

On stars and Steiner stars. II

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Abstract

A *Steiner star* for a set P of n points in \mathbb{R}^d connects an arbitrary center point to all points of P , while a *star* connects a point $p \in P$ to the remaining $n - 1$ points of P . All connections are realized by straight line segments. Fekete and Meijer showed that the minimum star is at most $\sqrt{2}$ times longer than the minimum Steiner star for any finite point configuration in \mathbb{R}^d . The maximum ratio between them, over all finite point configurations in \mathbb{R}^d , is called the *star Steiner ratio* in \mathbb{R}^d . It is conjectured that this ratio is $4/\pi = 1.2732\dots$ in the plane and $4/3 = 1.3333\dots$ in three dimensions. Here we give upper bounds of 1.3631 in the plane, and 1.3833 in 3-space, thereby substantially improving recent upper bounds of 1.3999, and $\sqrt{2} - 10^{-4}$, respectively. Our results also imply improved bounds on the maximum ratios between the minimum star and the maximum matching in two and three dimensions.

Our method exploits the connection with the classical problem of estimating the maximum sum of pairwise distances among n points on the unit sphere, first studied by László Fejes Tóth. It is quite general and yields the first non-trivial estimates below $\sqrt{2}$ on the star Steiner ratios in arbitrary dimensions. We show, however, that the star Steiner ratio in \mathbb{R}^d tends to $\sqrt{2}$, the upper bound given by Fekete and Meijer, as d goes to infinity. Our estimates on the star Steiner ratios are therefore much closer to the conjectured values in higher dimensions! As it turns out, our estimates as well as the conjectured values of the Steiner ratios (in the limit, for n going to infinity) are related to the classical infinite Wallis product: $\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{4n^2-1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \dots$

1 Introduction

Let P be a set of n points in \mathbb{R}^d . A *Steiner star* for P connects an arbitrary center point to all points of P , while a *star* connects a point $p \in P$ to the remaining $n - 1$ points of

P . All connections are realized by straight line segments. The study of minimum Steiner stars and minimum stars is motivated by applications in facility location and computational statistics [7, 8, 9, 13]. The *Weber point*, also known as the *Fermat-Torricelli point* or *Euclidean median*, is the point of the space that minimizes the sum of distances to n given points in \mathbb{R}^d . The problem of finding such a point can be asked in any metric space. It is known that even in the plane, the Weber point cannot be computed exactly, already for $n \geq 5$ [5, 10]. (For $n = 3$ and 4, resp., Torricelli and Fagnano gave algebraic solutions.) The Weber point can however be approximated with arbitrary precision [8, 9], based on Weiszfeld's algorithm [18]. The Fermat-Torricelli problem has been also studied in Minkowski spaces [17]. The reader can find more information on the problem in [11], and in the recent paper of the first two named authors [12].

The maximum ratio between the lengths of the minimum star and the minimum Steiner star, over all finite point configurations in \mathbb{R}^d , is called the *star Steiner ratio* in \mathbb{R}^d , denoted by ρ_d . The same ratio for a specific value of n is denoted by $\rho_d(n)$. Obviously $\rho_d(n) \leq \rho_d$, for each n . Fekete and Meijer [15] were the first to study the star Steiner ratio. They proved that $\rho_d \leq \sqrt{2}$ holds for any dimension d . It is conjectured that $\rho_2 = 4/\pi = 1.2732\dots$, and $\rho_3 = 4/3 = 1.3333\dots$, which are the limit ratios for a uniform mass distribution on a circle, and a sphere, respectively (in these cases the Weber point is the center of the circle, or sphere) [15]. By exploiting these bounds, Fekete and Meijer also established bounds on the maximum ratio η_d , between the length of the minimum star and that of a maximum matching on a set of n points (n even) in two and three dimensions ($d = 2, 3$).

In a recent paper, the upper bounds on the Steiner ratios for $d = 2, 3$ have been lowered to 1.3999 in the plane, and to $\sqrt{2} - 10^{-4}$ in 3-space [12]. The method of proof used there was not entirely satisfying, as it involved heavy use of linear programming. Besides that, the proofs were quite involved and the improvements were rather small, particularly in three-space. It was also shown in [12] that the bound $4/\pi$ holds in two special cases, corresponding to the lower bound construction; details at the end of Section 2.

In this paper we get closer to the core of the problem, obtain substantially better upper bounds, and moreover replace the use of linear programming by precise and much shorter

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Ratio	Lower bound	Old upper bound	New upper bound
$\rho_2 : (\min S)/SS^*$	$\frac{4}{\pi} = 1.2732\dots$	1.3999	1.3631 †
$\rho_3 : (\min S)/SS^*$	$\frac{4}{3} = 1.3333\dots$	$\sqrt{2} - 10^{-4} = 1.4141\dots$	1.3833 †
$\rho_4 : (\min S)/SS^*$	$\frac{64}{15\pi} = 1.3581\dots$ †	$\sqrt{2} = 1.4142\dots$	1.3923 †
$\rho_5 : (\min S)/SS^*$	$\frac{48}{35} = 1.3714\dots$ †	$\sqrt{2} = 1.4142\dots$	1.3973 †
$\rho_{100} : (\min S)/SS^*$	1.4124\dots †	$\sqrt{2} = 1.4142\dots$	1.4135 †
$\eta_2 : \min S / \max M$	$\frac{4}{3} = 1.3333\dots$	1.6165	1.5739 †
$\eta_3 : \min S / \max M$	$\frac{3}{2} = 1.5$	1.9999	1.9562 †

Table 1: Lower and upper bounds on star Steiner ratios ($\rho_2, \rho_3, \rho_4, \rho_5, \rho_{100}$), and matching ratios (η_2, η_3) for some small values of d . Those marked with † are new.

mathematical proofs. Here we prove that $\rho_2 \leq 1.3631$, and $\rho_3 \leq 1.3833$. Based on these estimates, we can then further improve the estimates given on η_2 and η_3 , using the method developed by Fekete and Meijer [15]. Our improvements are summarized in Table 1. Here $\min S$ denotes a minimum star and its length (with a slight abuse of notation), SS^* denotes a minimum Steiner star and its length, and $\max M$ denotes a maximum length matching and its length (for an even number of points). Finally, our method yields the first non-trivial estimates below $\sqrt{2}$ on the star Steiner ratios in arbitrary dimensions. Among others, we show that the upper bound $\sqrt{2}$ on the star Steiner ratio given Fekete and Meijer is in fact a very good approximation of this ratio for higher dimensions d ; thus in this sense the problem in the plane is the most interesting one.

If q_0, q_1, \dots, q_{n-1} are n variable points on the unit (radius) sphere in \mathbb{R}^d , let $G(d, n)$ denote the maximum value of the function $\sum_{i < j} |q_i q_j|$, i.e., the maximum value of the sum of pairwise distances among the points. It was shown by László Fejes Tóth [14], and rediscovered in [12], that $G(2, n)$ has a nice expression in closed form:

$$(1.1) \quad G(2, n) = \frac{n}{\tan \frac{\pi}{2n}} = \frac{2}{\pi} n^2 - \frac{\pi}{6} + O\left(\frac{1}{n^2}\right).$$

However, only the simpler inequality $G(2, n) \leq \frac{2}{\pi} n^2$ will be needed here. The exact determination of $G(d, n)$ for $d \geq 3$ is considered to be a difficult geometric discrepancy problem [3, pp. 298], however estimates of the form $G(d, n) \leq c_d n^2/2$ are known [6], where c_d is the ‘‘constant of uniform distribution’’ for the sphere [1, 3, 4]: c_d equals the average inter-point distance for a uniform mass distribution on the unit sphere in \mathbb{R}^d . In particular for $d = 3$, Alexander [1, 2] has shown that

$$(1.2) \quad \frac{2}{3} n^2 - 10n^{1/2} < G(3, n) < \frac{2}{3} n^2 - \frac{1}{2}.$$

Here $c_2 = 4/\pi$, and $c_3 = 4/3$. The connection with this problem is explained in the next section.

2 Stars in the plane

Fix an arbitrary coordinate system. Let $P = \{p_0, \dots, p_{n-1}\}$ be a set of n points in the Euclidean plane, and let $p_i = (x_i, y_i)$, for $i = 0, \dots, n-1$. Let SS^* be a minimal Steiner star for P , and assume that its center $c = (x, y)$ is not an element of P . As noted in [15], the minimality of the Steiner star implies that the sum of the unit vectors rooted at c and oriented to the points vanishes, i.e., $\sum_{i=0}^{n-1} \frac{\overrightarrow{cp_i}}{|\overrightarrow{cp_i}|} = 0$. For completeness, we include here the brief argument. The length of the star centered at an arbitrary point (x, y) is

$$L(x, y) = \sum_{i=0}^{n-1} \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

If (x, y) is the Weber point, we have

$$\frac{\partial}{\partial x} L(x, y) = \frac{\partial}{\partial y} L(x, y) = 0.$$

The two equations give

$$(2.3) \quad \begin{aligned} \sum_{i=0}^{n-1} \frac{x - x_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} &= 0 \quad \text{and} \\ \sum_{i=0}^{n-1} \frac{y - y_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} &= 0. \end{aligned}$$

Our setup is as follows. Refer to Figure 1. We may assume w.l.o.g. that the Weber point is the origin $o = (0, 0)$. We may also assume that the Weber point is not in P , since otherwise the ratio is 1. We can assume w.l.o.g. that the closest point in P to o is $p_0 = (1, 0)$, hence $SS^* = (1 + \delta)n$, for some $\delta \geq 0$. Let C be the unit radius circle centered at o . We denote by \vec{v} a vector v , and by $|\vec{v}|$ its length. Write $\vec{r}_i = \overrightarrow{op_i}$, for $i = 0, 1, \dots, n-1$, and let $\vec{oq}_i = \frac{\overrightarrow{op_i}}{|\overrightarrow{op_i}|}$ be the corresponding unit vector; i.e., q_i is the intersection between \vec{r}_i and the unit circle C . Let $a_i = |\vec{r}_i|$, $b_i = |\overrightarrow{p_0 q_i}|$, and $a'_i = |\overrightarrow{p_0 p_i}|$, for $i = 0, 1, \dots, n-1$. We have $SS^* = \sum_{i=0}^{n-1} a_i$. Finally, let $\alpha_i \in [0, 2\pi)$ be the angle between the positive x -axis and \vec{r}_i .

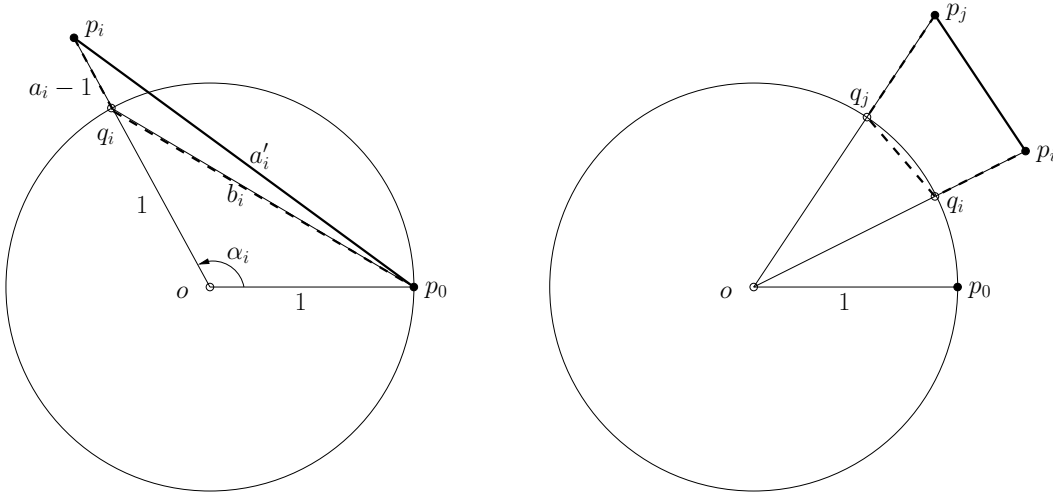


Figure 1: Left: estimating the length of a star centered at $p_0 = (1, 0) \in P$. Right: estimating pairwise distances.

Henceforth (2.3) can be rewritten in the more convenient form

$$(2.4) \quad \sum_{i=0}^{n-1} \cos \alpha_i = 0 \quad \text{and} \quad \sum_{i=0}^{n-1} \sin \alpha_i = 0.$$

Since the orientation of the coordinate system was chosen arbitrarily, such formulas hold for any other orientation. Note that this is also equivalent with the sum of unit vectors vanishing: $\sum_{i=0}^{n-1} \vec{oq}_i = 0$. Moreover, such formulas hold for any dimension $d \geq 2$.

Let S_i be the star (and its length) centered at p_i , for $i = 0, 1, \dots, n-1$, and let $\min S = \min_{0 \leq i \leq n-1} S_i$ denote the minimum star. Using the local optimality condition (2.4), Fekete and Meijer [15] show that if one moves the center of SS^* from the Weber point to a closest point of P , the sum of distances increases by a factor of at most $\sqrt{2}$; this bound which is best possible for this method implies that for any $d \geq 2$ (see [15] for details):

$$(2.5) \quad \rho_d(n) \leq \sqrt{2}, \quad \text{thus} \quad \rho_d \leq \sqrt{2}.$$

From the opposite direction, by considering a uniform mass distribution on the unit sphere in \mathbb{R}^d , one has for any $d \geq 2$:

$$(2.6) \quad \rho_d \geq c_d.$$

Our new argument is a nutshell is as follows. If δ is large, we consider the star centered at a point in P closest to the Weber point, as a good candidate for approximating the minimum star. If δ is small, we use an averaging argument to upper bound the length of the minimum star. In the end we balance the two estimates obtained. Applying the averaging argument (for small δ) leads naturally to the problem of maximizing the sum of pairwise distances among n points on the unit sphere (or unit circle).

THEOREM 2.1. *The star Steiner ratio in the plane is less than 1.3631. More precisely:*

$$\frac{4}{\pi} \leq \rho_2 \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} < 1.3631.$$

Proof. Consider an n -element point set P , and the previous setup. It is enough to show that

$$\frac{\min S}{SS^*} \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}}.$$

By the triangle inequality (see also Fig. 1(left)), we have $a'_i \leq b_i + a_i - 1$, for $i = 0, \dots, n-1$. Hence

$$S_0 = \sum_{i=0}^{n-1} a'_i \leq \sum_{i=0}^{n-1} (b_i + a_i - 1) = \sum_{i=0}^{n-1} b_i + \delta n.$$

By Lemma 4 in [15], the local optimality condition (2.4) implies $\sum_{i=0}^{n-1} b_i \leq n\sqrt{2}$. It follows then that $S_0 \leq (\sqrt{2} + \delta)n$. Hence the star Steiner ratio is at most

$$(2.7) \quad \frac{S_0}{SS^*} \leq \frac{\sqrt{2} + \delta}{1 + \delta}.$$

Since the local optimality condition $\sum_{i=0}^{n-1} \cos \alpha_i = 0$ holds for any d , (2.7) also holds for any d . Observe that

$$\lim_{\delta \rightarrow 0} \frac{\sqrt{2} + \delta}{1 + \delta} = \sqrt{2}, \quad \text{and} \quad \lim_{\delta \rightarrow \infty} \frac{\sqrt{2} + \delta}{1 + \delta} = 1,$$

with the above expression being a decreasing function of δ for $\delta \geq 0$.

Clearly, the sum of the lengths of the n stars (centered at each of the n points) equals twice the sum of pairwise

distances among the points.

$$\sum_{i=0}^{n-1} S_i = 2 \sum_{i<j} |p_i p_j|.$$

By the triangle inequality (see also Fig. 1(right))

$$\begin{aligned} |p_i p_j| &\leq |p_i q_i| + |q_i q_j| + |q_j p_j| \\ &= (a_i - 1) + |q_i q_j| + (a_j - 1). \end{aligned}$$

By summing up over all pairs $i < j$, we get

$$\begin{aligned} \sum_{i=0}^{n-1} S_i &\leq 2 \sum_{i<j} |q_i q_j| + 2(n-1) \sum_{i=0}^{n-1} (a_i - 1) \\ &= 2 \sum_{i<j} |q_i q_j| + 2\delta(n-1)n. \end{aligned}$$

Using the upper bound estimate on $G(2, n)$ in (1.1), we obtain

$$\sum_{i=0}^{n-1} S_i \leq \frac{2n}{\tan \frac{\pi}{2n}} + 2\delta(n-1)n \leq \frac{4}{\pi} n^2 + 2\delta n^2.$$

The minimum of the n stars, $\min S$, clearly satisfies:

$$\min S \leq \frac{\sum_{i=0}^{n-1} S_i}{n} \leq \left(\frac{4}{\pi} + 2\delta \right) n.$$

It follows that the star Steiner ratio is at most

$$(2.8) \quad \frac{\min S}{SS^*} \leq \frac{\frac{4}{\pi} + 2\delta}{1 + \delta}.$$

This estimate holds for any dimension d : the points q_i lie on the unit radius sphere B in \mathbb{R}^d centered at o (rather than on the unit circle C). Observe that

$$\lim_{\delta \rightarrow 0} \frac{\frac{4}{\pi} + 2\delta}{1 + \delta} = \frac{4}{\pi}, \text{ and } \lim_{\delta \rightarrow \infty} \frac{\frac{4}{\pi} + 2\delta}{1 + \delta} = 2,$$

with the above expression being an increasing function of δ for $\delta \geq 0$. Therefore, by combining this observation with the previous observation following (2.7), we get

$$\frac{\min S}{SS^*} \leq \max_{\delta \geq 0} \min \left(\frac{\sqrt{2} + \delta}{1 + \delta}, \frac{\frac{4}{\pi} + 2\delta}{1 + \delta} \right).$$

The maximum value is given by substituting for δ the solution of the equation $\sqrt{2} + \delta = \frac{4}{\pi} + 2\delta$ (that is, by balancing the two upper estimates on the Steiner ratio given by inequalities (2.7) and (2.8)). The solution $\delta_0 = \sqrt{2} - \frac{4}{\pi} = 0.1409 \dots$ yields

$$\rho_2(n) \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} = 1.3630 \dots,$$

thus also

$$(2.9) \quad \rho_2 \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} = 1.3630 \dots$$

This completes the proof of Theorem 2.1. \square

By the result in [15], $SS^* \leq \frac{2}{\sqrt{3}} \max M$. By our Theorem 2.1,

$$\min S \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} \cdot SS^*.$$

Combining the two upper bounds yields the following upper bound on η_2 :

COROLLARY 2.1. *The minimum star to maximum matching ratio in the plane (η_2) is less than 1.5739. That is, for any finite point set*

$$\frac{\min S}{\max M} \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} \cdot \frac{2}{\sqrt{3}} \leq 1.5739.$$

The best known lower bound for this ratio, is $4/3$, see [15].

According to [12, Theorem 2], the star Steiner ratio for a set of n points in the plane that lie on a circle centered at the Weber point is at most

$$\frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} \cdot \frac{4}{\pi} < \frac{4}{\pi}.$$

The same bound holds for any finite point set in the plane where the angles from the Weber point to the n points are uniformly distributed (that is, $\alpha_i = 2i\pi/n$, for $i = 0, 1, \dots, n-1$) [12, Theorem 3]. We think that this is always the case, and thereby venture a slightly stronger version of the conjecture proposed by Fekete and Meijer [15] (however, this is for specific values of n):

CONJECTURE 2.1. *The star Steiner ratio for n points in the plane is*

$$\rho_2(n) = \frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} \cdot \frac{4}{\pi}.$$

3 Stars in the space

In this section we give estimates on the star Steiner ratio in 3-space (Theorem 3.1), and in higher dimensions (Theorem 3.2).

THEOREM 3.1. *The star Steiner ratio in \mathbb{R}^3 is less than 1.3833. More precisely:*

$$\frac{4}{3} \leq \rho_3 \leq \frac{2}{17}(16 - 3\sqrt{2}) < 1.3833.$$

Proof. The method of proof and the setup is the same as in the planar case, so we omit the details. Let B be the unit radius sphere centered at o , analogous to the unit circle C . Now all the points q_i lie on B . Using the upper bound estimate on $G(3, n)$ in (1.2) we get the analogue of Equation (2.8):

$$(3.10) \quad \frac{\min S}{SS^*} \leq \frac{\frac{4}{3} + 2\delta}{1 + \delta}.$$

Taking also (2.7) into account, we have

$$\rho_3 \leq \max_{\delta \geq 0} \min \left(\frac{\sqrt{2} + \delta}{1 + \delta}, \frac{\frac{4}{3} + 2\delta}{1 + \delta} \right).$$

By balancing the two upper estimates in (2.7) and (3.10) as in the planar case yields $\delta_0 = \sqrt{2} - \frac{4}{3} = 0.0808\dots$, and

$$(3.11) \quad \begin{aligned} \rho_3 &\leq \frac{2\sqrt{2} - \frac{4}{3}}{1 + \sqrt{2} - \frac{4}{3}} = \frac{6\sqrt{2} - 4}{3\sqrt{2} - 1} \\ &= \frac{2}{17}(16 - 3\sqrt{2}) = 1.3832\dots \end{aligned}$$

This completes the proof of Theorem 3.1. \square

By the result in [15], $SS^* \leq \sqrt{2} \cdot \max M$. By our Theorem 3.1, $\min S \leq \frac{2}{17}(16 - 3\sqrt{2}) \cdot SS^*$. Combining the two yields the following upper bound on η_3 :

COROLLARY 3.1. *The minimum star to maximum matching ratio in 3-space (η_3) is less than 1.9562. That is, for any finite point set*

$$\frac{\min S}{\max M} \leq \frac{2}{17}(16 - 3\sqrt{2})\sqrt{2} = \frac{4}{17}(8\sqrt{2} - 3) < 1.9562.$$

The best known lower bound for this ratio, is $3/2$, see [15].

The same method we used in proving Theorem 2.1 and Theorem 3.1, together with various approximations yield the following estimates on the star Steiner ratio in \mathbb{R}^d :

THEOREM 3.2. *Let $1 < c_d < 2$ be the “constant of uniform distribution” for the sphere in \mathbb{R}^d , $d \geq 2$. The star Steiner ratio in \mathbb{R}^d is bounded as follows:*

$$c_d \leq \rho_d \leq \frac{2\sqrt{2} - c_d}{1 + \sqrt{2} - c_d}, \text{ where } \lim_{d \rightarrow \infty} c_d = \lim_{d \rightarrow \infty} \rho_d = \sqrt{2}.$$

The following closed formula approximations hold:

$$\sqrt{2}e^{-\frac{1}{4(2d-3)}} \leq \rho_d \leq \frac{2\sqrt{2} - \sqrt{2}e^{-\frac{1}{5(2d-1)}}}{1 + \sqrt{2} - \sqrt{2}e^{-\frac{1}{5(2d-1)}}}.$$

Proof. Note that by the same argument used in the proofs of Theorem 2.1 and Theorem 3.1 (equations (2.9) and (3.11)), we have

$$(3.12) \quad c_d \leq \rho_d \leq \frac{2\sqrt{2} - c_d}{1 + \sqrt{2} - c_d}.$$

In order to establish the limits, we start by computing the “constant of uniform distribution” c_d . Recall that c_d equals the average distance from a point on the unit sphere in \mathbb{R}^d to all the other points on the same sphere, for a uniform mass distribution. It is easy to verify that c_d is given by the following integral formula:

$$(3.13) \quad c_d = \frac{2 \int_0^{\pi/2} \sin^{d-2}(2\alpha) \cdot \sin \alpha \, d\alpha}{\int_0^{\pi/2} \sin^{d-2}(2\alpha) \, d\alpha}.$$

Some initial values are

$$c_1 = 1, \quad c_2 = \frac{\pi}{4} = 1.2732\dots, \quad c_3 = \frac{4}{3} = 1.3333\dots,$$

$$c_4 = \frac{64}{15\pi} = 1.3581\dots, \quad c_5 = \frac{48}{35} = 1.3714\dots$$

In order to establish a recurrence on c_d , define

$$a_{ij} = \int_0^{\pi/2} \sin^i \alpha \cdot \cos^j \alpha \, d\alpha, \quad i, j \geq 0.$$

Some initial values are

$$a_{00} = \frac{\pi}{2}, \quad a_{01} = a_{10} = 1, \quad a_{11} = \frac{1}{2}, \quad a_{02} = a_{20} = \frac{\pi}{4}.$$

Expanding $\sin 2\alpha$ yields then

$$c_d = \frac{2a_{d-1, d-2}}{a_{d-2, d-2}}.$$

Recall that integration by parts leads to the well-known recurrence relations for a_{ij} , for $i, j \geq 1$:

$$\begin{aligned} a_{ij} &= \int_0^{\pi/2} \sin^i \alpha \cdot \cos^j \alpha \, d\alpha \\ &= \left. -\frac{\sin^{i-1} \alpha \cdot \cos^{j+1} \alpha}{i+j} \right|_0^{\pi/2} \\ &\quad + \frac{i-1}{i+j} \int_0^{\pi/2} \sin^{i-2} \alpha \cdot \cos^j \alpha \, d\alpha \\ &= \left. \frac{\sin^{i+1} \alpha \cdot \cos^{j-1} \alpha}{i+j} \right|_0^{\pi/2} \\ &\quad + \frac{j-1}{i+j} \int_0^{\pi/2} \sin^i \alpha \cdot \cos^{j-2} \alpha \, d\alpha. \end{aligned}$$

Plugging these in the formula for c_d immediately gives a recurrence for c_d . For any $d \geq 1$:

$$\begin{aligned} c_{d+2} &= \frac{2 \cdot \frac{d}{2d+1} \cdot \frac{d-1}{2d-1} \cdot a_{d-1, d-2}}{\frac{d-1}{2d} \cdot \frac{d-1}{2d-2} \cdot a_{d-2, d-2}} \\ &= \frac{4d^2}{4d^2 - 1} c_d = \left(1 + \frac{1}{4d^2 - 1} \right) c_d. \end{aligned}$$

Recall at this point the infinite Wallis product from number theory [16]:

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \left(\frac{4k^2}{4k^2 - 1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

Let

$$W_n = \prod_{k=1}^n \left(\frac{4k^2}{4k^2 - 1} \right), \quad \text{and} \quad Z_n = \prod_{k=n+1}^{\infty} \left(\frac{4k^2}{4k^2 - 1} \right),$$

denote the partial finite and respectively partial infinite Wallis products, so that $W_n Z_n = \pi/2$, for every $n \geq 1$. Our recurrence for c_d yields that c_d is an increasing sequence satisfying also

$$(3.14) \quad c_{d+1} c_{d+2} = c_1 c_2 W_d, \quad d \geq 1.$$

Since c_d is bounded, it converges to some limit c . The value of c can be obtained by solving the equation

$$c^2 = c_1 c_2 \frac{\pi}{2} = 2.$$

We thus have $\lim_{d \rightarrow \infty} c_d = \sqrt{2}$. Since $c_d \leq \rho_d \leq \sqrt{2}$, we also have $\lim_{d \rightarrow \infty} \rho_d = \sqrt{2}$. From Equation (3.14), we also get that for $d \geq 3$

$$c_1 c_2 W_{d-2} \leq c_d^2 \leq c_1 c_2 W_{d-1}, \quad \text{or}$$

$$(3.15) \quad \sqrt{c_1 c_2 W_{d-2}} \leq c_d \leq \sqrt{c_1 c_2 W_{d-1}}.$$

Observe that

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{1}{4k^2 - 1} &= \frac{1}{2} \sum_{k=n+1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \\ &= \frac{1}{2(2n+1)}. \end{aligned}$$

Standard inequalities¹ $e^{4x/5} \leq 1+x \leq e^x$ for $x \in [0, 1/3]$ now imply that for each $n \geq 1$

$$Z_n = \prod_{k=n+1}^{\infty} \left(1 + \frac{1}{4k^2 - 1} \right) \leq e^{\sum_{k=n+1}^{\infty} \frac{1}{4k^2 - 1}} = e^{\frac{1}{2(2n+1)}},$$

and

$$Z_n = \prod_{k=n+1}^{\infty} \left(1 + \frac{1}{4k^2 - 1} \right) \geq e^{\frac{4}{5} \sum_{k=n+1}^{\infty} \frac{1}{4k^2 - 1}} = e^{\frac{2}{5(2n+1)}}.$$

Since $W_n = (\pi/2)/Z_n$, we have

$$\frac{\pi}{2} \cdot e^{-\frac{1}{2(2n+1)}} \leq W_n \leq \frac{\pi}{2} \cdot e^{-\frac{2}{5(2n+1)}},$$

¹Here we have chosen $4x/5$ to simplify the resulting expressions.

and consequently (3.15) gives

$$\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} \cdot e^{-\frac{1}{4(2d-3)}} \leq c_d \leq \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} \cdot e^{-\frac{1}{5(2d-1)}},$$

or equivalently

$$\sqrt{2} e^{-\frac{1}{4(2d-3)}} \leq c_d \leq \sqrt{2} e^{-\frac{1}{5(2d-1)}}.$$

Taking into account (3.12) and substituting the above upper bound on c_d , we finally get the estimate (for any $d \geq 4$):

$$\sqrt{2} e^{-\frac{1}{4(2d-3)}} \leq \rho_d \leq \frac{2\sqrt{2} - \sqrt{2} e^{-\frac{1}{5(2d-1)}}}{1 + \sqrt{2} - \sqrt{2} e^{-\frac{1}{5(2d-1)}}}.$$

This completes the proof of Theorem 3.2. \square

For the values of c_d given by (3.13), we can extend the conjecture of Fekete and Meijer to all dimensions $d \geq 2$:

CONJECTURE 3.1. *The star Steiner ratio in \mathbb{R}^d equals the “constant of uniform distribution” for the sphere in \mathbb{R}^d : that is, $\rho_d = c_d$ for any $d \geq 2$.*

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