Maximal Biconnected Subgraphs of Random Planar Graphs*

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Abstract

Let $\mathcal{P}_n$ be the class of simple labeled planar graphs with n vertices, and denote by P_n a graph drawn uniformly at random from this set. Basic properties of P_n were first investigated by Denise, Vasconcellos, and Welsh [7]. Since then, the random planar graph has attracted considerable attention, and is nowadays an important and challenging model for evaluating methods that are developed to study properties of random graphs from classes with structural side constraints.

In this paper we study closely the structure of P_n. More precisely, let $b(\ell; \mathsf{P}_n)$ be the number of blocks (i.e. maximal biconnected subgraphs) of P_n that contain exactly ℓ vertices, and let $lb(\mathsf{P}_n)$ be the number of vertices in the largest block of P_n. We show that with high probability P_n contains a *giant* block that includes up to lower order terms cn vertices, where $c \approx 0.959$ is an analytically given constant. Moreover, we show that the *second* largest block contains only $\widetilde{\Theta}(n^{2/3})$ vertices, and prove sharp concentration results for $b(\ell; \mathsf{P}_n)$, for all $2 \leq \ell \leq n^{2/3}$ (here $\widetilde{\Theta}(.)$ stands for "up to logarithmic factors").

In fact, we obtain this result as a consequence of a much more general result that we prove in this paper. Let $\mathcal{C}$ be a class of labeled connected graphs, and let C_n be a graph drawn uniformly at random from graphs in $\mathcal{C}$ that contain exactly n vertices. Under certain assumptions on $\mathcal{C}$, and depending on the behavior of the singularity of the generating function enumerating the elements of $\mathcal{C}$, C_n belongs with high probability to one of the following three categories, which differ vastly in complexity. C_n either

(1) behaves like a random planar graph, i.e. $lb(\mathsf{C}_n) \sim cn$, for some analytically given $c = c(\mathcal{C})$, and the second largest block is of order n^{α}, where $1 > \alpha = \alpha(\mathcal{C})$, or

(2) $lb(\mathsf{C}_n) = \mathcal{O}(\log n)$, i.e., all blocks contain at most logarithmically many vertices, or

(3) $lb(\mathsf{C}_n) = \widetilde{\mathcal{O}}(n^{\alpha})$, for some $\alpha = \alpha(\mathcal{C}) < 1$.

Planar graphs belong to category (1). In contrast to that, outerplanar and series-parallel graphs belong to category (2).

1 Introduction

Over the last decades the theory of analysis of algorithms was mainly developed from a *worst case* perspective. While this led to many lines of research with deep and beautiful results, it also turned out that from a more practical point of view a worst case analysis is often too restrictive, as many real world problems are $\mathcal{NP}$-hard and often also $\mathcal{NP}$-hard to approximate. This seems to call for an analysis from an *average case* point of view. As a first step in performing such an analysis for a real world problem one needs to describe a probability distribution on the set of input instances. Even if this point (that is often completely unclear and/or difficult to obtain) could be resolved, it is usually unclear how to proceed from there. This is due to the fact that up to now we are still lacking a powerful machinery that describes how to approach such average case analysis. Even in the case where the inputs come from a somewhat restricted graph class equipped with the uniform distribution we still lack an appropriate easy to use approach.

Over the last years the class of all planar graphs has evolved as a primary example for the development of methods for studying properties of restricted graph classes equipped with the uniform distribution. More precisely, we denote by $\mathcal{P}_n$ the class of all simple labeled planar graphs with n vertices, and use P_n to denote a graph drawn uniformly at random from this set. Basic properties of P_n were first investigated by Denise, Vasconcellos, and Welsh [7]. Using a crude counting argument, McDiarmid, Steger, and Welsh [15] showed that a random planar graph in fact has some properties that are quite different from the behaviour of a classical random graph in the Erdős-Rényi model. Namely, they showed that the probability that P_n is connected is, for n tending to infinity, bounded away from 0 *and* from 1. (Recall that an Erdős-Rényi graph $G_{n,p}$ satisfies a 0-1 law for all "natural" properties.) While the precise value of this probability is of course given by $\lim_{n\to\infty} |\mathcal{P}_n|/|\mathcal{C}_n|$, where $\mathcal{C}_n$ denotes the class of all labeled connected planar graphs with n vertices, it took quite a while and required deep methods from combinatorial counting and analytic combinatorics to determine the required values asymptotically.

THEOREM 1.1. ([2], [13]) *There exist analytically given constants $p, b, \rho > 0$ such that*

$$|\mathcal{P}_n| \sim pn^{-7/2}\rho^{-n}n! \quad \textit{and} \quad |\mathcal{B}_n| \sim bn^{-7/2}\rho^{-n}n!.$$

While the above result determines precisely the limit of the probability that a random planar graph is connected, the methods that were used in proving this result seem to be restricted to certain kind of questions. For example, even the seemingly "trivial" question of

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the value of the maximum degree in a random planar graph is extremely hard, if not impossible, to attack by such direct counting arguments.

Using the concept of Boltzmann samplers, cf. Section 2.2, Bernasconi, Panagiotou, and Steger [4, 5] obtained for certain subclasses of planar graphs not only the maximum degree, but also the degree distribution of an element drawn uniformly at random from this class. Their approach relied implicitly on the fact that the classes under consideration have a simple block structure, i.e. they used that all blocks are "small". In this paper we show that such a property is not true for planar graphs. More precisely, we show that a random planar graph exhibits a *block* structure whose phenomenology is similar to the *component* structure of classical Erdős-Rényi random graphs at the threshold $p = 1/n$ [9]. Namely, we show that in a random planar graph there is one giant block containing a linear number of vertices, while the second largest block is of much smaller order, namely $\tilde{\Theta}(n^{2/3})$. Moreover, we show that there are "many" blocks of "small" order.

Recall that the blocks of a graph G are the set of all maximum (induced) subgraphs of G that are biconnected, where with slight abuse of notation we assume that the graph consisting of a single edge is biconnected. For a graph G we denote by $b(\ell; G)$ the number of blocks in G that contain exactly k vertices. Moreover, let $lb(G) := \max\{\ell : b(\ell; G) \neq 0\}$, i.e., $lb(G)$ is the number of vertices in the largest biconnected subgraph of G.

It was shown in [15] that a random planar graph P_n contains with high probability a *giant (connected) component* that contains all but a constant number of vertices. We thus restrict our considerations in this paper to studying the block structure of random planar *connected* graphs. The theorem below summarizes a few "high-level" facts that we can show.

Theorem 1.2. *Let C_n be a graph drawn uniformly at random from the set of labeled connected planar graphs with n vertices. Then the following statements are simultaneously true asymptotically almost surely.*

1. *The largest block in C_n contains $\sim cn$ vertices, where $c \approx 0.959$ is analytically given.*
2. *For $n^{2/3} \ll \ell < lb(\mathsf{C}_n)$ we have $b(\ell; \mathsf{C}_n) = 0$.*
3. *The second largest block of C_n contains at least $\frac{n^{2/3}}{\log n}$ vertices.*
4. *For $2 \leq \ell \leq \frac{n^{2/3}}{\log n}$ the quantity $b(\ell; \mathsf{C}_n)$ is sharply concentrated around a known value.*

While Theorem 1.2 exhibits that random planar graphs have a reasonably complex block structure, this is not the case for some subclasses of planar graphs.

Theorem 1.3. *Let O_n be a graph drawn uniformly at random from the set of labeled outerplanar graphs. Then with probability tending to one for n tending to infinity $lb(\mathsf{O}_n) = \mathcal{O}(\log n)$. A similar result holds for the class of series-parallel graphs.*

In fact, we obtain Theorems 1.2 and 1.3 as special cases of a much more general theorem. In order to state it we start with a definition.

Definition 1.1. *Let $\mathcal{C}$ be a class of labeled connected graphs and let $\mathcal{B} \subset \mathcal{C}$ be the class of biconnected graphs in $\mathcal{C}$. The class $\mathcal{C}$ is called (α, β)-nice if it satisfies the following two properties.*

1. *For any graph $G \in \mathcal{C}$ the following is true. If we replace an arbitrary block of G by a graph from $\mathcal{B}$ on the same vertex set, then the resulting graph also belongs to $\mathcal{C}$.*
2. *There exist constants $c, b > 0$, $\rho_{\mathcal{C}}, \rho_{\mathcal{B}} > 0$ such that*
(1.1)
$$|\mathcal{C}_n| \sim cn^{-\alpha}\rho_{\mathcal{C}}^{-n}n! \text{ and } |\mathcal{B}_n| \sim bn^{-\beta}\rho_{\mathcal{B}}^{-n}n!.$$

Clearly, many natural graph classes are nice, as for example outerplanar, series-parallel, or planar graphs, and more generally, classes that are given in terms of excluded minors. Before we state our main theorem let us recall that the *exponential generating function* (egf) of a graph class $\mathcal{G}$ is defined as $G(x) = \sum_{n\geq 1} \frac{|\mathcal{G}_n|}{n!}x^n$, where $\mathcal{G}_n$ denotes the set of graphs in $\mathcal{G}$ that contain exactly n vertices. We will denote by $\rho_{\mathcal{G}}$ the dominant singularity of $G(x)$. With these definitions at hand we can now state our main theorem.

Theorem 1.4. *Let $\mathcal{C}$ be a class of labeled connected graphs that is (α, β)-nice, where $\alpha \geq \frac{5}{2}$ and $\beta \in \mathbf{R}$. Then the following is true asymptotically almost surely for a graph C_n drawn uniformly at random from $\mathcal{C}_n$.*

(i) If $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) > 1$, then $lb(\mathsf{C}_n) = \mathcal{O}(\log n)$.

(ii) If $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) < 1$, then $lb(\mathsf{C}_n) \sim (1 - \rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}))n$. Moreover, $\beta > 2$ and we have

1. *$b(\ell; \mathsf{C}_n) = 0$ for all $n^{1/(\beta-2)}\omega_n \leq \ell < lb(\mathsf{C}_n)$,*
2. *$b(\ell; \mathsf{C}_n) \sim b_\ell n$ for all $2 \leq \ell \leq (\frac{n}{\log^2 n})^{1/(\beta-1)}$, where*
(1.2)
$$b_\ell = [x^{\ell-1}]B'(x) \cdot \rho_{\mathcal{B}}^{\ell-1} \;\sim_\ell\; b\rho_{\mathcal{B}}^{-1}\ell^{-\beta+1},$$

3. $b(\ell \ldots \delta\ell; \mathsf{C}_n) \sim b_{\ell,\delta} n$ *for all* $\ell \le (\frac{n}{\log^2 n})^{1/(\beta-2)}$, *where*

$$b_{\ell,\delta} = \sum_{s=\ell}^{\delta\ell} [x^{s-1}]B'(x) \cdot \rho_{\mathcal{B}}^{s-1} \tag{1.3}$$

and

$$b_{\ell,\delta} \sim_\ell \frac{b}{\rho_{\mathcal{B}}(\beta-2)} \cdot (1-\delta^{-\beta+2})\ell^{-\beta+2}.$$

(iii) *If* $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) = 1$ *then asymptotoically almost surely there is a block in* C_n *that contains* $\widetilde{\Theta}(n^{1/(\beta-2)})$ *vertices. Moreover, if* $3 < \beta < 5$, *then* $lb(\mathsf{C}_n) = \widetilde{\mathcal{O}}(n^{2/(\beta-1)})$, *and otherwise* $lb(\mathsf{C}_n) = \widetilde{\mathcal{O}}(n^{1/2})$.

Theorems 1.2 and 1.3 are a consequence of Theorem 1.4, as the results of Giménez and Noy [13] and Bodirsky, Giménez, Kang, and Noy [6] imply that planar, outerplanar, and series-parallel graphs are nice.

Note that in the above theorem we require $\alpha \ge 5/2$. We will show in Section 4 that this is in fact a very mild restriction. More precisely, we will show that if we have sufficient information about the behavior of the egf $B(x)$ around its singularity $\rho_{\mathcal{B}}$, then the resulting class $\mathcal{C}$ is always nice with $\alpha \ge 5/2$.

LEMMA 1.1. *Let* $\mathcal{C}$ *be a class of labeled connected graphs and let* $\mathcal{B} \subset \mathcal{C}$ *be the class of biconnected graphs in* $\mathcal{C}$. *Assume that* $\mathcal{C}$ *satisfies condition (i) of Definition 1.1. Moreover, assume that the egf* $B(x)$ *of* $\mathcal{B}$ *has a unique finite real singularity* $\rho_{\mathcal{B}} > 0$ *and admits a full singular expansion of the form*

$$B(x) = \sum_{k\in\mathbf{Z}} s_k \left(1 - \frac{x}{\rho_{\mathcal{B}}}\right)^{k/\mu}, \tag{1.4}$$

where μ *is an integer* ≥ 2. *Then, if* $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) \ne 1$, *then* $\mathcal{C}$ *is* $(\alpha, \beta+1)$*-nice for some* $\alpha \ge 5/2$, *and* $\beta < \infty$ *is the* singularity type *of* $B(x)$, *i.e.,* $\beta = \min\{k/\mu \mid k/\mu \notin \mathbf{N}_0$ *and* $s_k \ne 0\}$.

Note that the assumption (1.4) is rather general, as it includes singularities arising both from algebraic as well as from meromorphic functions. Indeed, most generating functions encountered in modern analytic combinatorics are of this type.

The next paragraphs and sections contain some related results and address notational issues. Moreover, some of the proofs are ommited and will be contained in the full version of the paper.

Related Results In contrast to random planar graphs, random planar *maps* and random triangulations are well-studied and well-understood objects. A map is a graph together with an embedding in the plane. Gao and Wormald [12] and Bender, Richmond, and Wormald [3] derived very general results about the size of largest components and applied them to a variety of types of planar maps. Among other results, they showed that a random planar map has with high probability a linear size 2-connected submap, and that a 2-connected map has a linear size 3-connected core. Moreover, they showed that a random triangulation has typically a 4-connected component that contains roughly half of the vertices. These results were generalized and strengthened by Banderier, Flajolet, Schaeffer and Soria [1], who determined the exact distribution of the size of largest (multi-)connected components in several kinds of planar maps. In this context they discovered a universal phenomenon that is of the exponential-cubic type, and corresponds to distributions that involve the Airy function.

All the above results are based on completely analytic techniques, singularity analysis of generating functions, and saddle point analysis of complex functions. In the present paper the approach is orthogonal to that: we exploit recent progress in the development of sampling algorithms by Duchon, Flajolet, Louchard and Schaeffer [8] to reduce the problem to one to which we can apply elementary tools from probability theory. Moreover, our results indicate that random planar graphs belong to the same universality class as planar maps and triangulations, in the sense that they contain large components of higher connectivity. In contrast to that, random outerplanar and series-parallel graphs are "simpler" objects.

Notation In this paragraph we fix the notation that will be used in the article. All graph classes considered here consist of labeled graphs, and without loss of generality we will assume that the vertices of an n-vertex graph bear the labels $\{1, \ldots, n\}$. Let $\mathcal{G}$ be such a class of labeled graphs. We denote by $\mathcal{G}^\bullet$ the class of *vertex-rooted* graphs from $\mathcal{G}$, i.e. $\mathcal{G}^\bullet = \{(G, v) \mid G \in \mathcal{G}, v$ is a labeled vertex of $G\}$. In the remainder we shall write with slight abuse of notation "$G^\bullet \in \mathcal{G}^\bullet$", meaning that $G^\bullet$ is a graph from $\mathcal{G}$ with some distinguished vertex. Finally, we denote by $\mathcal{G}'$ the class of *derivated* graphs from $\mathcal{G}$: every graph $G' \in \mathcal{G}'$ is obtained from a graph $G \in \mathcal{G}$ by removing the label from the vertex with the maximum label in G. So, if G had n vertices, then G' has n vertices too, but only $n-1$ of them bear a label. We denote the unlabeled vertex in G' as the *virtual vertex* of G'.

For any (normal, vertex-rooted, or derivated) graph class $\mathcal{G}$ we denote by $\mathcal{G}_n \subset \mathcal{G}$ the class of graphs consisting of precisely n *labeled* vertices, and for any $G \in \mathcal{G}_n$ we write $|G| = n$ (i.e., $|G|$ is the number of labeled vertices in G). So, the above discussion implies that $|\mathcal{G}_n^\bullet| = n|\mathcal{G}_n|$, and moreover that $|\mathcal{G}_n'| = |\mathcal{G}_{n+1}|$. Finally, we note that the egf's for $\mathcal{G}^\bullet$ and $\mathcal{G}'$ are $G^\bullet(x) := x\frac{\partial}{\partial x}G(x)$ and $G'(x) = \frac{\partial}{\partial x}G(x)$.

2 Preliminaries

2.1 Asymptotic Estimates In our proofs we will often bound the probability that certain random variables assume values far away from their expectation. The next lemma states the well-known Chernoff bounds.

LEMMA 2.1. *Let* $X \sim \mathrm{Bin}\,(n,p)$. *For every* $0 < \varepsilon < 1$ *we have* $\Pr[X \notin (1 \pm \varepsilon)np] \le 2e^{-\varepsilon^2 np/3}$.

Similarly, we will make use of bounds for the tails of a Poisson distributed random variable.

LEMMA 2.2. *Let* $X \sim \mathrm{Po}\,(\mu)$. *For every* $0 < \varepsilon < 1$ *we have* $\Pr[X \notin (1 \pm \varepsilon)\mu] \le 2e^{-\varepsilon^2 \mu/3}$.

The next lemma states an elementary inequality.

PROPOSITION 2.1. *Let* $n_0 \ge 1$ *and* $t \ge 1$ *be integers. For any* $\beta > 2$ *there is a constant* $C = C(\beta) > 0$ *such that for any integer* $N \ge tn_0$ *we have*

$$\sum_{\substack{(s_1,\dots,s_t)\in\mathbf{N}^t \\ \sum s_i = N, s_i \ge n_0}} \prod_{i=1}^t s_i^{-\beta} \le C^{t-1} \cdot n_0^{-(\beta-1)(t-1)} \cdot N^{-\beta}.$$

2.2 Boltzmann Sampling By applying a standard decomposition of a graph into maximal biconnected subgraphs (see e.g. [14]) we obtain the following combinatorial relation for nice classes of graphs. Let us introduce some notation first. We denote by $\mathcal{Z}$ the graph class consisting of one single labeled vertex. Furthermore, for two graph classes $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{A} = \mathcal{X} \times \mathcal{Y}$ the *cartesian product* of $\mathcal{X}$ and $\mathcal{Y}$ followed by a relabeling step, so as to guarantee that all labels are distinct. Note that the relation "$\mathcal{A} = \mathcal{X} \times \mathcal{Y}$" expresses the fact that there is a bijection between the elements of $\mathcal{A}$ and pairs of elements from $\mathcal{X}$ and $\mathcal{Y}$, but it does not provide any information about how this bijection looks like, i.e., how to construct a graph in $\mathcal{A}$ from two graphs in $\mathcal{X}$ and $\mathcal{Y}$. The same is true for the operators described in the remainder. We denote by $\textsc{Set}(\mathcal{X})$ the graph class such that each object in it is an ordered collection of graphs in $\mathcal{X}$. Finally, the class $\mathcal{X} \circ \mathcal{Y}$ consists of all graphs that are obtained from graphs from $\mathcal{X}$, where each vertex is replaced (in a unique way) by a graph from $\mathcal{Y}$. This set of combinatorial operators (cartesian product, set, and substitution) appears frequently in modern theories of combinatorial analysis as well as in systematic approaches to random generation of combinatorial objects. For a very detailed description of these operators and numerous applications we refer to [10].

LEMMA 2.3. *Let* $\mathcal{C}$ *be a graph class that satisfies condition (i) of Definition 1.1, and let* $\mathcal{B} \subset \mathcal{C}$ *be the class of biconnected graphs in* $\mathcal{C}$. *Then* $\mathcal{C}^\bullet = \mathcal{Z} \times \textsc{Set}(\mathcal{B}' \circ \mathcal{C}^\bullet)$, *where* $\mathcal{Z}$ *denotes the graph class consisting of one single labeled vertex.*

In words, a rooted connected graph from $\mathcal{C}$ is an unordered collection of rooted *bi*connected graphs, which are merged at their roots, where every vertex different from the root is subsequently substituted again by a rooted connected graph.

A combinatorial relation as given in the above lemma has two important advantages: first, it can be used to obtain equations that relate the generating functions $C(x)$ and $B(x)$ enumerating connected and biconnected graphs. In particular, we get that $C^\bullet(x) = xe^{B'(C^\bullet(x))}$, see e.g. [14]. Moreover, the relation translates immediately to a randomized sampling algorithm that generates rooted graphs according to an appropriate probability measure defined over the whole class $\mathcal{C}^\bullet$, the so-called *Boltzmann model.* This framework was introduced by Duchon, Flajolet, Louchard and Schaeffer in [8], and was extended by Fusy [11]. Here we just present the basic ideas of this framework. Let $\mathcal{G}$ be a class of labeled graphs. In the Boltzmann model of parameter x we assign to a graph $\gamma \in \mathcal{G}$ the probability

$$\mathbf{P}_x[\gamma] = \frac{1}{G(x)} \frac{x^{|\gamma|}}{|\gamma|!}, \tag{2.5}$$

if the expression above is well-defined. It is straightforward to see that the expected size of an object in $\mathcal{G}$ under this probability distribution is $\frac{xG'(x)}{G(x)}$. A *Boltzmann sampler* $\Gamma G(x)$ for $\mathcal{G}$ is an algorithm that generates graphs from $\mathcal{G}$ according to (2.5). In [8, 11] several general procedures which translate common combinatorial construction rules like union, set, etc. into Boltzmann samplers are given. Observe that the probability in (2.5) depends only on the parameter x and on the size of γ, implying that every object of the same size has the *same* probability of being generated. If we thus condition on the output being of a certain size n, then the Boltzmann sampler $\Gamma G(x)$ is a *uniform* sampler for the class $\mathcal{G}_n$.

In the sequel we are going to demonstrate how we can exploit the rules in [8, 11] to obtain a sampler for $\mathcal{C}^\bullet$. Define

$$\lambda_{\mathcal{C}} := B'(C^\bullet(\rho_{\mathcal{C}})), \tag{2.6}$$

and note that this quantity is finite for (α, β)-nice classes with $\alpha > 2$ (cf. Lemma 2.6 below). Moreover, let $\Gamma B'(x)$ be a Boltzmann sampler for $\mathcal{B}'$, i.e. $\Gamma B'(x)$ samples graphs from $\mathcal{B}'$ according to the Boltzmann distribution (2.5) with parameter x. Note that here $0 \le x \le C^\bullet(\rho_\mathcal{C})$ is admissible, as $\lambda_\mathcal{C}$ is finite and hence $C^\bullet(\rho_\mathcal{C}) \le \rho_\mathcal{B}$. Then the sampler $\Gamma C^\bullet$ for $\mathcal{C}^\bullet$ with parameter $x = \rho_\mathcal{C}$ is given by the following algorithm.

$\Gamma C^\bullet$: $\gamma \leftarrow$ a single node r
    $k \leftarrow \mathrm{Po}(\lambda_\mathcal{C})$ $(\star)$
    for $j = 1, \dots, k$
        $\gamma' \leftarrow \Gamma B'(C^\bullet(\rho_\mathcal{C}))$,
            discard the labels of γ' $(\star\star)$
        $\gamma \leftarrow$ merge γ and γ' at their roots
    foreach vertex $v \ne r$ of γ
        $\gamma_v \leftarrow \Gamma C^\bullet$, discard the labels of γ_v
    replace all nodes $v \ne r$ of γ with γ_v
    return γ, where the vertices are
        labeled uniformly at random

Note that the above algorithm just reverses the decomposition given in Lemma 2.3: it starts with a single vertex, attaches to it a random set of biconnected graphs, and proceeds recursively to substitute every newly generated vertex by a rooted connected graph. The following lemma is an immediate consequence of the compilation rules in [8, 11].

LEMMA 2.4. *Let $\mathcal{C}$ be an (α, β)-nice class with $\alpha > 2$, and let $\gamma \in \mathcal{C}^\bullet$. Then we have* $\Pr[\Gamma C^\bullet = \gamma] = \frac{\rho_\mathcal{C}^{|\gamma|}}{|\gamma|! C^\bullet(\rho_\mathcal{C})}$.

COROLLARY 2.1. *Let $\mathcal{C}$ be an (α, β)-nice class with $\alpha > 2$. Then there is a positive constant $\hat{c}$ such that* $\Pr[\Gamma C^\bullet \in \mathcal{C}_n^\bullet] \sim \hat{c} n^{-\alpha+1}$.

Proof. The claim follows immediately from Lemma 2.4 and $|\mathcal{C}_n^\bullet| = n|\mathcal{C}_n| \sim c n^{-\alpha+1} \rho_\mathcal{C}^{-n} n!$.

For us it will be convenient to follow an approach first used in [4] and replace the sampler $\Gamma C^\bullet$ by a slightly different sampler with the property that the output distributions of both "algorithms" are the same. Observe that $\Gamma C^\bullet$ makes two kinds of random choices: first, when it chooses a random value according to a Poisson distribution in the line marked with $(\star)$, and second, when it calls the sampler $\Gamma B'$ in the line marked with $(\star\star)$ (the return value of this sampler is a random graph from $\mathcal{B}'$). We adapt the sampler $\Gamma C^\bullet$ by making the random choices in advance, and by providing them as part of the input. More precisely, let K be an infinite sequence of non-negative integers, each one chosen independently according to the distribution $\mathrm{Po}(\lambda_\mathcal{P})$, and let B' be an infinite sequence of graphs from $\mathcal{B}'$, drawn independently according to the Boltzmann distribution with parameter $C^\bullet(\rho_\mathcal{C})$. Then the sampler $\Gamma C^\bullet(K, B')$, which simulates the execution of $\Gamma C^\bullet$ by using the next unused value from the provided lists, generates obviously every graph from $\mathcal{C}^\bullet$ with the same probability as $\Gamma C^\bullet$. In the sequel we will therefore assume that the notation $\Gamma C^\bullet$ in fact denotes the sampler $\Gamma C^\bullet(K, B')$, where we often omit the lists (K, B') for ease of notation.

Recall that $b(\ell; G)$ denotes the number of blocks in a graph G that contain exactly ℓ vertices. Our first step in proving Theorem 1.4 is the following lemma, which relates facts about the block structure of a graph generated by $\Gamma C^\bullet(K, B')$ to properties of the lists (K, B'). As the latter consist of entries that are sampled *independently*, this lemma will be a key step in our analysis.

LEMMA 2.5. *Let $K = \{k_1, k_2, \dots\}$ be an infinite sequence of non-negative integers and let $B' = \{B'_1, B'_2, \dots\}$ be an infinite sequence of graphs from $\mathcal{B}'$. Suppose that $\Gamma C^\bullet(K, B')$ used the first n values in K and the first m graphs in B' to generate a graph $\gamma \in \mathcal{C}^\bullet$. Then the following statements are true.*

(1) $n = |\gamma|$.

(2) $m = \sum_{j=1}^{n} k_j$.

(3) $m = \sum_{\ell \ge 2} b(\ell; \gamma)$.

(4) For any $\ell \ge 2$ we have that $b(\ell; \gamma) = \left|\left\{1 \le i \le m \mid |B'_i| = \ell - 1\right\}\right|$.

Proof. Before we proceed with the details of the proof let us discuss the general "high-level" strategy. Recall how γ is constructed: in the initial call to the sampler a single vertex r is generated and the value k_1 is read. This determines the *number* of biconnected graphs that have r as a common vertex. Then the k_1 graphs $B'_1, \dots, B'_{k_1}$ are glued together at r. Denote by V the set of non-virtual vertices in $B'_1, \dots, B'_{k_1}$. Finally, the sampler is called recursively for each $v \in V$, where $|V|$ graphs $(\gamma_v)_{v \in V}$ from $\mathcal{C}^\bullet$ are generated. Note that for all $v \in V$ we have that $|\gamma_v| < |\gamma|$. This calls for a proof by induction on $|\gamma|$. Note that all statements are trivially true for the base case $|\gamma| = 1$.

First we perform the induction step for (1). Denote for $v \in V$ by n_v the number of variables from K that the sampler used to generate γ_v, and note that due to the induction hypothesis we have $n_v = |\gamma_v|$. But then, as in the construction of γ the root vertex of γ_v is identified with v for all $v \in V$, we have that

$$n = 1 + \sum_{v \in V} n_v = 1 + \sum_{v \in V} |\gamma_v| = |\gamma|.$$

To see (2) let $V = \{v_1, \dots, v_{|V|}\}$ and assume that the sampler is called recursively first for v_1, then for v_2, and so on. This means, by using (1), that there are indexes $2 = i_1 \le i_2 \le \cdots \le i_{|V|} \le i_{|V|+1} = n + 1$

such that γ_{v_j} is constructed by $\Gamma C^\bullet(K, B')$ by using the values $(k_s)_{i_j \le s < i_{j+1}}$ from K (note that if $i_j = i_{j+1}$, then γ_{v_j} is just the graph consisting of a single vertex). By the induction hypothesis the number m_j of graphs used from B' to construct γ_{v_j} is equal to $\sum_{s=i_j}^{i_{j+1}-1} k_s$. Hence, the number of graphs from B' used in the construction of γ is

$$k_1 + \sum_{j=1}^{|V|} m_j = k_1 + \sum_{j=1}^{|V|} \sum_{s=i_j}^{i_{j+1}-1} k_s = \sum_{j=1}^{n} k_j,$$

which completes the proof of (2).

As (3) follows immediately from (4), it suffices to prove (4). Observe that the total number $b(\ell;\gamma)$ of blocks with ℓ vertices in γ is the number of graphs among $B'_1, \ldots, B'_{k_1}$ with ℓ vertices (from which exactly $\ell-1$ are non-virtual vertices) plus the total number of blocks with ℓ vertices in γ_v for all $v \in V$. We deduce

$$\begin{aligned} b(\ell;\gamma) &= \sum_{i=1}^{k_1} b(\ell; B'_i) + \sum_{v \in V} b(\ell; \gamma_v) \\ &= \left|\left\{1 \le i \le k_1 \mid |B'_i| = \ell - 1\right\}\right| + \sum_{v \in V} b(\ell; \gamma_v), \end{aligned}$$

and the proof is completed with the induction hypothesis on $b(\ell;\gamma_v)$ for all $v \in V$ and (2).

Finally, we need a technical statement about the relation of the singularities of $\mathcal{C}$ and $\mathcal{B}$.

LEMMA 2.6. *Let $\mathcal{C}$ be a class that satisfies condition (i) of Definition 1.1, and let $\mathcal{B} \subset \mathcal{C}$ be the set of biconnected graphs in $\mathcal{C}$. Let $\rho_\mathcal{C}$ and $\rho_\mathcal{B}$ be the singularities of the egf's $C(x)$ and $B(x)$, and assume that $\rho_\mathcal{B} > 0$.*

- *If $\rho_\mathcal{B} B''(\rho_\mathcal{B}) > 1$, then $C^\bullet(\rho_\mathcal{C}) = \tau < \rho_\mathcal{B}$, where $\rho_\mathcal{C} = \tau e^{-B'(\tau)}$ and τ is the unique solution to $\tau B''(\tau) = 1$. Moreover, there is $g > 0$ such that $|\mathcal{C}_n| \sim g n^{-5/2} \rho_\mathcal{C}^{-n} n!$.*
- *If $\rho_\mathcal{B} B''(\rho_\mathcal{B}) \le 1$, then $C^\bullet(\rho_\mathcal{C}) = \rho_\mathcal{B}$. Moreover, $\rho_\mathcal{C} = \rho_\mathcal{B} e^{-B'(\rho_\mathcal{B})}$.*

Proof. We first consider the case $\rho_\mathcal{B} B''(\rho_\mathcal{B}) > 1$. As $B(x)$ has non-negative coefficients it follows that there is a unique $0 < \tau < \rho_\mathcal{B}$ such that $\tau B''(\tau) = 1$. Using Theorem VII.2 from [10], where we set $y(x) = C^\bullet(x)$ and $\phi(u) = e^{B'(u)}$, we infer that $\rho_\mathcal{C} = \frac{\tau}{\phi(\tau)}$, and that a full locally convergent expansion of $C^\bullet(x)$ exists, starting with

(2.7)
$$C^\bullet(x) \overset{x \to \rho_\mathcal{C}}{=} \tau - \sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}} \sqrt{1 - \frac{x}{\rho_\mathcal{C}}} + \mathcal{O}\left(1 - \frac{x}{\rho_\mathcal{C}}\right).$$

In particular, we have that $C^\bullet(\rho_\mathcal{C}) = \tau < \rho_\mathcal{B}$. Moreover, by applying Theorem VI.6 from [10] we deduce that there is a constant $g > 0$ such that $|\mathcal{C}_n^\bullet| \sim g n^{-3/2} \rho_\mathcal{C}^{-n} n!$, which concludes the first part of the proof.

We now consider the case $\rho_\mathcal{B} B''(\rho_\mathcal{B}) \le 1$. Here, there is no $0 < \tau < \rho_\mathcal{B}$ such that $\tau B''(\tau) = 1$, and it follows that the function $\psi(u) = u e^{-B'(u)}$ satisfies $\psi'(u) > 0$ for all $0 < u < \rho_\mathcal{B}$. As one easily checks that $\psi(u)$ is the functional inverse of $C^\bullet(x)$, it follows from the Analytic Inversion Lemma (see e.g. Lemmas IV.2 and IV.3 in [10]) that $\rho_\mathcal{C} \ge \rho := \psi(\rho_\mathcal{B})$. Moreover, note that $C^\bullet(\rho) = \rho_\mathcal{B}$, which implies that $\rho_\mathcal{C} \le \rho$, as $C^\bullet(x) = x e^{B'(C^\bullet(x))}$ is an increasing function of x. Putting everything together yields $\rho_\mathcal{C} = \rho_\mathcal{B} e^{-B'(\rho_\mathcal{B})}$, i.e., the dominant singularity of $C^\bullet(x)$ is determined by the singularity of $B(x)$.

3 Proof of Theorem 1.4

3.1 Small Blocks As a first application of Lemma 2.5 we deduce some information on the number of "small" blocks in a random graph C_n from a nice class $\mathcal{C}$.

LEMMA 3.1. *Let $\mathcal{C}$ be an (α, β)-nice class with $\alpha > 2$ and let $0 < \varepsilon = \varepsilon(n) < 1$. For $\ell \ge 2$ define the quantities*

$$b_\ell = [x^{\ell-1}] B'(x) \cdot (C^\bullet(\rho_\mathcal{C}))^{\ell-1}$$

and

$$\ell_0 = \ell_0(n, \varepsilon) = \max\left\{\ell \mid b_\ell n \ge 50 \varepsilon^{-2} \alpha \log n\right\}.$$

Then we have for all $2 \le \ell \le \ell_0$ and sufficiently large n

(3.8)
$$\Pr\left[b(\ell; \mathsf{C}_n) \in (1 \pm \varepsilon) b_\ell n\right] \ge 1 - e^{-\frac{\varepsilon^2}{40} b_\ell n}.$$

Proof. In the proof we argue via the Boltzmann sampler $\Gamma C^\bullet$ introduced in the previous section. Fix an ℓ such that $2 \le \ell \le \ell_0$ and let $\mathcal{S}_n \subset \mathcal{C}_n$ denote the set of labeled graphs in $\mathcal{C}_n$ whose number of blocks of size ℓ is *not* in the interval $(1 \pm \varepsilon) b_\ell n$. Using Corollary 2.1 we obtain that there exists a constant $\hat{c} > 0$ such that for all large enough n we have

(3.9)
$$\begin{aligned} \Pr\left[\mathsf{C}_n \in \mathcal{S}_n\right] &= \Pr\left[\Gamma C^\bullet \in \mathcal{S}_n \mid \Gamma C^\bullet \in \mathcal{C}_n^\bullet\right] \\ &\le \hat{c} n^{\alpha-1} \Pr\left[\Gamma C^\bullet \in \mathcal{S}_n \text{ and } \Gamma C^\bullet \in \mathcal{C}_n^\bullet\right]. \end{aligned}$$

In order to estimate the last probability we write $\mathcal{S}_n = \mathcal{S}_n^{(1)} \cup \mathcal{S}_n^{(2)}$, where $\mathcal{S}_n^{(1)}$ contains all graphs G in $\mathcal{S}_n$ that have the property $\sum_{\ell \ge 2} b(\ell; G) \notin (1 \pm \frac{\varepsilon}{3}) \lambda_\mathcal{C} n$, and $\mathcal{S}_n^{(2)} = \mathcal{S}_n \setminus \mathcal{S}_n^{(1)}$. In words, the total number of blocks of every graph in $\mathcal{S}_n^{(1)}$ is less than $(1 - \frac{\varepsilon}{3}) \lambda_\mathcal{C} n$ or greater than $(1+\frac{\varepsilon}{3})\lambda_{\mathcal{C}} n$, where $\lambda_{\mathcal{C}} = B'(C^\bullet(\rho_{\mathcal{C}}))$ is the constant defined in (2.6).

By using Lemma 2.5, statements (1)-(3), the event "$\Gamma C^\bullet \in \mathcal{S}_n^{(1)}$ and $\Gamma C^\bullet \in \mathcal{C}_n^\bullet$" implies that the sum of n independent variables distributed like $\mathrm{Po}(\lambda_{\mathcal{C}})$ is not in $(1\pm\frac{\varepsilon}{3})\lambda_{\mathcal{C}} n$. But this probability is at most $e^{-\frac{\varepsilon^2}{30}\lambda_{\mathcal{C}} n}$, due to Lemma 2.2.

Moreover, again by Lemma 2.5, Statement (4), the event "$\Gamma C^\bullet \in \mathcal{S}_n^{(2)}$ and $\Gamma C^\bullet \in \mathcal{C}_n^\bullet$" implies that a sequence of $N = (1\pm\frac{\varepsilon}{3})\lambda_{\mathcal{C}} n$ independent random graphs, which are drawn from $\mathcal{B}'$ according to the Boltzmann distribution with parameter $C^\bullet(\rho_{\mathcal{C}})$, contains less than $(1-\varepsilon)b_\ell n$ or more than $(1+\varepsilon)b_\ell n$ graphs with $\ell-1$ non-virtual vertices. From (2.5) we deduce that the probability that a single such random graph has exactly $\ell-1$ non-virtual vertices is precisely $t_\ell := [x^{\ell-1}]B'(x)\cdot\frac{(C^\bullet(\rho_{\mathcal{C}}))^{\ell-1}}{B'(C^\bullet(\rho_{\mathcal{C}}))}$. Hence, by applying the Chernoff bounds from Lemma 2.1 we deduce that the number of graphs with $\ell-1$ non-virtual vertices among N independently drawn random graphs is less than $(1-\frac{\varepsilon}{3})t_\ell N$ or more than $(1+\frac{\varepsilon}{3})t_\ell N$ with probability at most $e^{-\frac{\varepsilon^2}{30}t_\ell N}$. The proof completes with $N \in (1\pm\frac{\varepsilon}{3})\lambda_{\mathcal{C}} n$ (due to $\Gamma C^\bullet \in \mathcal{S}_n^{(2)}$) and the assumption $\lambda_{\mathcal{C}} t_\ell n = b_\ell n \geq 50\varepsilon^{-2}\alpha\log n$.

Note that the term $b_\ell n$ in Lemma 3.1 denotes the expected number of blocks of size exactly ℓ in a graph generated by $\Gamma C^\bullet$ that has exactly n vertices. The definition of $\ell_0 = \ell_0(n,\varepsilon)$ reflects the fact that we need this expectation to be large enough so that the Chernoff bound yields a probability of deviation that is asymptotically smaller than the probability that $\Gamma C^\bullet$ does *not* return a graph of size n (which is in the order of $n^{-\alpha+1}$ due to Corollary 2.1). For ℓ larger than $\ell_0(n,\varepsilon)$ the argument of Lemma 3.1 can no longer be applied directly. Here we instead make use of the following two approaches. If ℓ is such that $b_\ell n = o(1)$ we can use Markov's inequality to deduce that the a random graph C_n w.h.p. does not have any block of this size. For intermediate sizes we consider sets of block sizes. Let $b(\ell_1\ldots\ell_2; G)$ denote the number of blocks of size at least ℓ_1 and at most ℓ_2 in the graph G. Then it is conceivable that for $b_2 \gg b_1$ the expected number of blocks $b(\ell_1\ldots\ell_2;\mathsf{C}_n)$ is again large enough so that Chernoff bounds can be applied. For ease of argumentation we treat the two cases $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) \leq 1$ and $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) > 1$ separately.

LEMMA 3.2. *Let $\mathcal{C}$ be an (α,β)-nice class with $\alpha > 2$ such that $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) \leq 1$, and let $0 < \varepsilon = \varepsilon(n) < 1$. Define for $\ell \gg 1$ and $\delta = \delta(n) > 1$ the quantities*

$$b_{\ell,\delta} = \sum_{s=\ell}^{\delta\ell} [x^{s-1}]B'(x)\cdot\rho_{\mathcal{B}}^{s-1}$$

and

$$\ell_0 = \ell_0(\delta) = \max\left\{\ell \mid b_{\ell,\delta} n \geq 50\varepsilon^{-2}\alpha(\beta-2)\log n\right\}.$$

Then we have for all $1 \ll \ell \leq \ell_0$ and sufficiently large n for a graph C_n drawn uniformly at random from $\mathcal{C}_n$

$$\Pr\left[b(\ell\ldots\delta\ell;\mathsf{C}_n) \in (1\pm\varepsilon)b_{\ell,\delta} n\right] \geq 1 - e^{-\frac{\varepsilon^2}{40}b_{\ell,\delta} n},$$

and

$$b_{\ell,\delta} \sim_\ell \frac{b}{\rho_{\mathcal{B}}(\beta-2)}\cdot(1-\delta^{-\beta+2})\ell^{-\beta+2},$$

where b is the constant from the definition of (α,β)-nice.

Proof. As before, we deduce that the probability that a random graph from $\mathcal{B}'$, drawn according to the Boltzmann distribution with parameter $x = C^\bullet(\rho_{\mathcal{C}})$, is precisely

$$t_\ell := \sum_{s=\ell}^{\delta\ell}[x^{s-1}]B'(x)\cdot\frac{(C^\bullet(\rho_{\mathcal{C}}))^{s-1}}{B'(C^\bullet(\rho_{\mathcal{C}}))} = \sum_{s=\ell}^{\delta\ell}\frac{|\mathcal{B}_s|}{(s-1)!}\cdot\frac{(C^\bullet(\rho_{\mathcal{C}}))^{s-1}}{B'(C^\bullet(\rho_{\mathcal{C}}))}.$$

Using that $C^\bullet(\rho_{\mathcal{C}}) = \rho_{\mathcal{B}}$ due to $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) \leq 1$ and Lemma 2.6, the assumption $\ell \gg 1$, the asymptotic estimate $|\mathcal{B}_n| \sim bn^{-\beta}\rho_{\mathcal{B}}^{-n} n!$, and the Euler-MacLaurin summation formula, we obtain that

$$(3.10)\qquad t_\ell \sim_\ell \frac{b}{B'(\rho_{\mathcal{B}})\rho_{\mathcal{B}}}\cdot\sum_{s=\ell}^{\delta\ell} s^{-\beta+1} \sim_\ell \frac{b\cdot(1-\delta^{-\beta+2})\ell^{-\beta+2}}{B'(\rho_{\mathcal{B}})\rho_{\mathcal{B}}\cdot(\beta-2)}.$$

Having this fact at hand we can easily replicate the proof of Lemma 3.1 to obtain the claimed statement. Indeed, let $\mathcal{S}_n$ be the set of all graphs in $\mathcal{C}_n$ that have at most $(1-\varepsilon)b_{\ell,\delta} n$ or at least $(1+\varepsilon)b_{\ell,\delta} n$ blocks of size in the interval $[\ell,\delta\ell]$, and write $\mathcal{S}_n = \mathcal{S}_n^{(1)} \cup \mathcal{S}_n^{(2)}$, where $\mathcal{S}_n^{(1)}$ is defined as in Lemma 3.1, and $\mathcal{S}_n^{(2)} = \mathcal{S}_n \setminus \mathcal{S}_n^{(1)}$. With this notation we can imitate the derivation in (3.9) and estimate the probability for the event "$\Gamma C^\bullet \in \mathcal{S}_n^{(1)}$ and $\Gamma C^\bullet \in \mathcal{C}_n^\bullet$" in exactly the same way as in Lemma 3.1. Finally, the probability for the event "$\Gamma C^\bullet \in \mathcal{S}_n^{(2)}$ and $\Gamma C^\bullet \in \mathcal{C}_n^\bullet$" can be estimated analogously, where we use the value t_ℓ derived above. This completes the proof.

Now consider the case $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) > 1$. Here we obtain the following statement for block sizes not covered by Lemma 3.1.

LEMMA 3.3. *Let $\mathcal{C}$ be an (α,β)-nice class with $\alpha > 2$ such that $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) > 1$. Then, for any $\xi = \xi(n) \geq 0$*

$$\Pr\Big[lb(\mathsf{C}_n) \geq (\alpha+\xi)\log_{\rho_{\mathcal{B}}/\tau} n\Big] = o(n^{-\xi}),$$

where $0 < \tau < 1$ is given by $\tau B''(\tau) = 1$. Moreover, for any fixed $\varepsilon > 0$ we have asymptotically almost surely that

$$b\big((1-\varepsilon)\log_{\rho_{\mathcal{B}}/\tau} n \ldots \alpha\log_{\rho_{\mathcal{B}}/\tau} n;\, \mathsf{C}_n\big) \leq n^{3\varepsilon/2}.$$

Proof. By applying Lemma 2.6 we infer that $\tau = C^\bullet(\rho_{\mathcal{C}}) < \rho_{\mathcal{B}}$. Note that in this case the probability that a random graph B' from $\mathcal{B}'$ drawn according to the Boltzmann distribution with parameter τ has ℓ vertices is *exponentially* small in ℓ. More precisely, for large ℓ we have

$$\begin{aligned}\Pr[|\mathsf{B}'| = \ell] &= \frac{|\mathcal{B}'_\ell|\tau^\ell}{B'(\tau)\ell!} = \frac{|\mathcal{B}_{\ell+1}|\tau^\ell}{B'(\tau)\ell!}\\ &\sim_\ell \frac{b}{\rho_{\mathcal{B}}B'(\tau)}\ell^{-\beta+1}\left(\frac{\tau}{\rho_{\mathcal{B}}}\right)^\ell.\end{aligned} \tag{3.11}$$

In order to prove the statements of the lemma we again replicate the proof of Lemma 3.1. To deduce the first statement we let $\mathcal{S}_n$ denote the set of all graphs in $\mathcal{C}_n^\bullet$ that contain a block with at least $\ell_1 = (\alpha+\xi)\log_{\rho_{\mathcal{B}}/\tau} n$ vertices, and write $\mathcal{S}_n = \mathcal{S}_n^{(1)} \cup \mathcal{S}_n^{(2)}$, where $\mathcal{S}_n^{(1)}$ is defined as in Lemma 3.1, and $\mathcal{S}_n^{(2)} = \mathcal{S}_n \setminus \mathcal{S}_n^{(1)}$. Again we estimate the probability for the event "$\Gamma C^\bullet \in \mathcal{S}_n^{(1)}$ and $\Gamma C^\bullet \in \mathcal{C}_n^\bullet$" in exactly the same way as in Lemma 3.1. To bound the probability for the event "$\Gamma C^\bullet \in \mathcal{S}_n^{(2)}$ and $\Gamma C^\bullet \in \mathcal{C}_n^\bullet$" we observe that the expected number of blocks of size at least ℓ_1 among $N = (1\pm\frac{\varepsilon}{3})\lambda_{\mathcal{C}}n$ randomly chosen blocks is given by N times the sum of the values in (3.11) for $\ell \geq \ell_1$. The latter is easily seen to be $o(n^{-\alpha-\xi})$, and thus the expected number of blocks is $o(n^{-\alpha-\xi+1})$, which completes the proof with the observation that $\Pr[X>0] \leq \mathbf{E}[X]$ (valid for any random variable X that obtains only non-negative integers, cf. Markov's inequality).

To deduce the second statement we let $\mathcal{S}_n$ denote the set of all graphs $G \in \mathcal{C}_n$ so that $b\big((1-\varepsilon)\log_{\rho_{\mathcal{B}}/\tau} n \ldots \alpha\log_{\rho_{\mathcal{B}}/\tau} n;\, G\big) \geq n^{3\varepsilon/2}$ and proceed as before. Note that (3.11) implies

$$\Pr\Big[|\mathsf{B}'| \geq (1-\varepsilon)\log_{\rho_{\mathcal{B}}/\tau} n\Big] = \Theta(n^{-1+\varepsilon}).$$

Hence, among $N = (1\pm\frac{\varepsilon}{3})\lambda_{\mathcal{C}}n$ blocks we expect to see $\Theta(n^{-1+\varepsilon}\cdot\lambda_{\mathcal{C}}n)$ blocks of size at least $(1-\varepsilon)\log_{\rho_{\mathcal{B}}/\tau} n$. A straightforward application of the Chernoff bounds thus completes the proof in the same way as before.

With these lemmas we are now ready to deduce some of the statements of Theorem 1.4.

Proof. [Proof of Theorem 1.4, statement (i) and parts 2 and 3 of statement (ii)] The statements follow immediately from Lemmas 3.1, 3.2, and 3.3. To deduce the asymptotic estimates for b_ℓ recall that $[x^{\ell-1}]B'(x) = |\mathcal{B}_\ell|/(\ell-1)!$ and use that $|\mathcal{B}_\ell| \sim_\ell b\ell^{-\beta}\rho_{\mathcal{B}}^{-\ell}\ell!$ due to the definition of (α,β)-nice.

3.2 Large Blocks in C_n

Recall that $b(\ell; G)$ denotes the number of blocks in a graph G that contain exactly ℓ vertices, and that $lb(G)$ is the maximum number of vertices in a block of G.

LEMMA 3.4. *Let $\mathcal{C}$ be an (α,β)-nice class such that $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) < 1$. For sufficiently large n we have asymptotically almost surely for a graph C_n drawn uniformly at random from $\mathcal{C}_n$ that $lb(\mathsf{C}_n) \sim cn$, where*

$$c = 1 - \rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}).$$

Moreover, let ω_n be any function having the property $\lim_{n\to\infty}\omega_n = \infty$. Then, for all $n^{1/(\beta-2)}\omega_n \leq \ell < lb(\mathsf{C}_n)$ we have $b(\ell; \mathsf{C}_n) = 0$.

Proof. We give here a rough sketch of the proof and refer the reader to the full version for more details. Our proof strategy consists of two main steps. First, we apply Lemma 3.1 and Lemma 3.2, and obtain sharp bounds for the number of blocks in $\mathcal{S}_n$ that contain ℓ vertices, where $2 \leq \ell \leq n^{1/(\beta-2)}\omega_n$. By using these bounds we deduce that the number of vertices in blocks of size $\leq n^{1/(\beta-2)}\omega_n$ is cn, up to smaller order error terms. Then, a careful counting argument shows that the probability that the remaining vertices are contained in a single block is $1-o(1)$.

4 Remaining Proofs

Proof. [Proof of Lemma 1.1] First, note that the terms $(1-\frac{x}{\rho_{\mathcal{B}}})^m$, where $m \in \mathbf{N}_0$ and $m < \beta$, do not contribute anything to $[x^n]B(x) = |\mathcal{B}_n|$ for large n. Moreover, by applying Theorem VI.1 and Corollary VI.1 from [10] to the singular expansion of $B(x)$ we get

$$|\mathcal{B}_n| \sim bn^{-\beta-1}\rho_{\mathcal{B}}^{-n}n!, \tag{4.12}$$

for an appropriate $b > 0$. To complete the proof we show $|\mathcal{C}_n^\bullet| \sim cn^{-\alpha}\rho_{\mathcal{C}}^{-n}n!$. For the case $\rho_{\mathcal{B}}B''(\rho_{\mathcal{B}}) > 1$ this follows immediately from the first conclusion of Lemma 2.6, where in particular $\alpha = 5/2$.

Next we consider the more involved case $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) < 1$. By applying Lemma 2.6 we obtain that $C^\bullet(\rho_{\mathcal{C}}) = \rho_{\mathcal{B}}$ and that $\rho_{\mathcal{C}} = \rho_{\mathcal{B}} e^{-B'(\rho_{\mathcal{B}})}$. Note that as the singularity type of $B(x)$ is β, the singularity type of $B'(x)$ is $\beta - 1$, and similarly, the singularity type of $B''(x)$ is $\beta - 2$. Hence, as $B''(\rho_{\mathcal{B}})$ exists, we infer that $\beta > 2$, which implies that $B'(x)$ is of singularity type > 1. Now, having the singular expansion (1.4) of $B(x)$ at hand, we can readily derive the singular expansions for $B'(x)$ and $B''(x)$, which start with the terms

$$B'(x) = -\frac{s_\mu}{\rho_{\mathcal{B}}} - \frac{2s_{2\mu}}{\rho_{\mathcal{B}}}\left(1 - \frac{x}{\rho_{\mathcal{B}}}\right) + \dots$$

and

$$B''(x) = \frac{2s_{2\mu}}{\rho_{\mathcal{B}}^2} + \dots \quad ,$$

where the "..." stand for terms of the form $\left(1 - \frac{x}{\rho_{\mathcal{B}}}\right)^\xi$ for $\xi \neq 0$, which depend on the expansion of $B(x)$. But then we have that $s_\mu \neq 0$, as otherwise $B'(\rho_{\mathcal{B}}) = 0$, which contradicts the positivity of the coefficients of $B(x)$. Moreover, due to our assumption $\rho_{\mathcal{B}} B''(\rho_{\mathcal{B}}) < 1$ we may assume that $\frac{2s_{2\mu}}{\rho_{\mathcal{B}}} - 1 \neq 0$. By exploiting this information we can derive straightforwardly by functional composition the singular expansion of the functional inverse $\psi(u) = ue^{-B'(u)}$ of $C^\bullet(x)$ by plugging in the singular expansion of $B'(u)$. A straightforward calculation shows that that the singularity type of ψ is also $\beta - 1$ and that the singular expansion of $\psi(u)$ starts with

$$\psi(u) = \rho_{\mathcal{C}} + \rho_{\mathcal{C}}\left(\frac{2s_{2\mu}}{\rho_{\mathcal{B}}} - 1\right)\left(1 - \frac{u}{\rho_{\mathcal{B}}}\right) + \dots \quad .$$

Given that $C^\bullet(x)$ and $\psi(u)$ are functional inverses and that $\psi(\rho_{\mathcal{B}}) = \rho_{\mathcal{C}}$ we can determine with the above information by indeterminate coefficients the singular expansion of $C^\bullet(x)$ at $x = \rho_{\mathcal{C}}$. Again, a straightforward calculation shows that this expansion is of the same singular type as the expansion of $\psi(u)$ at $u = \rho_{\mathcal{B}}$. That is, the singular type of $C^\bullet(x)$ is $\beta - 1 > 1$. By applying the Standard Function Scale Theorem (see e.g. Theorem VI.1 from [10]), we readily deduce that this implies that there is a $c > 0$ such that $|\mathcal{C}_n^\bullet| \sim cn^{-\beta}\rho_{\mathcal{C}}^{-n} n!$, as we wanted to show.

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