(a) *For* $k \in K_0$, $D(k) = 0$. *For* $k \in K_1$, $D(k) \leq (\log n)^4$, *and*

$$\min\{k \in K_1\} \geq (\log n)^{1/2} \textit{and } |K_1| = O(\log\log n).$$

If $k \in K_2$ *then* $\frac{1}{2}\overline{D}(k) \leq D(k) \leq 2\overline{D}(k)$.

(b) *Let* $k_1 = (c-1)\log n$. *Let* $\gamma_c = (c-1)\log(c/(c-1))$. *There are* $\sim (nep/k_1)^{k_1}n^{1-c} = n^{\gamma_c + o(1)}$ *vertices* v *with* $d(v) = k_1$.

Proof An identical calculation is made in [7] for the degree sequence of the random graph $G_{n,p}$. □

Let

$$h_a = \epsilon_a r \textit{ and } h_b = L_1 h_a$$

where $\epsilon_a \leq 1/4d$ is a small positive constant. We assume that $\ell_a = 1/h_a$ is an even integer and L_1 is a large odd integer constant which divides ℓ_a, and thus ℓ_a/L_1 is even. The size of L_1 is determined by the inequalities (2.5) and (2.6), and Γ_d is given by (2.3). The parameter L of (2.6) satisfies (2.4).

We partition $I(d)$ into grids K_a, K_b where K_a, K_b are made up of cubes of side h_a, h_b and K_a is a refinement of K_b. Note that if x, y are in K_a-cubes that share a $(d-1)$-dimensional face then x, y are adjacent in G.

Each K_a-cube is labeled by a d-tuple in $[\ell_a]^d$. Given a K_b-cube B we define a *line* of B to be a set of L_1 K_a-cubes of B, where the labels are constant except for exactly one index. A *slice* of B is a set of L_1^{d-1} K_a-cubes of B, where the labels are constant on exactly one index. A slice is *extreme* if meets a $(d-1)$-dimensional face of B. Given a line of B, its two *ends* are the two K_a-cubes lying in extreme slices.

Given a K_a-cube A, let $K_L(A)$ be a cube of side Lh_a with A at its centre (assuming that L is an odd integer). Consider $K_L(A)$ to be partitioned into L^d K_a-cubes. If we fix a K_a-cube A, then the number of points in V that are in A is distributed as $Bin(n, \alpha\log n/n)$ where $\alpha = c\epsilon_a^d/\Upsilon_d$. A cube is *light* if it contains fewer than $\epsilon_\ell\alpha\log n$ points in V where ϵ_ℓ is a small positive constant, otherwise it is *heavy*. If C is an arbitrary union of K_a-cubes $A_1, A_2, \ldots, A_k$ then $heavy(C) = \{A_i : A_i \textit{ is heavy}\}$.

Let

$$\Gamma_d = 20\Upsilon_d\epsilon_a^{-d}. \tag{2.3}$$

Lemma 2.3. *Suppose that $L = O(1)$. Then* **whp** *there does not exist a K_a-cube A such that $K_L(A)$ contains Γ_d light K_a-cubes*

Proof omitted. ☐

We use the following result, which is part of Lemma 9.9 of Penrose [17]: Let $B_Z(n) = [n]^d$ and let A be a subset of $B_Z(n)$. We assume a graph structure with vertices $[n]^d$, and where two vertices are adjacent if there Hamming distance is one. The *external vertex boundary* $\partial^+_{B(n)}A$ is the set of vertices in $B_Z(n) \setminus A$ which are adjacent to some $x \in A$.

Lemma 2.4. *If $A \subseteq [n]^d$ and $|A| \leq 2n^d/3$ then*

$$|\partial^+_{B(n)}A| \geq (2d)^{-1}(1 - (2/3)^{1/d})|A|^{(d-1)/d}.$$

☐

Fix a cube C that is the union of K_a-cubes and is of side Lh_a, $L = O(\log n)$. Consider the graph H_C that consists of *heavy*(C). Two vertices are adjacent if the corresponding cubes share a $(d-1)$-dimensional face. Let $\kappa_1(H_C)$ denote the size of the largest component of H_C. We also somewhat loosely refer to "the largest component of C".

Lemma 2.5. **Whp**

$$\kappa_1(H_C) \geq L^d - \gamma\Gamma_d^2$$

for some $\gamma = \gamma(d) \geq 1$.

Proof omitted. ☐

Recall that $h_b = L_1 h_a$ defines the side length of the K_b-cubes. For arbitrary L, we define the L-centre $K_L^* = K_L^*(B)$ of a K_b-cube B to be $K_L(A)$ where A is the centre K_a-cube of B. If F is an extreme slice of B then we can define its L-centre as follows: If X is the centre K_a-cube of F then $K_L^*(F) = F \cap K_L(X)$.

A line Λ of B containing a cube $\hat{A} \in K_L^*$ is *good* if it satisfies the following conditions:

1. All its K_a-cubes are heavy.
2. Let A_1, A_2 be the K_a-cubes at the ends of Λ. Let F_i be the extreme slice containing A_i. Let $F_i^* = K_L^*(F_i)$. Let H_i^* be the sub-graph of H_B induced by F_i^*. We require that $A_i \in \kappa_1(H_i^*)$ for $i = 1, 2$.

We say that a K_a-cube $\hat{A}$ is *good* if $\hat{A} \in K_L^* \cap \kappa_1(H_B)$ and if the d lines through $\hat{A}$ are good. A good cube $\hat{A}$ is *useable* if all K_a-cubes within distance 10 of $\hat{A}$ are good.

Lemma 2.6. *If $L > \gamma\Gamma_d^2$ then* **whp** *K_L^* contains at least $L^d - (2d)^{11}L\gamma\Gamma_d^2$ useable cubes.*

Proof omitted. ☐

We make L large enough so that

$$L > \gamma\Gamma_d^2 \text{ and } L^d - (2d)^{11}L\gamma\Gamma_d^2 > L^d/2 \tag{2.4}$$

and so Lemma 2.6 holds **whp**. We also assume that

$$L_1 \geq L^d. \tag{2.5}$$

Finally, Lemma 2.8 below, requires the following lower bound on L_1:

$$L_1 \geq 3^d(\Gamma_d + \gamma\Gamma_d^2)/\epsilon_a. \tag{2.6}$$

Let $B_1, B_2, \ldots, B_M$, $M = \Omega(n/\log n)$ be an enumeration of the K_b-cubes. Our grids define bipartite graphs, this is why we chose $\ell_a, \ell_a/L_1$ even. Thus each cube will have a parity, with neighbouring cubes having different parity. Similarly, the K_a sub-cubes of each B_i have a parity. We choose a useable cube A_i in each B_i, $i = 1, 2, \ldots, M$. A_i is chosen to have the same parity as B_i.

Lemma 2.7. *If B_i, B_j share a $(d-1)$-dimensional face F then there is a path $P(i,j)$ of length $3L_1$ of heavy cubes joining A_i and A_j, where A_i, A_j are useable cubes, as above. These paths are pair-wise internally vertex (cube) disjoint. By a path, we mean a sequence of K_a-cubes with consecutive cubes sharing a $(d-1)$-face.*

Proof omitted. ☐

We now consider points that do not lie in a cube of H_B for any B in the K_b dissection of $I(d)$.

Lemma 2.8. *For $v \in V$, let C_v be the K_b-cube containing v in the K_b dissection of $I(d)$ and let A_v be the K_a-cube containing v in the K_a dissection of C_v. Then, there exist a K_b-cube B_i such that*

- **(a)** *v is at G-distance $\leq 3^d(\Gamma_d + \gamma\Gamma_d^2)$ from $\kappa_1(H_{B_i})$.*
- **(b)** *v is at G-distance $O(1)$ from any point $w \in V$ which lies in any sub-cube A of B_i.*

Proof omitted. ☐

3 Estimating first visit probabilities

We use the approach of [5, 6, 8, 9]. Let G denote a fixed connected graph, and u is some arbitrary vertex from which a walk $\mathcal{W}_u$ is started. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk, and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. Let π be the steady state distribution of the random walk $\mathcal{W}_u$. Let $\pi_v = \pi(v)$ denote the stationary distribution of the vertex v. For an unbiased ergodic random walk on a graph G with $m = m(G)$ edges, $\pi_v = \frac{d(v)}{2m}$, where $d(v)$ denotes the degree of v in G.

Let $d(t) = \max_{u,x\in V} |P_u^{(t)}(x) - \pi_x|$, and let T be such that, for $t \geq T$

$$\max_{u,x\in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3}. \tag{3.7}$$

It follows from e.g. Aldous and Fill [1] that $d(s+t) \leq 2d(s)d(t)$ and so for $k \geq 1$,

$$\max_{u,x\in V} |P_u^{(kT)}(x) - \pi_x| \leq \frac{2^{k-1}}{n^{3k}}. \tag{3.8}$$

Now fix $u \neq v \in V$. Next, let $r_t = \mathbf{Pr}(\mathcal{W}_v(t) = v)$ be the probability that this walk returns to v at step t. Let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \tag{3.9}$$

For a large constant $K > 0$, let

$$\lambda = \frac{1}{KT}. \tag{3.10}$$

For $t \geq 0$, let $\mathcal{A}_t(v)$ be the event that $\mathcal{W}_u$ does not visit v in steps $T, T+1, \ldots, t$ The vertex u will have to be implicit in this definition. The following was proved in [8].

Lemma 3.1. *Suppose that*

(a) *For some constant $\theta > 0$, we have*

$$\min_{|z|\leq 1+\lambda} |R_T(z)| \geq \theta.$$

(b) *$T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$.*

Let

$$p_v = \frac{\pi_v}{R_T(1)(1 + O(T\pi_v))},$$

where $R_T(1)$ is from (3.9).

Then for all $t \geq T$,

$$\mathbf{Pr}(\mathcal{A}_t(v)) = \frac{(1 + O(T\pi_v))}{(1+p_v)^t} + O(Te^{-\lambda t/2}). \tag{3.11}$$

The evaluation of $R_T(z)$ at $z = 1$ occurs frequently in our calculations in this paper. For the rest of the paper u, v will not be fixed and it is appropriate to replace the notation $R_T(1)$ by something dependent on v. We use the notation R_v. For $u \neq v$ we let $R_{u,v}$ denote the expected visits by $\mathcal{W}_u$ to v up to time T.

4 Mixing time of the random walk

We need two basic results on mixing times. First let $\lambda_{\max}$ be the second largest eigenvalue of the transition matrix P. Then,

$$|P_u^{(t)}(x) - \pi_x| \leq \left(\frac{\pi_x}{\pi_u}\right)^{1/2} \lambda_{\max}^t \leq (4c/c_0)^{1/2} \lambda_{\max}^t. \tag{4.12}$$

See for example, Jerrum and Sinclair [16].

Next, for each $x \neq y \in V$ let $Q(x,y)$ be a *canonical* path from x to y in G. Then, see for example Sinclair [18], we have that

$$\lambda_{\max} \leq 1 - \frac{1}{\rho}, \tag{4.13}$$

where

$$\rho = \max_{e=\{x,y\}\in E(G)} \frac{1}{\pi(x)P(x,y)} \sum_{\gamma_{ab}\ni e} \pi(a)\pi(b)|\gamma_{ab}|, \tag{4.14}$$

and $|\gamma_{ab}|$ is the length of the canonical path $Q(a,b)$ from a to b.

Here is an example.

Lemma 4.1.

$$1 - \lambda_{\max} = \tilde{\Omega}\left(n^{-2/d}\right).$$

Proof Consider the K_b-grid of Section 2. Arbitrarily choose $x_i \in C_i$ for $i = 1, 2, \ldots, M$. We first define canonical paths between the x_i. We can in a natural way express $x_i = y(j_1, j_2, \ldots, j_d)$ where $0 \leq j_t < 1/h_b$ for $1 \leq t \leq d$. The path from $y(j_1, j_2, \ldots, j_d)$ to $y(k_1, k_2, \ldots, k_d)$ goes

$$\begin{aligned} y(j_1, j_2, \ldots, j_d) &\Longleftrightarrow y(j_1+1, j_2, \ldots, j_d) \Longleftrightarrow \cdots \\ y(k_1, j_2, \ldots, j_d) &\Longleftrightarrow y(k_1, j_2+1, \ldots, j_d) \\ &\Longleftrightarrow \cdots \Longleftrightarrow y(k_1, k_2, \ldots, k_d). \end{aligned}$$

The $\Longleftrightarrow$ represents a path in G that follows a $P(i,j)$, choosing one vertex from each K_a-cube as necessary.

Thus we first increase the first component (mod $1/h_b$) until it is k_1 and then do the same for the second and subsequent components. Each such path has length at most $3dL_1/h_b = O(1/r)$. If we fix a grid edge e (really an edge of a path $\Longleftrightarrow$) joining $y(j_1, \ldots, j_t, \ldots, j_d)$ to $y(j_1, \ldots, j_t+1, \ldots, j_d)$ then the number of paths through e is $O(h_b^{-d-1})$; any such path starts at $y(l_1, \ldots, l_t, j_{t+1}, \ldots, j_d)$ and ends at $y(j_1, \ldots, j_{t-1}, l_t', \ldots, l_d')$ for some $l_1, \ldots, l_t, l_t', \ldots, l_d$.

We obtain canonical paths for every pair of vertices by using Lemma 2.8 i.e. we connect each point x of V to its closest $x_i = \phi(x)$. Each x_i is chosen by $O(\log n)$

points in this way. (Using Chernoff bounds, we bound the number of points in V at G-distance $O(1)$ (Lemma 2.8) from any x_i). Our path from x to y goes x to $\phi(x)$ to $\phi(y)$ to y. After this we find that each path has length $O(1/r)$ and each edge is in $\tilde{O}\left(1/r^{d+1}\right)$ paths. It follows from (4.14) that

$$\rho = \tilde{O}\left(n \cdot 1 \cdot r^{-d-1} \cdot n^{-2} \cdot r^{-1}\right) = \tilde{O}\left(n^{2/d}\right)$$

and the lemma follows from (4.13). □

Applying (4.12) we see that we can take

$$T = \tilde{O}\left(n^{2/d}\right) \tag{4.15}$$

when we use Lemma 3.1.

5 Upper bound on the number of returns during the mixing time

Having obtained a good enough bound on T, we now show that **whp** $R_v = 1 + o(1)$ for all $v \in V$.

If we distinguish two vertices a, b, the escape probability $p_{\rm esc} = p_{\rm esc}(a, b)$, is the probability that a random walk leaving a does not return to a before reaching b. This probability is given by

$$p_{\rm esc} = \frac{1}{d(a)\mathrm{R_{EFF}}}, \tag{5.16}$$

where $\mathrm{R_{EFF}} = \mathrm{R_{EFF}}(a, b)$ is the *effective resistance* between a and b in an electrical network with all edges having resistance one, see for example Doyle and Snell [10]. If we wish to calculate the escape probability $p_{\rm esc}(a, B)$, where B is a set of vertices, then we can join the vertices of B to an additional vertex b by edges of resistance zero, and calculate $p_{\rm esc}(a, b)$.

Let $a = v$, then $R_v(B)$, the expected number of returns to v before reaching B is given by

$$R_v(B) = \frac{1}{p_{\rm esc}} = d(v)\mathrm{R_{EFF}}. \tag{5.17}$$

Raleigh's Theorem (see e.g. [10]), states that deleting edges increases effective resistance. Provided we do not prune edges incident with v, edge deletion increases $R_v(B)$.

The following lemma gives a crude bound for $R_{EFF}(a, b)$.

Lemma 5.1. **Whp** *for fixed* $d \geq 3$

$$\max_{a,b} R_{EFF}(a, b) = O(1).$$

Proof Chandra et al [4] showed that, regardless of the number of vertices, the effective resistance of a d-dimensional toroidal grid is $O(1/d)$ for $d \geq 3$. Lemma 2.7 implies that such a grid can be embedded into G by taking the chosen vertices $x_1, x_2, \ldots, x_M$ together with paths joining x_i, x_j when C_i, C_j are adjacent. This gives a subgraph of G for which the effective resistance between any two vertices is $O(1)$. Then note that we have shown that any vertex can be joined to an x_i by a path of length $O(1)$, Lemma 2.8. □

5.1 The number of vertices likely to visit one fixed vertex

For $x \neq y \in V$ and $t > 0$ we let

$$\eta(x, y) = \mathbf{Pr}(\exists\, 1 \leq t \leq T : \mathcal{W}_x(t) = y). \tag{5.18}$$

We aim to show that if y is fixed, then $\eta(x, y) = o(1)$ for almost all choices of x.

For $\epsilon > 0$ let

$$B_\epsilon(x) = \left\{y \in V : \ \eta(y, x) \geq \epsilon\right\}.$$

By stationarity, for fixed t,

$$\sum_{y \in V} \pi_y \mathbf{Pr}(\mathcal{W}_y(t) = x) = \pi_x.$$

Thus

$$\begin{aligned} T\pi_x &= \sum_{1 \leq t \leq T} \sum_{y \in V} \pi_y P_y^{(t)}(x) \\ &= \sum_{y \in V} \pi_y \sum_{1 \leq t \leq T} P_y^{(t)}(x) \\ &\geq \sum_{y \in V} \pi_y \eta(y, x) \\ &\geq \sum_{y \in B_x(\epsilon)} \pi_y \eta(y, x) \\ &\geq \pi_{min}\epsilon |B_x(\epsilon)|. \end{aligned}$$

where $\pi_{min} = \min\left\{\pi_y : y \in V\right\}$.

Consequently,

$$|B_\epsilon(x)| \leq \frac{T\pi_x}{\epsilon \pi_{min}}.$$

It follows that if

$$U_v = \left\{x : \ \eta(x, v) \geq \frac{1}{(\log n)^2}\right\}$$

then **whp**

$$|U_v| = \tilde{O}\left(T\right). \tag{5.19}$$

We prove next that

Lemma 5.2. **Whp** $R_v = 1 + O(1/\log n)$ *for all* $v \in V$.

Proof Fix $v \in V$ and make v the centre of a K_b-cube C_v of side $L_1 h_a$ and partition $I(d)$ into K_b-cubes with C_v as one of the cubes. Let $\bar{U}_v = V \setminus U_v$.

If p_v is the probability of a first return to v by $\mathcal{W}_v$ within time T then

$$\begin{aligned} &(5.20) \qquad p_v \;\le\; 1 - p_{\text{esc}}(v, \bar{U}_v) + 1/(\log n)^2. \\ &(5.21) \qquad R_v \;\le\; \frac{1}{1 - p_v}. \end{aligned}$$

Given (5.20) and (5.21) it is sufficient to prove that **whp**

$$(5.22) \quad p_{\text{esc}}(v, \bar{U}_v) = 1 - O(1/\log n) \qquad \text{for all } v \in V.$$

We focus on proving (5.22). We first construct the following sub-graph $G_v^* = (V^*, E_v^*)$ of G. Let S denote the set of heavy K_a-cubes that make up all the paths $P(i,j)$ of Section 2. For each $A \in S$, we choose an arbitrary subset of vertices of size $\epsilon_\ell \alpha \log n$, $\alpha = c/(\Upsilon_d \epsilon_a^d)$ and place these vertices in V^* (see definition of *heavy*). The edges E_v^* consist of those edges (x,y) where x, y come from K_a-cubes which are adjacent on some $P(i,j)$. We obtain G_v^* by adding vertex v and all of the edges of G that are incident with v. We then take each heavy cube of C_v and choose $\epsilon_\ell \alpha \log n$ vertices and add the edges between each adjacent pair of heavy cubes. This construction is equally valid when v is in any of the A_i or $P(i,j)$.

The degree of v in G^* is the same as its degree in G and G^* is a sub-graph of G. From Raleigh's Theorem, we see that $\text{R}_{\text{EFF}}(v, \bar{U}_v) \le \text{R}^*_{\text{EFF}}(v, \bar{U}_v)$. So if $p^*_{\text{esc}}(v, \bar{U}_v)$ is the probability that the random walk $\mathcal{W}_v^*$ on G^* visits $\bar{U}_v$ before returning to v then

$$p_{\text{esc}}(v, \bar{U}_v) \ge p^*_{\text{esc}}(v, \bar{U}_v).$$

So to prove (5.22), it suffices to prove

$$(5.23) \qquad p^*_{\text{esc}}(v, \bar{U}_v) = 1 - O(1/\log n)$$

A random walk $\mathcal{W}_v^*$ on G^* can be coupled with a random walk $\tilde{\mathcal{W}}$ on a d-dimensional grid $\tilde{\Gamma}$ with M vertices as follows: Let $V_i^* = V^* \cap A_i$ where A_i is defined prior to Lemma 2.7. When $\mathcal{W}_v^*$ is inside V_i^*, $\tilde{\mathcal{W}}$ will be at the ith vertex of $\tilde{\Gamma}$. If $\mathcal{W}_v^*$ is on a vertex of a path $P(i,j)$ then $\tilde{\Gamma}$ stays at its current vertex. Since the paths $P(i,j)$ are all of the same length and since the V_i^* are all the same size the next vertex that $\mathcal{W}_v^*$ visits is equally likely to be any neighbour of the current vertex.

Now consider the random walk $\tilde{\mathcal{W}}$ on the $[N]^d$, $N = M^{1/d}$, where $M = \Omega(n/\log n)$. We can assume w.l.o.g. that $\tilde{\mathcal{W}}$ starts at the origin. Let $\mathcal{J}_V^* = \left\{ i \in [M] : V_i^* \subseteq \bar{U}_v \right\}$.

We will prove that with probability bounded below by a constant $\gamma > 0$, the random walk $\tilde{\mathcal{W}}$ from the origin which is of length $T \ge c_1 N^2$, c_1 small, will visit $\mathcal{J}_V^*$ before returning to the origin. We will also show that

$$\begin{aligned} (5.24) \quad \mathbf{Pr}(\mathcal{W}_v^* \text{ revisits } v \mid \mathcal{W}_v^* \text{ visits } V \cap C_v) \\ \le \epsilon_v = O(1/\log n). \end{aligned}$$

This includes the probability of a re-visit to v before $\mathcal{W}^*$ first leaves C_v.

Thus,

$$1 - p^*_{\text{esc}}(v, \bar{U}_v) \le \epsilon_v \sum_{i=0}^{\infty} (1-\gamma)^i = \frac{\epsilon_v}{\gamma} = O(1/\log n)$$

and this completes the proof of (5.23).

Now because $d \ge 3$ there is a positive probability γ' such that we have $\hat{\mathcal{W}}(t') \ne 0$ for $1 \le t' \le t = c_1 N^2$. This is because the random walk $\hat{\mathcal{W}}$ on the infinite d-dimensional lattice is non-recurrent i.e. there is a positive probability ζ_d that it does not return to the origin. If c_1 is small, then there is a greater than $1 - \zeta_d/2$ chance that $\hat{\mathcal{W}}$ stays inside the box $[-N/3, N/3]^d$ for the first t steps and this implies that $\hat{\mathcal{W}}$ does not return to the origin with probability at least $\zeta_d - \zeta_d/2$. Assume w.l.o.g. that $\hat{\mathcal{W}}(t)_d \ne 0$, this happens with probability $\Omega(1/d)$. Now for a fixed $x \in N^d$ with $x_d \ne 0$ we have

$$\mathbf{Pr}(\mathcal{W}_v^*(t) = x \mid x_d \ne 0) = O(dt^{-d/2}) = O(N^{-d})$$

as, to be at x, each component has to be correct and for a single component, $O(t^{-1/2})$ is the right probability. So

$$\begin{aligned} \mathbf{Pr}(\mathcal{W}_v^*(t) \notin \mathcal{J}_V^*) &= \tilde{O}\left(|U_v| N^{-d}\right) \\ &= \tilde{O}\left(T N^{-d}\right) = O(1/\log n). \end{aligned}$$

Now any constant $\gamma < \gamma'$ will suffice.

5.2 Proof of (5.24)**:** For this we consider the graph H with vertex set equal to the set of heavy K_a-cubes C. Two heavy cubes C_1, C_2 are defined to be adjacent if the centres of C_1, C_2 are no more than $r_1 = r - 2d^{1/2} h_a$ apart. In which case, $v_i \in C_i \cap V$, $i = 1, 2$ implies that $(v_1, v_2) \in G$.

CLAIM 1. *The ball $D(v,r)$ contains $\Upsilon_d \epsilon_a^{-d}(1 - \epsilon_B)$ K_a-cubes, where $0 \le \epsilon_B \le (1 - 2\epsilon_a d^{1/2})^d$.*

Proof omitted. □

CLAIM 2. **Whp**, *for every $v \in V$, $D(v,r)$ contains at least one heavy cube.*

Proof omitted. □

The following claim is somewhat crude, but will prove sufficient.

Claim 3. **Whp** *H contains no component of size $\kappa \le \log\log n$ vertices.*

Proof omitted. □

Now consider a random walk on G^*. Note firstly that, when at a neighbour of v, there is only an $O(1/\log n)$ chance of returning to v at the next step. Secondly, at any vertex, there is at least the chance $\epsilon_h = \frac{\epsilon_\ell \alpha}{c+10}$ of moving to a heavy cube (Claim 2). Then there is at least the chance $\epsilon^* = \epsilon_h^{3^d(\Gamma_d + c_b\Gamma_d^2)+L_1^d+1}$ of leaving C_1 by going along the path promised by Lemma 2.8 to a giant component and then going through this giant component and leaving C_1. (This is obviously a ridiculously small estimate, but there is not much point in trying to improve it). Thus the chance of returning to v is $O(1/\log n)$ either when starting at v or when returning to C_1.

This completes the proof of (5.24) and the lemma. □

5.3 Conditions of Lemma 3.1

It is clear from (4.15) that Lemma 3.1(b) holds. To check condition (a) we see that if $|z| \le 1 + \lambda$ then

$$\left|\sum_{j=1}^{T-1} r_j z^j\right| \le (1+\lambda)^T \sum_{j=1}^{T-1} r_t = o(1).$$

By Lemma 5.2, $R_v = 1 + O(1/\log n)$ and thus $R_T(z) \ge 1 - o(1)$ for $|z| \le 1 + \lambda$.

6 Cover time

From (3.11) of Lemma 3.1 we have that for all $t \ge T$,

$$\mathbf{Pr}(\mathcal{A}_t(v)) = \frac{1+o(1)}{\left(1 + \frac{\pi_v}{R_v(1+O(T\pi_v))}\right)^{t+1}} + O(Te^{-\lambda t/2}). \tag{6.25}$$

An upper bound is obtained as follows: Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$\mathbf{Pr}(T_G(u) > t) = \mathbf{Pr}(U_t > 0) \le \min\{1, \mathbf{E}U_t\}, \tag{6.26}$$

$$C_u = \mathbf{E}T_G(u) = \sum_{t>0} \mathbf{Pr}(T_G(u) > t) \tag{6.27}$$

It follows from (6.25,6.26) that for all t

$$C_u \le t+1+\sum_{s>t} \mathbf{E}U_s = t+1+\sum_{v\in V}\sum_{s>t} \mathbf{Pr}(\mathcal{A}_s^*(v)), \tag{6.28}$$

where $\mathbf{Pr}(\mathcal{A}_s(v))$ serves as an upper bound on the probability that v is unvisited by step s.

To lower bound the cover time of the graph, we use the Chebychev inequality.

Let

$$t^* = \left(c\log\left(\frac{c}{c-1}\right)\right) n\log n$$

and

$$t_0 = (1-\delta)t^* \text{ and } t_1 = (1+\delta)t^*$$

where $\delta = o(1)$ but grows sufficiently slowly that inequalities below are satisfied.

6.1 Upper bound on the cover time

For $v \in V$ we have

$$\mathbf{Pr}(\boldsymbol{A}_s(v)) = (1+o(1))\exp\{-(1+o(1/\log n))\pi_v s\} + O(Te^{-\Omega(s/T)})$$

and

$$\pi_v = \frac{d(v)}{2m}.$$

Then we find, using the **whp** bounds in Lemma 2.2,

$$C_u \le t_0 + 1 + S_1 + S_2 + O(nTe^{-\Omega(s/T)}) \tag{6.29}$$

where

$$\begin{aligned} S_i &= \sum_{k\in K_i} D(k) \sum_{s\ge t_0} \exp\left\{-\frac{(1-o(1))ks}{2m}\right\} \\ &\le 3m \sum_{k\in K_i} \frac{D(k)}{k} e^{-(1-o(1))kt_0/2m} \\ &\le 3m \sum_{k\in K_i} \frac{D(k)}{k} \left(\frac{c-1}{c}\right)^{(1+\delta/2)k}. \end{aligned}$$

For the main term,

$$\begin{aligned} S_2 &\le \frac{6m}{n^{c-1}} \sum_{k\in I_c} \left(\frac{nep}{k}\right)^k \left(\frac{c-1}{c}\right)^{(1+\delta/2)k} \\ &\le 6m \sum_{k\in I_c} e^{-\delta k/2c} \\ &= o(t_0), \end{aligned} \tag{6.30}$$

where we have used the fact that $(nep(c-1))/(kc))^k$ is maximized at $k = np(c-1)/c$, and $\delta k = \Omega(1)$.

Continuing we get

$$\begin{aligned} S_1 &\le 3m \sum_{k\in K_1} \frac{D(k)}{k} \left(\frac{c-1}{c}\right)^{(1+\delta/2)k} \\ &\le 3m \sum_{k\in K_1} \frac{(\log n)^4}{k} \left(\frac{c-1}{c}\right)^{(1+\delta/2)k} \\ &= o(t_0) \end{aligned} \tag{6.31}$$

since $D(k) \le (\log n)^4$ and $\min\{k \in K_2\} \ge (\log n)^{1/2}$.

It now follows from (6.29) – (6.31) that $C_u \le t_0 + o(t_0)$.

6.2 Lower bound on the cover time We can find a vertex u and a set of vertices S_0 such that at time t_0, the probability the set S_0 is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_0$ **whp** which implies that $C_G \geq (1-o(1))t^*$.

We construct S_0 as follows. Let k_1 be as defined in Lemma 2.2. Let $S_1 = \{v : d(v) = k_1\}$ and let $A = \big\{(u,v) : \ u \notin S_1, v \in S_1, \eta(u,v) \geq 1/(\log n)^2\big\}$ (see (5.18) for the definition of $\eta(u,v)$). It follows from (5.19) that **whp** $|A| = \tilde{O}\,(T|S_1|)$. By simple counting, we see that there exists $u \notin S_1$ such that $|\,\{v \in S_1 : \ (u,v) \in A\}\,| = \tilde{O}\,(T|S_1|/n) = o(|S_1|)$. We choose such a u and let $S_0 = \{v \in S_1 : \ (u,v) \notin A\}$

Let $\mathcal{B}_v$ be the event that $\mathcal{W}_u$ does not visit v in the time interval $[1,T]$. Then, by our choice of u, we see that for $v \in S_0$,

$$\mathbf{Pr}(\mathcal{B}_v) \geq 1 - 1/(\log n)^2. \tag{6.32}$$

We need to prove that

$$\mathbf{Pr}(\mathcal{A}_t(v) \mid \mathcal{B}_v) \sim \exp\left\{-\frac{(1+o(1))(c-1)t_0}{cn\log n R_v(1+O(T\pi_v))}\right\} \sim \mathbf{Pr}(\mathcal{A}_t(v)). \tag{6.33}$$

The proof of this requires just a small change to the proof of Lemma 3.1.

Then **whp**, if Z_0 is the number of vertices in S_0 that are not visited in time $[1,t_0]$,

$$\begin{aligned}\mathbf{E}(Z_0) \geq{}& A_1 n^{-o(1)+(c-1)\log(c/(c-1))}\exp\left\{-(1+o(1))\frac{(c-1)t_0}{cn\log n}\right\}\\ \geq{}& A_2 n^{\frac{1}{2}\delta(c-1)\log(c/(c-1))} \to \infty\end{aligned} \tag{6.34}$$

for some constants $A_1, A_2 > 0$.

We show next, for all $v, w \in S_0$, that

$$\eta(v,w) = O(1/\log n). \tag{6.35}$$

We define a new graph G_ψ by identifying v, w and replacing them with a new node ψ. The proof of Lemma 4.1 can be modified to show that mixing time T_ψ of G_ψ will satisfy (4.15). Indeed, we can assume that our choice of x_i's excludes v, w and then v, w can only appear as endpoints of canonical paths. For a path from x to ψ we can then choose one of the already constructed canonical paths from x to v or x to w.

Similarly, the proof of Lemma 5.2 can be modified to show that

$$p_{\mathrm{esc}}(\psi, \bar{U}_\psi) = 1 - O(1/\log n). \tag{6.36}$$

Here $\bar{U}_\psi = \bar{U}_v \cap \bar{U}_w$ and the probability is for a random walk in G_ψ starting at ψ. Our modification of Lemma 5.2 requires a random walk on the d-dimensional lattice, starting at point x (a surrogate for v's cube), to have positive probability of not returning to x or some other fixed vertex y (a surrogate for w's cube) and vice-versa. This is a simple consequence of Polya's classic result.

Now

$$\eta(v,w) \leq 2(1 - p_{\mathrm{esc}}(\psi, \bar{B}_\psi)) + O(1/(\log n)^2)$$

since the RHS above is at least the probability that the random walk $\mathcal{W}_v$ (in G) reaches w within T steps. The factor two accounts for forcing the walk to move to a neighbour of v at the start. This verifies (6.35).

Now for $v, w \neq u$ let

$$\mathcal{A}_t(v,w) = \mathcal{A}_t(v) \wedge \mathcal{A}_t(w) \textit{ and } \mathcal{B}_{v,w} = \mathcal{B}_v \wedge \mathcal{B}_w.$$

Now for $v \neq w$,

$$\mathbf{Pr}(\mathcal{B}_{v,w}) \geq 1 - \mathbf{Pr}(\bar{\mathcal{B}}_v) - \mathbf{Pr}(\bar{\mathcal{B}}_w) = 1 - 2/(\log n)^2 \tag{6.37}$$

and we will show that

$$\mathbf{Pr}(\mathcal{A}_{t_1}(v,w) \mid \mathcal{B}_{v,w}) \leq A_0 \mathbf{Pr}(\mathcal{A}_{t_1}(v))\mathbf{Pr}(\mathcal{A}_{t_1}(w)). \tag{6.38}$$

for all $v, w \in S_0$, for some absolute constant A_0 and

$$\begin{aligned}\mathbf{Pr}(\mathcal{A}_{t_1}(v,w) \mid \mathcal{B}_{v,w}) ={}& (1+o(1))\mathbf{Pr}(\mathcal{A}_{t_1}(v,w)) =\\ & (1+o(1))\mathbf{Pr}(\mathcal{A}_{t_1}(v))\mathbf{Pr}(\mathcal{A}_{t_1}(w))\end{aligned} \tag{6.39}$$

for almost all pairs $(v,w) \in S_0$.

It then follows that

$$\mathbf{E}(Z_0(Z_0-1)) \leq (1+o(1))\mathbf{E}(Z_0)^2$$

and so

$$\begin{aligned}\mathbf{Pr}(Z_0 \neq 0) \geq{}& \frac{\mathbf{E}(Z_0)^2}{\mathbf{E}(Z_0^2)}\\ ={}& \frac{1}{\frac{\mathbf{E}(Z_0(Z_0-1))}{\mathbf{E}(Z_0)^2} + (\mathbf{E}Z_0)^{-1}} = 1 - o(1)\end{aligned}$$

from (6.34) and (6.39).

6.2.1 Proof of (6.38) We argue next that

$$R_\psi \leq \frac{R_v + R_w}{2} + O(1/\log n) \tag{6.40}$$

Walks in G_ψ can be mapped to walks in G in a natural way. If the walk is not at ψ then it chooses its successor with the same probability. This includes neighbours of

v, w, since they are non-adjacent in v. When at ψ, with probability 1/2 it moves to a neighbour of v and with probability 1/2 it moves to a neighbour of w. Returns to v, w account for the term $\frac{R_v+R_w}{2}$. We must also account for returns to ψ that come from walks from v to w and vice-versa. This can be overestimated by $R_v\eta(v,w) + R_w\eta(w,v)$, giving the $O(1/\log n)$ term.

Putting $\pi_v = \pi_w = \pi_0$, this implies that

$$\begin{aligned}
&\frac{\pi_\psi}{R_\psi} - \frac{\pi_v}{R_v} - \frac{\pi_w}{R_w}\\
&= \frac{\pi_0}{R_\psi R_v R_w}\left(2R_vR_w - R_\psi(R_v + R_w)\right)\\
&\geq \frac{\pi_0}{R_\psi R_v R_w}\times\\
&\left(2R_vR_w - \left(\frac{R_v + R_w}{2} + O\left(\frac{1}{\log n}\right)\right)(R_v + R_w)\right)\\
&= \frac{\pi_0}{2R_\psi R_v R_w}\left((R_v - R_w)^2 + O(1/\log n)\right)\\
&= O\left(\frac{1}{n \log n}\right). \qquad (6.41)
\end{aligned}$$

So, with $\mathbf{Pr}_\psi$ referring to probability in the space of random walks on G_ψ,

$$\begin{aligned}
&\mathbf{Pr}_\psi(\mathcal{A}_{t_0}(\psi)) \sim \exp\left\{-\frac{t_0\pi_\psi}{(1 + O(T\pi_\psi))R_\psi}\right\}\\
&\sim \exp\left\{-\frac{t_0\pi_v}{R_v}\right\}\exp\left\{-\frac{t_0\pi_w}{R_w}\right\}\exp\left\{O\left(\frac{t_0}{n\log n}\right)\right\}\\
&= O(\mathbf{Pr}(\mathcal{A}_{t_0}(v))\mathbf{Pr}(\mathcal{A}_{t_0}(w))). \qquad (6.42)
\end{aligned}$$

But, using rapid mixing in G_ψ,

$$\begin{aligned}
&\mathbf{Pr}_\psi(\mathcal{A}_{t_0}(\psi))\\
&= \sum_{x\neq\psi} P_{\psi,u}^{T_\psi}(x)\mathbf{Pr}_\psi(\mathcal{W}_x(t - T_\psi) \neq \psi,\ T_\psi \leq t \leq t_0)\\
&= \sum_{x\neq\psi}\left(\frac{d(x)}{2m} + O(n^{-3})\right)\times\\
&\qquad\qquad \mathbf{Pr}_\psi(\mathcal{W}_x(t - T_\psi) \neq \psi,\ T_\psi \leq t \leq t_0)\\
&= \sum_{x\neq v,w}\left(P_u^{T_\psi}(x) + O(n^{-3})\right)\times \qquad (6.43)\\
&\qquad\qquad \mathbf{Pr}(\mathcal{W}_x(t - T_\psi) \neq v, w,\ T_\psi \leq t \leq t_0)\\
&= \mathbf{Pr}(\mathcal{W}_u(t) \neq v, w,\ T_\psi \leq t \leq t_0) + O(n^{-3})\\
&= \mathbf{Pr}(\mathcal{A}_{t_0}(v,w)) + O(n^{-3}). \qquad (6.44)
\end{aligned}$$

Equation (6.43) follows because there is a natural measure preserving map ϕ between walks in G that start at $x \neq v, w$ and avoid v, w and walks in G_ψ that avoid ψ. The map ϕ also shows that

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{A}_{t_0}(v,w) \wedge \mathcal{B}_{v,w}) =\\
&\quad \mathbf{Pr}_\psi(\mathcal{A}_{t_0}(\psi) \wedge \mathcal{B}_\psi) = (1 + o(1))\mathbf{Pr}_\psi(\mathcal{A}_{t_0}(\psi) \mid \mathcal{B}_\psi).
\end{aligned}$$

But the argument for (6.33) can be used to show that

$$\mathbf{Pr}_\psi(\mathcal{A}_{t_0}(\psi) \mid \mathcal{B}_\psi) = (1 + o(1))\mathbf{Pr}_\psi(\mathcal{A}_{t_0}(\psi)). \tag{6.45}$$

Equation (6.38) follows from (6.42)–(6.45).

6.3 Proof of (6.39) We get this sharpening of (6.38) whenever we can replace the $O(1/\log n)$ in (6.40) by $o(1/\log n)$. This replacement can be done whenever we can replace $O(1/\log n)$ in (6.35) by $o(1/\log n)$. We show that this can be done for almost all pairs $v, w \in S_0$.

There is a very simple argument when c is sufficiently large. The size of S_0 is $n^{\gamma_c + o(1)}$ **whp** where $\gamma_c = (c-1)\log\left(\frac{c}{c-1}\right)$. For any fixed $v \in S_0$ there are at most $T(\log n)^2$ vertices w such that $\eta(v,w) \geq 1/(\log n)^2$. If $\gamma_c > 2/d$ then **whp** $T(\log n)^2 = o(|S_0|)$ and (6.39) holds. For example, if $c \geq 2$ then $\gamma_c \geq \log 2 = .69314718 > 2/d$ for $d \geq 3$.

So now we must consider the case where $1 < c \leq 2$. Let A denote the set of *unordered* pairs $v, w \in S_0$ such that either $\eta(v,w) \geq 1/(\log n)^2$ or $\eta(w,v) \geq 1/(\log n)^2$. To prove (6.39) it is enough to show that

$$\begin{aligned}
&\mathbf{Pr}(\eta(v,w) \geq 1/(\log n)^2 \mid v, w \in S_0,\ |v - w| \geq r^{1/2})\\
&\qquad\qquad = o(1).
\end{aligned} \tag{6.46}$$

Here, if $v = (v_1, v_2, \ldots, v_d)$ then $|v| = (v_1^2 + v_2^2 + \cdots + v_d^2)^{1/2}$.

Note that the expected number of pairs $v, w \in S_0$ such that $|v - w| \leq r^{1/2}$ can be bounded by $\tilde{O}\left(\max\left\{n^{2\gamma_c - 1/2 + o(1)}, 1\right\}\right)$. So **whp** there are at most $\log n$ times this quantity. These pairs can therefore be ignored in our verification of (6.39).

To prove (6.46) we choose two points v, w for which $|v - w| \geq r^{1/2}$, condition on $v, w \in S_0$ and then bound $\mathbf{Pr}(\eta(v,w))$ from below. We condition on $v, w \in S_0$ by randomly placing k_1 points into each of $D(v,r), D(w,r)$. We then couple part of the remaining construction of G along with the first T steps of the random walk $\mathcal{W}_v$. Let $P_t = (x_0 = v, x_1, \ldots, x_{\phi(t)})$ be the unique path obtained from the walk $(\mathcal{W}_v(0) = v, \mathcal{W}_v(1), \ldots, \mathcal{W}_v(t))$ by repeatedly removing paths between repeated vertices. If $\mathcal{W}_v$ reaches w within T steps, then there exists $t \geq r^{-1/2}$ such that $x_{\phi(t)} = w$. We weaken this to $x_{\phi(t)} \in D(w,r)$.

Let $i_v = \max\{i : x_i \in D(v,2r)\}$. Notice next that for $i > i_v$, x_{i+1} is randomly chosen from $D(x_i, r)$ and these choices are made independently, at least until the walk reaches $D(w,r)$, if at all. Suppose now that $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,d})$ and let $y_{i,j} = x_{i,j} - x_{i-1,j}$. The $y_{i,j}$, $j = 1, 2, \dots, d$ are not independent. Their sum of squares is at most r^2. On the other hand, if $B(x_i)$ is the cube of side $2r/d^{1/2}$ with centre x_i and we condition on $x_i \in B(x_{i-1})$ then the $y_{i,j}$, $j = 1, 2, \dots, d$ are independent. So let $I_t = \{i_v < i < t : x_i \in B(x_{i-1})\}$. The size of I_t is $Bin(t - i_v, q)$ where q is bounded away from 0. So, by use of the Chernoff bounds, we can assume that $|I_t| \geq tq/2$. Now fix t and condition on the values $I_t, y_{i,j}, i \notin I_t$ and let $Z_j = \sum_{i \in I_t} y_{i,j}$, $j = 1, 2, \dots, d$. Now we have $Z_1, Z_2, \dots, Z_d$ independent. Fix j. Then $Z_j = \sum_{l=1}^{s} \xi_l$ where **whp** $s \geq tq/2$ and ξ_l is uniform in $[-r/d^{1/2}, r/d^{1/2}]$. As such it is well approximated by a normal distribution. In particular we can use the Berry-Esseen inequality, see for example [11]:

Let $X_1, X_2, \dots, X_n$ *be i.i.d. with* $\mathbf{E}(X_i) = 0, \mathbf{E}(X_i^2) = \sigma^2$ *and* $\mathbf{E}(|X_i|^3) = \rho < \infty$. *If* $F_n(x)$ *is the distribution of* $(X_1 + X_2 + \cdots + X_n)/(\sigma\sqrt{n})$ *and* $\mathcal{N}(x)$ *is the standard normal distribution, then*

$$|F_n(x) - \mathcal{N}(x)| \leq \frac{3\rho}{\sigma^3\sqrt{n}}.$$

To have $x_{\phi(t)} \in D(w,r)$ each Z_j will have to have to take a value in an interval A_j of length at most $2r$. This interval being determined by the values $x_i, i \notin I_t$. It follows from the Berry-Esseen inequality that $\mathbf{Pr}(Z_j \in A_j) = O(t^{-1/2})$. (We have $\sigma = \Omega(r)$ and $\rho = O(r^3)$). Hence, for some constant C,

$$\mathbf{E}(\eta(v,w)) \leq C \sum_{t=r^{-1/2}}^{T} t^{-d/2} = O(r^{1/4})$$

and (6.46) and (6.39) follow.

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