Sequential cavity method for computing limits of the log-partition function for lattice models

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Abstract

One of the key computational problems in combinatorics/statistical physics is the problem of computing limits of the log-partition functions for various statistical mechanics models on lattices. In combinatorics this limit corresponds to the exponent of various arrangements on lattices, for example the exponents of the number of independent sets, proper colorings or matchings on a lattice. In statistical physics this limit is called free energy.

We propose a new method, sequential cavity, which beats the best known existing methods, such as transfer matrix method, in obtaining sharper bounds on the limits of the log-partition function for two models: independent sets (hard-core) and matchings (monomer-dimer). Our method is based on a surprisingly simple representation of the log-partition function limit in terms of a certain marginal probability of a suitably modified lattice, and using recent deterministic approximation counting algorithms for these two models. Our method also has a provably better theoretical performance compared with the transfer matrix method.

1 Introduction

It is a classical fact in statistical physics that the logarithm of the partition function of a general statistical mechanics model on $[-n,n]^d \subset \mathbb{Z}^d$, appropriately rescaled, has a well-defined limit as $n \to \infty$ [Geo88],[Sim93]. In combinatorics this object in a special case is the limit of the logarithm of the number of different combinatorial arrangements on $[-n,n]^d$, for example the number of independent sets, coloring or partial matchings. The limit is called free energy or pressure (the difference is in normalizing constant) in statistical physics [Sim93] and entropy in combinatorics [FP05] (the latter is due to the uniform distribution on the arrangements usually considered in combinatorics). It is a different matter to compute this limit, and this question interested researchers both in the statistical physics and combinatorics communities. In very special cases, free energy can be computed analytically. The most widely known example is Fisher-Kasteleyn-Temperley's formula for dimer model on $\mathbb{Z}^2$ [Fis61], [Kas61],[TF61]. Another example of an exactly solvable model is hard-core model on a hexagonal lattice [Bax80].

Short of these special cases, the existing methods for computing free energy mostly rely on numerical approximations. These include randomized methods such as Monte-Carlo [JS97],[KRS96] and deterministic methods such as transfer matrices. The Monte-Carlo method can be used to estimate free energy in finite graphs, say $[-n,n]^d$, with some probabilistic approximation guarantee, provided that some underlying Markov chain is rapidly mixing. One then has to relate finite graph to infinite lattices to approximate free energy for an infinite lattice. The drawback of this method is its dependence on the sampling error. Transfer matrix method, on the other hand is a deterministic method and provides rigorous bounds on the free energy. It is based on considering an infinite strip $[-n,n]^{d-1} \times \mathbb{Z}$ and then identifying the number of different ways two configurations on $[-n,n]^{d-1}$ can match. One then constructs an $[-n,n]^{d-1}$ by $[-n,n]^{d-1}$ matrix and the spectral radius of this transfer matrix can be related to the growth rate of the partition function on $[-n,n]^{d-1} \times [-N,N]$ as a function of N, namely the partition function on $[-n,n]^{d-1} \times \mathbb{Z}$. Then by making n sufficiently large, bounds on the free energy on the entire lattice $\mathbb{Z}^d$ can be obtained. The construction of such transfer matrix requires time $\exp(O(n^{d-1}))$ time. Since the convergence rate of the free energy to its limit wrt n is known to be $O(1/n)$ [Sim93], then in order to get a target additive error ϵ, the transfer matrix method requires time $\exp(O((1/\epsilon)^{d-1})$. Namely, this is time required to construct an ϵ-length interval containing the actual value of the free energy. This is why the method stops being effective for large d. Some additional computational savings can be achieved using underlying automorphisms group structure, see Friedland and Peled [FP05], but we are not aware of any formal analysis of the computational savings produced by this method.

The transfer matrix method was used to obtain some of the best known bounds. For the hard-core model (see Section 4) with activity $\lambda = 1$, which cor-

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responds to counting independent sets in $\mathbb{Z}^d$, the exponent of the free energy is known to be in the range $[1.503047782, 1.5035148]$, see Calkin and Wilf [CW98]. A far more accurate but non-rigorous estimate was obtained by Baxter [Bax99]. Similarly, the transfer matrix method was used to compute the free energy of the monomer-dimer model (see Section 5). The problem has a long history. Earlier studies include Hammersley [Ham66a], [Ham66b], Hammersley and Menon [HM70], Baxter [Bax68], where some non-rigorous estimates and crude bounds were obtained. Recently Friedland and Peled [FP05] obtained rigorously a range $0.6627989727 \pm 0.0000000001$ for $d = 2$ and $[0.7653, 0.7863]$ for $d = 3$ using the transfer matrix method. The lower bound was later tightened to 0.7845 using the Friedland-Tveberg inequality which provides a bound for general regular graphs. This bound, while quite accurate for the case $\mathbb{Z}^3$, is not improvable by running some numerical procedure longer or on a faster machine.

In this paper we propose a completely new approach for the problem of computing numerically free energy. Our approach takes advantage of the fact that some of these models, including the two models above, are in the so-called uniqueness regime. Namely, the Gibbs measure on the infinite lattice is unique. This is an implication of the Strong Spatial Mixing (SSM) property [BMP99], see Subsection 2.4 for the definition. This property asserts that the marginal probability that a node v attains a particular spin value σ_v is asymptotically independent from the spin configurations for nodes u which are far away from v. Our main theoretical result is Theorem 3.1 which provides a surprisingly simple representation of the free energy in terms of such a marginal probability, provided that the SSM holds. Such representation is called cavity method in statistical physics and has been used heavily for analyzing statistical models on sparse random graphs and more generally locally tree-like graphs [Ald01],[RBMM04], [MP03],[MP87], where one can write certain recursive distributional equations satisfied by cavity values. Here due to a particular sequencing of removed vertices, we call our approach *sequential cavity method*.

We note that our results provide an independent proof of the existence of the free energy, not relying on the sub-additivity arguments, albeit in the special case of SSM. Theorem 3.1 reduces the problem of computing free energy to the one of computing marginal probabilities, and this is done using recent deterministic algorithms for computing such marginals in certain models satisfying SSM property [Wei06],[GK07],[BGK$^+$07]. These results are based on establishing even stronger property, namely correlation decay on a computation tree and lead to efficient algorithms for computing such marginals. We have implemented these algorithm for the hard-core [Wei06] and monomer-dimer [BGK$^+$07] models in the special case of $\mathbb{Z}^d$. Using our approach we improve existing bounds for these models. For example we show that the exponent of the free-energy of the hard-core model in two dimensions is in the range $[1.503034, 1.503058]$. While our lower bound is weaker than the previous best known bound 1.503047782 [CW98] (see above), which is already quite close to a believed estimate [Bax99], our upper bound improves the earlier best bound 1.5035148 [CW98]. For the case of monomer-dimer model in three dimensions we obtained a range $[0.78595, 0.78599]$, substantially improving earlier bounds [FG08] (see above). Further we show that the numerical complexity of obtaining ϵ-additive approximation using our approach is $(1/\epsilon)^{O(1)}$. The constant in $O(1)$ may depend on model parameters and dimensions. This is a substantial improvement over the computation effort $\exp(O((1/\epsilon)^{d-1})$ of the transfer matrix method for computing free energy.

The rest of the paper is organized as follows. In the following section we provide a necessary background on Gibbs measures on general graphs and lattices, free energy, and define an important notion – (exponential) Strong Spatial Mixing. Our main theoretical result is Theorem 3.1 which represents free energy in terms of marginal probabilities. This result and its several variations are stated and proven in Section 3. Sections 4 and 5 are devoted to application of Theorem 3.1 and its variations to the problem of numerically estimating free energy for hard-core and monomer-dimer models specifically. Additionally, in these sections we compare the algorithmic complexity of our method with the complexity of the transfer matrix method. Most of the proofs are in the Appendix section and all the figures are in the Appendix.

2 Model, assumptions and notations

2.1 Finite and locally finite graphs Consider a finite or infinite locally finite simple undirected graph $\mathbb{G}$ with node set V and edge set $E \subset V \times V$. The locally-finite property means every node is connected to only finitely many neighbors. The graph is undirected and simple (no loops, no multiple edges). For every $v, u \in V$ let $d(u, v)$ be the length (number of edges) in the shortest path connecting u and v. For every node v, we let $N(v)$ stand for the set of neighbors of v: $N(v) = \{u : (u, v) \in E\}$. We will write $N_{\mathbb{G}}(v)$ when we need to emphasize the underlying graph $\mathbb{G}$. The quantity $\Delta = \Delta_{\mathbb{G}} \triangleq \max_v |N(v)|$ is called the degree of the graph. For every $r \in \mathbb{Z}_+$ let $B_r(v) = \{u : d(v, u) \leq r\}$. For every set $A \subset V$, let $B_r(A) = \cup_{v \in A} B_r(v)$.

Thus locally finite property means $|B_r(v)| < \infty$ for all v, r. Given $A \subset V$, let $\partial A = \{v \in A : N(v) \cap A^c \neq \emptyset\}$ and let $\partial_r A = \partial(B_r(A))$. For every subset $A \subset V$, we have an induced subgraph obtained by taking nodes in A and all edges $(v, u) \in E \cap A^2$. The corresponding edge set of the induced graph is denoted by $E(A)$.

Let $\mathbb{R}_+$ ($\mathbb{Z}_+$) denote the set of all non-negative real (integer) values. Le $\mathbb{R}_{>0}(\mathbb{Z}_{>0})$ be the set of all positive real (integer) values. Our main example of an infinite locally-finite graph is the d-dimensional lattice $\mathbb{Z}^d$ with $V = \{(v_1, \ldots, v_d) : v_i \in \mathbb{Z}$ and $E = \{(v, u) \in V^2 : \|v - u\| = 1\}$, where $\|w\| = \sum_{1 \leq j \leq d} |w_j|$. We denote the origin $(0, 0, \ldots, 0)$ by 0 for short. Given a vector $a = (a_1, \ldots, a_d) \in \mathbb{R}^d_+$, and $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, let $B_{an}(v) = \{u \in \mathbb{Z}^d : |u_j - v_j| \leq a_j n,\ j = 1, 2, \ldots, d\}$. In the special case $\mathbb{G} = \mathbb{Z}^d, v = 0$, we write B_r instead of $B_r(0)$ and B_{an} instead of $B_{an}(0)$. Each of these is a d-dimensional half-plane in $\mathbb{Z}^d$. Let $\prec$ denote a lexicographic full order on $\mathbb{Z}^d$. Namely, $v \prec u$ iff either $u_d > v_d$ or $\exists k \in \{1, 2, \ldots, d-1\}$ such that $u_k > v_k$ and $v_j = u_j, j = k+1, \ldots, d$. For every $v \in \mathbb{Z}^d$ and $k = 1, 2, \ldots, d$, let $\mathbb{Z}^d_{\prec v} = \{u \in \mathbb{Z}^d : u \prec v\} \cup \{v\}$.

Throughout the paper we write $f(n) = O(g(n))$ and $f(n) = o(g(n))$, $n \in \mathbb{Z}_+$ if $f(n) \leq Cg(n)$, respectively $f(n)/g(n) \to 0$, for all n, for some constant C. This constant may in general depend on model parameters such as H, h (see the next section) or dimension d. However, in some places the constant is universal, namely independent from any model parameters. We will explicitly say so if this is the case.

2.2 Gibbs measures Consider a finite set of spin values $\chi = \{s_1, \ldots, s_q\}$, a Hamiltonian function $H : \chi^2 \to \mathbb{R} \cup \{\infty\}$, and an external field $h : \chi \to \mathbb{R}$. Given a graph $\mathbb{G} = (V, E)$ we consider the associated spin configuration space $\Omega = \chi^{|V|}$ equipped with product σ-field $\mathcal{F}$. If the graph $\mathbb{G}$ is finite, a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is defined to be Gibbs measure if for every spin assignment $(s_v) \in \chi^V$

$$\begin{aligned}&\mathbb{P}\left(\sigma_v = s_v,\ \forall\ v \in V\right)\\ &= Z^{-1} \exp\big(-\sum_{(v,u) \in E} H(s_v, s_u) - \sum_{v \in V} h(s_v)\big),\end{aligned} \tag{2.1}$$

where Z is the normalizing partition function:

$$Z = \sum_{(s_v) \in \chi^{|V|}} \exp\big(-\sum_{(v,u) \in E} H(s_v, s_u) - \sum_{v \in V} h(s_v)\big).$$

We will often write $\mathbb{P}_{\mathbb{G}}$ and $Z_{\mathbb{G}}$ in order to emphasize the underlying graph. The case $H(a, a') = \infty$ corresponds to a hard-core constraint prohibiting assigning $a \in \chi$ and $a' \in \chi$ to neighbors.

When $\mathbb{G}$ is infinite, a probability measure $\mathbb{P}(\cdot)$ is defined to be Gibbs measure, if it satisfies certain spatial Markovian property, see [Sim93],[Geo88] for details. Generally there are multiple Gibbs measures and the space of Gibbs measures of $\mathbb{G}$ is denoted by $\mathcal{M}$ or $\mathcal{M}_{\mathbb{G}}$. We say that the model $(\mathbb{G}, H, h)$ is in the uniqueness regime if $\mathcal{M}$ consists of a unique measure $\mathcal{M} = \{\mathbb{P}\}$. Most of the results in this paper correspond to the uniqueness case, and more specifically to the case of Strong Spatial Mixing (SSM) defined below. Given a model $(\mathbb{G}, H, h)$ and a subset $A \subset V$, we have a naturally defined induced submodel on the the induced subgraph $\mathbb{G}(A) = (V(A), E(A))$, given by the same χ, H and h (but different partition function). In order to emphasize the underlying subgraph $\mathbb{G}$ we write, with abuse of notation, $\mathbb{Z}_A$ and $\mathbb{P}_A$ for the partition function and the Gibbs measure on the subsystem $(\mathbb{G}(A), H, h)$, when it is unique. By default we drop the subscript when the underlying graph is the entire lattice $\mathbb{Z}^d$.

The definitions above is ”node” centered: the spins are associated with nodes of a graph. In order to study the monomer-dimer model we need to consider a similar ”edge” model where spins are associated with edges. Thus given a finite graph $\mathbb{G} = (V, E)$, and a finite set of spin values χ, we consider a probability space $\Omega = \chi^{|E|}$. Then (2.1) is restated as follows. Write $e \sim e'$ if edges e, e' are distinct and incident (share a node). For every spin assignment $(s_e) \in \chi^{|E|}$ we assign probability measure

$$\begin{aligned}&\mathbb{P}\left(\sigma_e = s_e,\ \forall\ e \in E\right)\\ &= Z^{-1} \exp\big(-\sum_{e \sim e' \in E} H(s_e, s_{e'}) - \sum_{e \in E} h(s_e)\big),\end{aligned} \tag{2.2}$$

2.3 Free energy and pressure Given $\mathbb{G} = \mathbb{Z}^d$, Hamiltonian H and an external field h, consider an arbitrary infinite sequence of finite subsets $0 \in \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathbb{Z}^d$, such that the sequence $r_n \triangleq \max\{r : B_r \subset \Lambda_i\}$ diverges to infinity as $n \to \infty$. Consider the corresponding sequence of Gibbs measures $\mathbb{P}_{\Lambda_n}$ and partition functions Z_{Λ_n} on the graphs induced by Λ_n. It follows from sub-additivity property of partition functions that the limit

$$\mathcal{P}(d, H, h) \triangleq \lim_{n \to \infty} \frac{\log Z_{\Lambda_n}}{|\Lambda_n|} \tag{2.3}$$

exists and is independent from the choice of the sequence of subsets [Sim93], [Geo88]. This quantity is called *free energy* (or sometimes *pressure*).

2.4 Strong Spatial Mixing The following technical assumption is needed for our analysis.

Assumption 1. *There exist $s^* \in \chi$ such that $\max_{s \in \chi} |H(s, s^*)| < \infty$.*

The assumption implies that under any Gibbs measure, for every node v

$$(2.4)\qquad c^* \triangleq \min_{(s_u)} \mathbb{P}(\sigma_v = s^* | \sigma_u = s_u, \forall u \in N(v)) > 0$$

where the minimum is over all possible spin assignments $(s_u) \in \chi^{|N(v)|}$ of the neighbors of v. We now introduce the Strong Spatial Mixing notion.

Definition 1. *$(\mathbb{G}, H, h)$ satisfies the Strong Spatial Mixing property if for every finite set $X \subset V(\mathbb{G})$, possibly infinite set $Y \in V(\mathbb{G})$ there exists a function $R(r)$ satisfying $\lim_{r\to\infty} R(r) = 0$, such that for every positive integer r*

$$\max \Big| \mathbb{P}(\sigma_v = s_v^1, v \in X | \sigma_v = s_v^2, v \in Y; \sigma_v = s_v^3, v \in \partial_r X) - \mathbb{P}(\sigma_v = s_v^1, v \in X | \sigma_v = s_v^2, v \in Y; \sigma_v = s_v^4, v \in \partial_r X) \Big| \le R(r),$$

where the maximum is over all possible spin assignments $(s_v^1) \in \chi^{|X|}, (s_v^2) \in \chi^{|Y|}, (s_v^3), (s_v^4) \in \chi^{|\partial_r X|}$. $(\mathbb{G}, H, h)$ satisfies exponential strong spatial mixing if there exist $\kappa, \gamma > 0$ such that $R(r) \le \kappa \exp(-\gamma r)$ for all $r \ge 0$.

It is known that if $(\mathbb{G}, H, h)$ exhibits SSM, then the Gibbs measure is unique: $\mathcal{M} = \{\mathbb{P}\}$, [BMP99].

3 Sequential cavity method

In this section we present our main theoretical result: the representation of the free energy on $\mathbb{Z}^d$ in terms of some conditional marginal probabilities $\mathbb{P}(\sigma_0 = s|\cdot)$ defined on suitably modified subset of $\mathbb{Z}^d$. The idea is to sequentially remove nodes from a rectangle B_{an} one by one and observe that the log-partition function can be written in terms of the log-marginal probabilities (cavity) of the removed nodes.

3.1 Representation theorem and variations

Given a graph $\mathbb{G} = (V, E)$ with some full order $\succ$, a node v and $s \in \chi$, let $\mathcal{E}_{v,s}$ denote the event "$\sigma_u = s$ for all $u \succ v, u \in V$". By convention we assume that $\mathcal{E}_{v,s}$ is the full event Ω, if the set $u \succ v, u \in V$ is empty. In the special case when $\mathbb{G} = \mathbb{Z}^d, v = 0$ and $\succ$ is the lexicographic order, this event is denoted by $\mathcal{E}_s$, see Figure 2 for the case $d = 2$. We remind the reader, that we drop the subscripts in $\mathbb{P}_{\mathbb{G}}, Z_{\mathbb{G}}$ when the underlying graph is $\mathbb{Z}^d$ and SSM holds guaranteeing uniqueness of the Gibbs measure.

Theorem 3.1. *Suppose $(\mathbb{Z}^d, H, h)$ satisfies the SSM property and the Assumption 1 holds. Then*

$$(3.5)\qquad \mathcal{P}(d, H, h) = -\log \mathbb{P}(\sigma_0 = s^* | \mathcal{E}_{s^*}) - dH(s^*, s^*) - h(s^*).$$

Remark : We stress that our proof does not rely on the existence of a limit in (2.3) and thus provides an independent proof for it.

Let us establish some variations of Theorem 3.1. First we extend identity (3.5) for the case when we do not necessarily have SSM, but instead have an upper or lower bound on marginal probability $\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*})$. In this case we obtain an analogue of (3.5) in the form of inequalities.

Given $r > 0$ consider an arbitrary spin assignment $(s_u) \in \chi^{|\partial B_r|}$ which is consistent with event $\mathcal{E}_{s^*}$. Namely, $s_u = s^*$ for all $u \succ 0, u \in \partial B_r$. Let $p_{\max}(r)$ ($p_{\min}(r)$) be the maximum (minimum) of $\mathbb{P}_{B_r}(\sigma_0 = s^*|\mathcal{E}_{s^*}, (a_u))$ when we vary over all such assignments.

Corollary 1. *For every $(\mathbb{Z}^d, H, h)$ and $r \ge 0$*

$$(3.6)\qquad \begin{aligned} &-\log p_{max}(r) - dH(s^*, s^*) - h(s^*) \\ &\le \mathcal{P}(d, H, h) \\ &\le -\log p_{min}(r) - dH(s^*, s^*) - h(s^*). \end{aligned}$$

While the result holds for arbitrary r the quality of the bounds presumably improves with increasing r. We will see in Section 4 that in some cases $p_{\max}(r)$ and $p_{\min}(r)$ are fairly close for large r even though the model is outside of provably exponential SSM regime.

Proof. The proof is a minor variation of the proof of (3.5). Instead of estimate (5.22) we use $p_{\min}(r) \le \mathbb{P}_{B_n}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) \le p_{\max}(r)$ for every $n > r$.

Now let us present a version of Theorem 3.1 for the model (2.2) where spins are assigned to edges rather than nodes. We need this for application to the monomer-dimer model. Let $\mathcal{E}_{v,s,\text{edges}}$ denote the event $\sigma_{u,w} = s, \forall u \succ v, w \in N(u)$. Let $\mathcal{E}_{s,\text{edges}}$ denote the same event when $v = 0$.

Theorem 3.2. *Consider a model $(\mathbb{Z}^d, H, h)$ given by (2.2) with spins assigned to edges. Suppose $(\mathbb{Z}^d, H, h)$ satisfies the SSM property and the Assumption 1 holds. Then*

$$(3.7)\qquad \begin{aligned} \mathcal{P}(d, H, h) &= \\ &= -\log \mathbb{P}(\sigma_{(0,v)} = s^*, \forall v \in N(0)|\mathcal{E}_{s^*,edges}) \\ &\quad - d(2d-1)H(s^*, s^*) - dh(s^*). \end{aligned}$$

4 Hard-core (independent set) model

The hard-core lattice gas model, commonly known as independent set model in combinatorics, is given by $\chi = \{0, 1\}, H(0, 0) = H(0, 1) = H(1, 0) = 0, H(1, 1) = \infty, h(0) = 0, h(1) = \beta$ for some parameter β. Choosing $s^* = 0$ we obtain that Assumption 1 holds.

It is common to set $\lambda = \exp(-\beta) > 0$ and let λ (called activity) be the parameter of the hard-core model. Note that in terms of λ for every finite graph $\mathbb{G}$, $Z_{\mathbb{G}} = \sum \lambda^{|\{v \in V : \sigma_v = 1\}|}$, where the sum is over all spin configurations $(\sigma_v) \in \{0,1\}^V$ such that $\sigma_v \sigma_u = 0$ for all $(v,u) \in E$. Equivalently, a subset of nodes $I \subset V(\mathbb{G})$ is called an independent (also sometimes called a stable set) if for no edge (u,v) we have both u and v belong to I. Then we may write $Z_{\mathbb{G}} = \sum_I \lambda^{|I|}$ where the summation is over all independent sets of $\mathbb{G}$. From now on we let $\boldsymbol{I}$ denote the random independent set selected according to a Gibbs measure (multiply defined if there are many Gibbs measures). In the case of finite graph $\mathbb{G}$, for every independent set I we have $\mathbb{P}_{\mathbb{G}}(\boldsymbol{I} = I) = Z_{\mathbb{G}}^{-1}\lambda^{|I|}$. The special case $\lambda = 1$ corresponds to a uniform distribution on the set of all independent sets in $\mathbb{G}$. We denote the free energy on $\mathbb{Z}^d$ by $\mathcal{P}(d,\lambda)$. Conditioning on spins taking value $s^* = 0$ simplifies significantly in the context of hard-core model:

Proposition 1. *If a hard-core model on a graph $\mathbb{G} = (V,E)$ satisfies SSM for some λ, then so does any subgraph of $\mathbb{G}$. The same assertion applies to exponential SSM. Moreover, for every $W_1, W_2 \subset V(\mathbb{G})$ the following identity holds with respect to the unique Gibbs measure.*

$$\mathbb{P}_{\mathbb{G}}(v \in \boldsymbol{I} | W_1 \cap \boldsymbol{I} = \emptyset, W_2 \subset \boldsymbol{I}) = \mathbb{P}_{\hat{\mathbb{G}}}(v \in \boldsymbol{I}), \tag{4.8}$$

for every $v \in V \setminus (W_1 \cup B_1(W_2))$, where $\hat{\mathbb{G}}$ is the subgraph induced by nodes in $V \setminus (W_1 \cup B_1(W_2))$.

In light of Propoposition 1 we obtain the following simplification of Theorem 3.1.

Corollary 2. *Suppose the hard-core model on $\mathbb{Z}^d$ satisfies SSM for a given λ. Then*

$$\mathcal{P}(d,\lambda) = -\log \mathbb{P}_{\mathbb{Z}^d_{\prec 0}}(0 \notin \boldsymbol{I}). \tag{4.9}$$

Thus we focus on developing an algorithm for estimating marginal probabilities in (4.9).

4.1 Recursion, sequential cavity algorithm and correlation decay Let us now introduce a recursion satisfied by the hard-core model. This identity in a different form using a self-avoiding tree construction was established recently by Weitz [Wei06]. We repeat here some of the developments in [Wei06], with some minor modifications, which are indicated as necessary.

Theorem 4.1. *Given a finite graph $\mathbb{G} = (V,E)$ and $v \in V$, let $N(v) = \{v_1, \ldots, v_k\}$. Then*

$$\mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{I}) = \frac{1}{1 + \lambda \prod_{1 \le i \le k} \mathbb{P}_{\mathbb{G}_{i-1}}(v_i \notin \boldsymbol{I})} \tag{4.10}$$

where $\mathbb{G}_i$ is the graph induced by $V \setminus \{v, v_1, \ldots, v_i\}$, $\mathbb{G}_0$ is induced by $V \setminus \{v\}$ and $\prod_{1 \le i \le k} = 1$ when $k = 0$.

Given a finite graph $\mathbb{G}$, for every subgraph $\hat{\mathbb{G}} = (\hat{V}, \hat{E})$ of $\mathbb{G}$, every vertex $v \in \hat{V}$ and every $t \in \mathbb{Z}_+$ we introduce a quantity $\Phi_{\hat{\mathbb{G}}}(v,t)$ defined inductively as follows.

(4.11)

$$\Phi_{\hat{\mathbb{G}}}(v,t) =$$
$$\begin{cases} 1, & t = 0; \\ (1+\lambda)^{-1}, & t > 0,\ N(v) = \emptyset, \\ (1 + \prod_{1 \le i \le k} \Phi_{\hat{\mathbb{G}}_{i-1}}(v_i, t-1))^{-1}, & t > 0,\ N(v) \neq \emptyset. \end{cases}$$

where in the third case we let $N(v) = \{v_1, \ldots, v_k\} \neq \emptyset$. Here again $\hat{\mathbb{G}}_0$ is induced by $\hat{V} \setminus \{v\}$ and $\hat{\mathbb{G}}_i$ is induced by $\hat{V} \setminus \{v, v_1, \ldots, v_i\}$. The recursion (4.11) is naturally related to the identity (4.10). Specifically if $\Phi_{\hat{\mathbb{G}}_{i-1}}(v_i, t-1) = \mathbb{P}_{\hat{\mathbb{G}}_{i-1}}(v_i \notin \boldsymbol{I})$ for all i then $\Phi_{\hat{\mathbb{G}}}(v,t) = \mathbb{P}_{\hat{\mathbb{G}}}(v \notin \boldsymbol{I})$. However, this will not occur in general, as we set $\Phi_{\hat{\mathbb{G}}}(v,0) = 1$, due to the lack of knowledge of actual values of the corresponding probabilities.

Similarly, we introduce values $\Psi_{\hat{G}}(v,t)$ with the only exception that $\Psi_{\hat{\mathbb{G}}}(v,0) = 0$ for all $\hat{\mathbb{G}} = (\hat{V}, \hat{E})$ and $v \in \hat{V}$. The following lemma follows from Theorem 4.1 and the definitions of Φ and Ψ using a simple induction argument

Lemma 4.1. *For every v and $t \in \mathbb{Z}_+$*

$$\Psi_{\mathbb{G}}(v,2t) \le \mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{I}) \le \Phi_{\mathbb{G}}(v,2t),$$
$$\Phi_{\mathbb{G}}(v,2t+1) \le \mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{I}) \le \Psi_{\mathbb{G}}(v,2t+1)$$

Lemma 4.2. *For every finite graph $\mathbb{G}$ with degree $\le \Delta$, $v \in \mathbb{G}$ and t, the values $\Phi_{\mathbb{G}}(v,t), \Psi_{\mathbb{G}}(v,t)$ can be computed in time $\exp(O(t \log \Delta))$, where the constant in $O(\cdot)$ is universal.*

Proof. The result follows immediately from the recursive definitions of Φ and Ψ.

The crucial correlation decay property is formulated in the following proposition.

Theorem 4.2. ([Wei06]) *For every $\Delta \ge 3$ and for every*

$$\lambda < (\Delta - 1)^{\Delta - 1}/(\Delta - 2)^{\Delta}, \tag{4.12}$$

there exists $C = C(\lambda,\Delta), \rho = \rho(\lambda,\Delta) < 1$ such that for every finite graph $\mathbb{G} = (V,E)$ with degree at most Δ and every $v \in V, t \in \mathbb{Z}_+$:

$$|\log \Phi_{\mathbb{G}}(v,t) - \log \Psi_{\mathbb{G}}(v,t)| \le C\rho^t. \tag{4.13}$$

As a result, $\mathbb{G}$ satisfies exponential SSM for λ satisfying (4.12).

4.2 Free energy and surface pressure on $\mathbb{Z}^d$. Numerical results We are now equipped to obtain bounds on the free energy for the hard-core model on $\mathbb{Z}^d$.

Denote by $\Phi(t)$ and $\Psi(t)$ the values of $\Phi_{\mathbb{G}}(v,t), \Psi_{\mathbb{G}}(v,t)$ when applied to a graph $\mathbb{G} = \mathbb{Z}^d_{\prec 0} \cap B_n$, for sufficiently large n and $v = 0$. Observe that the values $\Phi_{\mathbb{G}}(v,t), \Psi_{\mathbb{G}}(v,t)$ are the same for all values of n sufficiently larger than t (for example $n \geq t+1$ suffices). Thus the notations are well-defined.

Corollary 3. *For every $t \in \mathbb{Z}_+$ and λ satisfying (4.12)*

$$-\log \Phi(2t) \leq \mathcal{P}(d,\lambda) \leq -\log \Psi(2t) \tag{4.14}$$

$$-\log \Psi(2t+1) \leq \mathcal{P}(d,\lambda) \leq -\log \Phi(2t+1). \tag{4.15}$$

Proof. By Theorem 4.2 we have exponential SSM. Thus

$$\mathbb{P}_{\mathbb{Z}^d_{\prec 0}}(0 \notin I) = \lim_{n\to\infty} \mathbb{P}_{B_n \cap \mathbb{Z}^d_{\prec 0}}(0 \notin I).$$

$B_n \cap \mathbb{Z}^d_{\prec 0}$ is finite graph for which bounds from Lemma 4.1 are applicable.

Our algorithm for computing $\mathcal{P}(d,\lambda)$ is based on (4.14) and (4.15), and will be called *Sequential Cavity Algorithm* (SCA).

We have numerically computed values $\Phi(t), \Psi(t)$ for the cases $d = 2,3,4$. Our results provide the following bounds on the free energy for the case $\lambda = 1$. Since previous bounds were stated in terms of $\exp(\mathcal{P}(d,1)$, we do the same here:

$$1.503034 \leq \exp(\mathcal{P}(2,1)) \leq 1.503058$$
$$1.434493 \leq \exp(\mathcal{P}(3,1)) \leq 1.449698$$
$$1.417583 \leq \exp(\mathcal{P}(4,1)) \leq 1.444713$$

Our lower bound for the case $d = 2$ is weaker than the previous best known 1.503047782 [CW98], which is already very close to the presumably correct but unproven value stated in [Bax99]. However, our upper bound is stronger than the previous best known 1.5035148 [CW98]. We are not aware of any estimates for the case $d = 3$. Thus we believe our bounds are the best known. Note that in the case $d = 3$ we have $\Delta = 6$, and $\lambda = 1$ no longer satisfies (4.12). Thus we have no guarantee that SCA will provide converging estimates as t increases. The correctness of our bounds for this case is guaranteed by Corollary 1. It is encouraging to see that the bounds are close and based on this fact we conjecture that $\lambda = 1$ corresponds to the uniqueness regime. A significant speed up in the computation was achieved by using a certain chess-board modification of the representation Theorem 3.1 which is based on removing every second node as opposed to every node $v \succ 0$. The details of this approach can be found in [GK08].

Let us now analyze the computation effort required to obtain a particular accuracy level.

Proposition 2. *For every $d, \lambda < (d-1)^{d-1}/(d-2)^d$ and $\epsilon > 0$ SCA produces an ϵ-additive estimate of $\mathcal{P}(d,\lambda)$ in time $(\frac{1}{\epsilon})^{O(1)}$, where the constant in $O(\cdot)$ depends on λ and d.*

5 Monomer-dimer model

The monomer-dimer model is defined by spin values $S = \{0,1\}$ assigned to edges of a graph $G = (V,E)$. A set of edges $M \subset E$ is a matching if no two edges in M are incident. The edges of M are called dimers and nodes in V which are not incident to any edge in M are called monomers. We set $H(0,0) = H(0,1) = H(1,0) = h(0) = 0, H(1,1) = \infty, h(1) = \beta$. The Gibbs measure is defined via (2.2). Introduce $\lambda = \exp(-\beta) > 0$. For finite graphs $\mathbb{G}$ we have $Z_{\mathbb{G}} = \sum_M \lambda^{|M|}$ were the summation is over all matchings in $\mathbb{G}$. We denote by $\boldsymbol{M}$ a random matching chosen according to the Gibbs measure, when it is unique. In the case of finite graphs $\mathbb{P}_{\mathbb{G}}(\boldsymbol{M} = M) = Z_{\mathbb{G}}^{-1}\lambda^{|M|}$. The monomer-dimer model is a close relative of the hard-core model, even though its properties are substantially different. For example this model does not exhibit a phase transition and is always in the uniqueness regime [HL72]. Moreover, it satisfies the SSM property for all activities λ as we shall shortly see.

5.1 Recursion, sequential cavity algorithm and correlation decay We now establish an analogue of (4.10) for the monomer-dimer model. The proof of this result can be found in [BGK$^+$07] and is omitted. It is similar to the proof of (4.10). In the following, with a slight abuse of notation we write $v \in M$ if matching M contains an edge incident to v.

Theorem 5.1. *[BGK$^+$07] For every finite graph $\mathbb{G} = (V,E)$ and $v \in V$*

$$\mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{M}) = \frac{1}{1 + \lambda \sum_{u \in N_{\mathbb{G}}(v)} \mathbb{P}_{\mathbb{G}_0}(u \notin \boldsymbol{M})} \tag{5.16}$$

where $\mathbb{G}_0$ is induced by $V \setminus \{v\}$ and $\sum_{u\in N_{\mathbb{G}}(v)} = 0$ when v is an isolated node.

Given a finite graph $\mathbb{G}$, for every subgraph $\hat{\mathbb{G}} = (\hat{V}, \hat{E})$ of $\mathbb{G}$, every node $v \in \hat{\mathbb{G}}$ and every $t \in \mathbb{Z}_+$ we introduce a quantity $\Phi_{\hat{\mathbb{G}}}(v,t)$ defined inductively as follows. In the context of monomer-dimer model this quantity stands for (approximate) probability that

$v \notin \boldsymbol{M}$ in the subgraph $\hat{\mathbb{G}}$.

(5.17)

$$\Phi_{\hat{\mathbb{G}}}(v,t) = \begin{cases} 1, & t=0 \text{ or } N(v)=\emptyset; \\ (1+\lambda\sum_{1\le i\le k}\Phi_{\mathbb{G}_0}(v_i,t-1))^{-1}, & t>0 \text{ and } N(v)\neq\emptyset \end{cases}$$

where in the last case we let $N(v) = \{v_1, \dots, v_k\} \neq \emptyset$. Here $\hat{\mathbb{G}}_0$ is induced by $\hat{V} \setminus \{v\}$. If $\Phi_{\mathbb{G}_0}(v_i, t-1) = \mathbb{P}_{\mathbb{G}_0}(v_i \notin \boldsymbol{M})$ for all i then $\Phi_{\mathbb{G}}(v,t) = \mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{M})$.

Similarly, introduce $\Psi_{\hat{G}}(v,t)$ with the only exception that $\Psi_{\hat{\mathbb{G}}}(v,0) = 0$ for all $\hat{\mathbb{G}}$ and v. The following proposition follows from Theorem 5.1 and the definitions of Φ and Ψ using a simple induction argument.

Lemma 5.1. *For every $v \in V, t \in \mathbb{Z}_+$*

$$\Psi_{\mathbb{G}}(v,2t) \le \mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{M}) \le \Phi_{\mathbb{G}}(v,2t)$$
$$\Phi_{\mathbb{G}}(v,2t+1) \le \mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{M}) \le \Psi_{\mathbb{G}}(v,2t+1).$$

Lemma 5.2. *For every finite graph $\mathbb{G}$ with degree Δ, $v \in \mathbb{G}$ and t, the values $\Phi_{\mathbb{G}}(v,t), \Psi_{\mathbb{G}}(v,t)$ can be computed in time $\exp(O(t \log \Delta))$, where the constant in $O(\cdot)$ is universal.*

Proof. The result follows immediately from the recursive definitions of Φ and Ψ.

The correlation decay property is formulated in the following proposition which is proved in [BGK$^+$07]. Let $\rho = \left(1 - \frac{2}{\sqrt{1+\lambda\Delta}+1}\right)^{1/2}$.

Theorem 5.2. ([BGK$^+$07]) *For every $\Delta \ge 2, \lambda > 0$, for every graph $\mathbb{G}$ with degree at most Δ and every node v*

$$|\log \Phi_{\mathbb{G}}(v,t) - \log \Psi_{\mathbb{G}}(v,t)| \le \rho^t \log(1+\lambda\Delta). \tag{5.18}$$

As a consequence, every graph $\mathbb{G}$ satisfies exponential SSM for all $\lambda > 0$.

We now state the analogues of Proposition 1 and Corollary 2. The proofs are very similar and omitted. Given any set $W \subset E$, let $N(W) = W \cup \{e : \exists e' \in W, e \sim e'\}$.

Proposition 3. *Given a graph $\mathbb{G} = (V,E)$, for every mutually exclusive sets $A, W_1, W_2 \subset E$ and spin assignment $(s_e), e \in A$ on A the following identity holds with respect to the unique Gibbs measure.*

$$\mathbb{P}_{\mathbb{G}}(1\{e \in \boldsymbol{M}\} = s_e, \forall e \in A | W_1 \cap \boldsymbol{M} = \emptyset, W_2 \subset \boldsymbol{M}) =$$
$$\mathbb{P}_{\hat{\mathbb{G}}}(1\{e \in \boldsymbol{M}\} = s_e, \forall e \in A), \tag{5.19}$$

where $\hat{\mathbb{G}}$ is the subgraph obtained from $\mathbb{G}$ by removing edges $W_1 \cup N(W_2)$.

Let $\mathcal{P}(d,\lambda)$ denote the free energy for the monomer-dimer model on $\mathbb{Z}^d$ as well. As a corollary of Theorem 3.2, and Proposition 3 we obtain

Corollary 4. *For every $\lambda > 0$ and d, $\mathcal{P}(d,\lambda) = -\log \mathbb{P}_{\mathbb{Z}^d_{\prec 0}}(0 \notin \boldsymbol{M})$.*

5.2 Free energy on $\mathbb{Z}^d$. Numerical results We now obtain bounds on the free energy for the monomer-dimer model on $\mathbb{Z}^d$. Again denote by $\Phi(t)$ and $\Psi(t)$ the values of $\Phi_{\mathbb{G}}(v,t), \Psi_{\mathbb{G}}(v,t)$ when applied to any graph $\mathbb{G} = \mathbb{Z}^d_{\prec 0} \cap B_n$ in the monomer-dimer context, for sufficiently large n and $v = 0$. The relations (4.14) and (4.15) hold as well and the proof is very similar.

Our algorithm for computing $\mathcal{P}(d,\lambda)$ and $s\mathcal{P}(\mathbb{Z}^d,\lambda)$ is again based on computing $\Phi(t)$ and $\Psi(t)$ and is again called *Sequential Cavity Algorithm* (SCA). We now report numerical results on computing $\mathcal{P}(d,\lambda)$. We have computed values $\Phi(t), \Psi(t)$ for $d = 2,3,4$ and a range of values λ using the chessboard pattern method [GK08]. Our upper and lower bounds for free energy can be found in a full version of the paper [GK08]. The depth levels $t = 14, 9, 7$ were used for the cases $d = 2,3,4$. As expected, our bounds are high quality for lower λ and then degrade as $\lambda \to \infty$. Each computation run took about 3 minutes on a workstation and we have not made an attempt to obtain very accurate bounds for each value λ. However for the case of interest $\lambda = 1$ we ran our algorithm for larger depths. For the case $d = 2$ a very accurate rigorous estimate $0.6627989727 \pm 0.0000000001$ is due to Friedland and Peled [FP05], shown non-rigorously earlier by Baxter [Bax68]. We have not made an attempt to improve this bound. However for the case $d = 3$ we can significantly improve the best known bound $0.7845 \le \mathcal{P}(3,1) \le 0.7863$ due to Friedland and Gurvits [FG08]. At depth $t = 19$ we obtained estimates $0.78595 \le \mathcal{P}(3,1) \le 0.78599$, which is two orders of magnitude improvement. One should also note that the lower bound 0.7845 in [FG08] was obtained using Friedland-Tveberg inequality which provides a bound for general regular graphs. Thus, while highly accurate for the case $\mathbb{Z}^3$, this bound is not improvable by running some numerical procedure longer or on a faster machine. The previous best known *numerical* estimate $\mathcal{P}(3,1) \ge 0.7653$ [FP05], is based on the transfer matrix method is weaker. We have also obtained bounds for $d = 4$ for which no prior computations are available. We obtained $0.8797 \le \mathcal{P}(4,1) \le 0.8812$. The computations were done at depth $t = 14$.

The following proposition gives a bound on the numerical complexity of SCA.

Proposition 4. *For every $d \geq 2, \lambda > 0, \epsilon > 0$ SCA produces an ϵ-additive estimate of $\mathcal{P}(d, \lambda)$ in time $\log(1 + 2\lambda d)(1/\epsilon)^{O\left((\lambda d)^{\frac{1}{2}} \log d\right)}$, where the constant in $O(\cdot)$ is universal.*

Again for constant λ, d we obtain performance $(1/\epsilon)^{O(1)}$, which is a qualitative improvement over the numerical effort $\exp(O((1/\epsilon)^{d-1}))$ of the transfer matrix method.

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Appendix

Proof. [Proof of Theorem 3.1] Let $v_1 \succ v_2 \succ \ldots \succ v_{|B_n|}$ be the labeling of nodes in B_n according to the lexicographic order. Note that the number of edges in B_n is $d|B_n| + o(|B_n|)$. We have

$$\begin{aligned}&\mathbb{P}_{B_n}(\sigma_v = s^*, \forall v \in B_n)\\&= Z_{B_n}^{-1} \exp\big(- d|B_n|H(s^*, s^*) - o(|B_n|)H(s^*, s^*)\\&- |B_n|h(s^*)\big),\end{aligned}$$

from which we infer that

$$\begin{aligned} Z_{B_n}^{-1} &= \exp\big(d|B_n|H(s^*,s^*) + o(|B_n|)H(s^*,s^*)+ \\ &\quad + |B_n|h(s^*)\big)\mathbb{P}_{B_n}(\sigma_v = s^*, \forall v \in B_n). \end{aligned} \tag{5.20}$$

On the other hand, by telescoping property

$$\begin{aligned} &\mathbb{P}_{B_n}(\sigma_v = s^*, \forall v \in B_n) = \\ &= \prod_{v\in B_n} \mathbb{P}_{B_n}(\sigma_v = s^* | \sigma_u = s^*,\ \forall\ u \succ v, u \in B_n) \\ &= \prod_{v\in B_n} \mathbb{P}_{B_n}(\sigma_v = s^* | \mathcal{E}_{v,s^*}). \end{aligned} \tag{5.21}$$

Fix $\epsilon > 0$ and find $r = r(\epsilon)$ such that according to Definition 1, $R(r) < \epsilon$ for $X = \{0\}, Y = \{u \succ 0\}$. Let $B^o = \{v \in B_n : B_r(v) \subset B_n\}$. Observe that $|B^o|/|B_n| \to 1$ as $n \to \infty$. By the choice of r we have for every $v \in B^o$

$$\Big|\mathbb{P}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) - \mathbb{P}_{B_n}(\sigma_v = s^*|\mathcal{E}_{v,s^*})\Big| \le \epsilon. \tag{5.22}$$

By translation invariance we have

$$\mathbb{P}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) = \mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*})$$

For every $v \in B_n \setminus B^o$ we have the generic lower bound (2.4)

$$\mathbb{P}_{B_n}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) \ge c^* \tag{5.23}$$

which is strictly positive by Assumption 1. A similar inequality with the same constant holds for $\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*})$. Combining this with (5.20) and (5.21) we obtain

$$\begin{aligned} &Z_{B_n}^{-1} = \\ &\exp\big(d|B_n|H(s^*,s^*) + o(|B_n|)H(s^*,s^*) + |B_n|h(s^*)\big)\times \\ &\times \prod_{v\in B^o} \mathbb{P}_{B_n}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) \prod_{v\in B_n\setminus B^o} \mathbb{P}_{B_n}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) \\ &\ge \exp\big(d|B_n|H(s^*,s^*) + o(|B_n|)H(s^*,s^*) + |B_n|h(s^*)\big) \\ &\big(\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*}) - \epsilon\big)^{|B^o|}(c^*)^{|B_n\setminus B^o|} \end{aligned}$$

Since $|B^o|/|B_n| \to 1$ as $n \to \infty$ and $c^* > 0$, then we obtain

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log Z_{B_n}^{-1}}{|B_n|} &\ge dH(s^*,s^*) + h(s^*) + \\ &\quad + \log(\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*}) - \epsilon) \end{aligned}$$

Recalling $\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*}) \ge c^* > 0$, since ϵ was arbitrary, we conclude

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log Z_{B_n}^{-1}}{|B_n|} &\ge H(s^*,s^*)d + h(s^*) \\ &\quad + \log\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*}). \end{aligned}$$

Similarly we show

$$\limsup_{n\to\infty} \frac{\log Z_{B_n}^{-1}}{|B_n|} \le H(s^*,s^*)d + h(s^*) + \log\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*}),$$

where for the case $v \in B_n \backslash B_n^o$ we use a trivial inequality $\mathbb{P}_{B_n}(\sigma_v = s^*|\mathcal{E}_{v,s^*}) \le 1$ in place of (5.23). We obtain

$$\lim_{n\to\infty} \frac{\log Z_{B_n}^{-1}}{|B_n|} = H(s^*,s^*)d + h(s^*) + \log\mathbb{P}(\sigma_0 = s^*|\mathcal{E}_{s^*}), \tag{5.24}$$

This concludes the proof of (3.5).

Proof. [Proof of Theorem 3.2] Let $E(B_n)$ denote the edge set of B_n. Note that $E(B_n) = d|B_n| + o(|B_n|)$ and the the number of edges with $2d-2$ incident edges is also $d|B_n| + o(|B_n|)$. The number of pairs of incident edges is then $d(2d-1)|B_n| + o(|B_n|)$. We have

$$\begin{aligned} &\mathbb{P}_{B_n}(\sigma_e = s^*, \forall e \in E(B_n)) \\ &\quad = Z_{B_n}^{-1}\exp\big(-d(2d-1)|B_n|H(s^*,s^*) \\ &\quad - o(|B_n|)H(s^*,s^*) - d|B_n|h(s^*)\big). \end{aligned}$$

On the other hand, by telescoping property

$$\begin{aligned} &\mathbb{P}_{B_n}(\sigma_e = s^*, \forall e \in E(B_n)) \\ &\quad = \prod_{v\in B_n} \mathbb{P}_{B_n}(\sigma_{(v,u)} = s^*, \forall u \in N(v)| \\ &\quad \sigma_{(u,w)} = s^*,\ \forall\ u \succ v, w \in N(u),\ u,w \in B_n) \\ &\quad = \prod_{v\in B_n} \mathbb{P}_{B_n}(\sigma_{(v,u)} = s^*, \forall u \in N(v)|\mathcal{E}_{v,s^*,\text{edges}}). \end{aligned}$$

The remainder of the proof of (3.7) is similar to the one of Theorem 3.1 and is omitted.

Proof. [Proof of Theorem 4.1] We have

$$\begin{aligned} Z_{\mathbb{G}} &= \sum_{I: v\notin I} \lambda^{|I|} + \sum_{I: v\in I} \lambda^{|I|} \\ &= \sum_{I: I\subset V\backslash\{v\}} \lambda^{|I|} + \lambda \sum_{I: I\subset V\backslash\{v,v_1,\dots,v_k\}} \lambda^{|I|}. \end{aligned} \tag{5.25}$$

where everywhere the sums are over independent sets I. Note that $\sum_{I:I\subset V\backslash\{v\}} \lambda^{|I|} = Z_{\mathbb{G}_0}$ and $\sum_{I:I\subset V\backslash\{v,v_1,\dots,v_k\}} \lambda^{|I|} = Z_{\mathbb{G}_k}$. Dividing both sides of the identity (5.25) by $Z_{\mathbb{G}_0}$ we obtain

$$\frac{Z_{\mathbb{G}}}{Z_{\mathbb{G}_0}} = 1 + \lambda\frac{Z_{\mathbb{G}_k}}{Z_{\mathbb{G}_0}}$$

It is immediate that $Z_{\mathbb{G}_0}/Z_{\mathbb{G}} = \mathbb{P}_{\mathbb{G}}(v \notin \boldsymbol{I})$. In order to interpret $Z_{\mathbb{G}_k}/Z_{\mathbb{G}_0}$ similarly we rewrite it as

$$\frac{Z_{\mathbb{G}_k}}{Z_{\mathbb{G}_0}} = \prod_{i=1}^{k} \frac{Z_{\mathbb{G}_i}}{Z_{\mathbb{G}_{i-1}}}$$

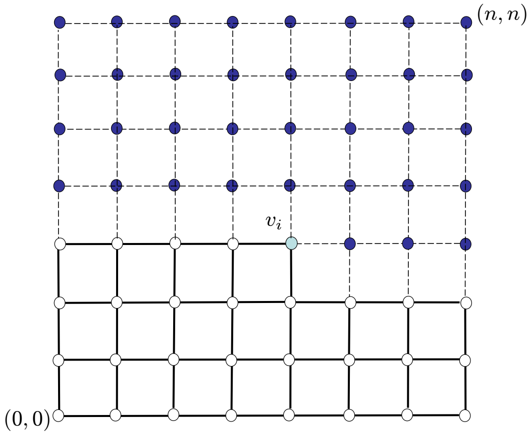

Figure 1: $\mathbb{G}_i$ is obtained by removing all dark nodes

and note that $Z_{\mathbb{G}_i}/Z_{\mathbb{G}_{i-1}} = \mathbb{P}_{\mathbb{G}_{i-1}}(v_i \notin \boldsymbol{I})$. Combining these observations we obtain (4.10).

Proof. [Proof of Proposition 2] Applying Theorem 4.2, an additive error ϵ is achieved provided that $C\rho^t < \epsilon$ or $t \geq \log(C/\epsilon)\log^{-1}(1/\rho) = O(\log(1/\epsilon))$. Hence the result for free energy follows from Lemma 4.2.

Proof. [Proof of Proposition 4] We have $\Delta = 2d$. Applying Theorem 5.2, an additive error ϵ is achieved provided that $\rho^t \log(1 + 2\lambda d) < \epsilon$ or

$$t \geq (-\log\rho)^{-1}\Big(\log\frac{1}{\epsilon} + \log\log(1 + 2\lambda d)\Big).$$

Applying the definition of ρ, we have $\log\rho = O(\frac{1}{\sqrt{1+2\lambda d}+1}) = O(\frac{1}{\sqrt{\lambda d}})$. The result for $\mathcal{P}(d, \lambda)$ then follows from this estimate and Lemma 5.2.

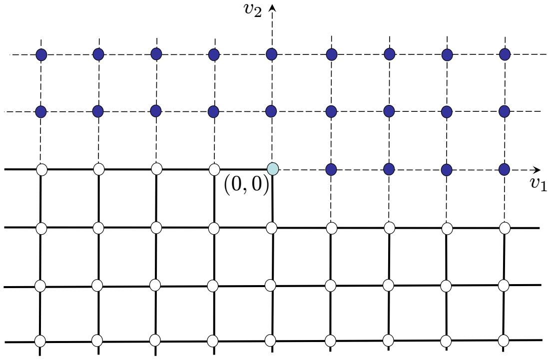

Figure 2: Event $\mathcal{E}_{s^*}$ on $\mathbb{Z}^2$. Every dark node is assigned spin s^*