Hardness of embedding simplicial complexes in $\mathbb{R}^d$

Jiří Matoušek[a,b,c] Martin Tancer[a,b,c] Uli Wagner[c,d]

Abstract

Let $\text{EMBED}_{k\to d}$ be the following algorithmic problem: Given a finite simplicial complex K of dimension at most k, does there exist a (piecewise linear) embedding of K into $\mathbb{R}^d$? Known results easily imply polynomiality of $\text{EMBED}_{k\to 2}$ ($k = 1, 2$; the case $k = 1$, $d = 2$ is graph planarity) and of $\text{EMBED}_{k\to 2k}$ for all $k \geq 3$ (even if k is not considered fixed).

We show that the celebrated result of Novikov on the algorithmic unsolvability of recognizing the 5-sphere implies that $\text{EMBED}_{d\to d}$ and $\text{EMBED}_{(d-1)\to d}$ are undecidable for each $d \geq 5$. Our main result is NP-hardness of $\text{EMBED}_{2\to 4}$ and, more generally, of $\text{EMBED}_{k\to d}$ for all k, d with $d \geq 4$ and $d \geq k \geq (2d-2)/3$.

1 Introduction

Does a given (finite) simplicial complex[1] K of dimension at most k admit an embedding into $\mathbb{R}^d$? We consider the computational complexity of this question, regarding k and d as fixed integers. To our surprise, this question has apparently not been explicitly addressed before (with the exception of $k = 1$, $d = 2$ which is graph planarity), as far we could find.

For algorithmic embeddability problems, we consider *piecewise linear* (PL) embeddings. Let us remark that there are at least two other natural notions of embeddings of simplicial complexes in $\mathbb{R}^d$: *linear embeddings* (also called *geometric realizations*), which are more restricted than PL embeddings, and *topological embeddings*, which give us more freedom than PL embeddings. We will recall the definitions in Section 2; here we quickly illustrate the differences with a familiar example: embeddings of 1-dimensional simplicial complexes, a.k.a. simple graphs, into $\mathbb{R}^2$. For a topological embedding, the image of each edge can be an arbitrary (curved) arc, for a PL embedding it has to be a polygonal arc (made of finitely many straight segments), and for a linear embedding, it must be a single straight segment. For this particular case ($k = 1$, $d = 2$), all three notions happen to give the same class of embeddable complexes, namely, all planar graphs (by Fáry's theorem). For higher dimensions there are significant differences, though, which we also discuss in Section 2.

Here we are interested mainly in embeddability in the topological sense, but since it seems problematic to deal with arbitrary topological embeddings effectively, we stick to PL embeddings, which can easily be represented in a computer.

[a]Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

[b]Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

[c]Institute of Theoretical Computer Science, ETH Zurich, 8092 Zurich, Switzerland

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[1]We assume that the reader is somewhat familiar with basic notions of combinatorial topology (introductory chapters of books like [25, 17, 22] should provide a sufficient background). Here, for convenience, we briefly recall basic definitions and facts concerning simplicial complexes. Later on, in some of the proofs, we will need other, slightly more advanced topological notions and results, which would take too much space to define properly. We hope that the main ideas can be followed even when such things are skipped.

We will formally regard a simplicial complex as a geometric object. That is, a simplicial complex is a collection K (a finite collection in our case) of closed simplices in some Euclidean space $\mathbb{R}^n$ such that if $\sigma \in K$ and σ' is a face of σ, then $\sigma' \in K$ as well, and if $\sigma, \tau \in K$, then $\sigma \cap \tau \in K$, too. The *vertex set* $V(K)$ is the set of all 0-dimensional simplices of K. The *polyhedron* of K, denoted by $|K|$, is the union of all simplices in K. Often we do not strictly distinguish between a simplicial complex and its polyhedron; for example, by an embedding of K in $\mathbb{R}^d$ we really mean an embedding of $|K|$ into $\mathbb{R}^d$.

Two simplicial complexes K and L are *isomorphic* if there is a face-preserving bijection $\varphi\colon V(K) \to V(L)$ of the vertex sets (that is, $F \subseteq V(K)$ is the vertex set of a simplex of K iff $\varphi(F)$ is the vertex set of a simplex of L). Isomorphic complexes have homeomorphic polyhedra. Up to isomorphism, a simplicial complex K can be described purely combinatorially, by specifying which subsets of $V(K)$ form vertex sets of simplices of K. We assume that the input to the embeddability problem is given in this form, i.e., as an abstract finite set system.

The *dimension* of a simplicial complex K is the maximum of the dimensions of its simplices. The *k-skeleton* of K consists of all simplices of K of dimension at most K. A *subcomplex* of K is a subset $L \subseteq K$ that is a simplicial complex. A simplicial complex K' is a *subdivision* of K if $|K'| = |K|$ and each simplex of K' is contained in some simplex of K.

We thus introduce the decision problem $\text{EMBED}_{k\to d}$, whose input is a simplicial complex K of dimension at most k, and where the output should be YES or NO depending on whether K admits a PL embedding into $\mathbb{R}^d$.

We assume $k \le d$, since a k-simplex cannot be embedded in $\mathbb{R}^{k-1}$. For $d \ge 2k+1$ the problem becomes trivial, since it is well known that *every* finite k-dimensional simplicial complex embeds in $\mathbb{R}^{2k+1}$, even linearly (this result goes back to Menger). In all other cases, i.e., $k \le d \le 2k$, there are both YES and NO instances; for the NO instances one can use, e.g., examples of k-dimensional complexes not embeddable in $\mathbb{R}^{2k}$ due to Van Kampen [45] and Flores [11].

Let us also note that the complexity of this problem is monotone in k by definition, since an algorithm for $\text{EMBED}_{k\to d}$ also solves $\text{EMBED}_{k'\to d}$ for all $k' \le k$.

Tractable cases. It is well known that $\text{EMBED}_{1\to 2}$ (graph planarity) is linear-time solvable. Based on planarity algorithms and on a characterization of complexes embeddable in $\mathbb{R}^2$ due to Halin and Jung [15], it is not hard to come up with a polynomial-time decision algorithm for $\text{EMBED}_{2\to 2}$. Since we do not know of a reference, we outline such an algorithm in Appendix A.

There are many problems in computational topology that are easy for low dimensions (say up to dimension 2 or 3) and become intractable from some dimension on (say 4 or 5); we will mention some of them later. For the embeddability problem, the situation is subtler, since there are tractable cases in arbitrarily high dimensions, namely, $\text{EMBED}_{k\to 2k}$ for every $k \ge 3$.

The algorithm is based on ideas of Van Kampen [45], which were made precise by Shapiro [41] and independently by Wu [47]. We refer to the full version of this paper [23] for an elementary exposition of such an algorithm.

Hardness. According to a celebrated result of Novikov ([46]; also see, e.g., [26] for an exposition), the following problem is algorithmically unsolvable: Given a d-dimensional simplicial complex, $d \ge 5$, decide whether it is homeomorphic to S^d, the d-dimensional sphere. By a simple reduction we obtain the following result:

THEOREM 1.1. $\text{EMBED}_{(d-1)\to d}$ *(and hence also* $\text{EMBED}_{d\to d}$*) is algorithmically undecidable for every* $d \ge 5$.

Our main result is hardness for cases where $d \ge 4$ and k is larger than roughly $\frac{2}{3}d$.

THEOREM 1.2. $\text{EMBED}_{k\to d}$ *is* NP*-hard for every pair* (k, d) *with* $d \ge 4$ *and* $d \ge k \ge \frac{2d-2}{3}$.

In this extended abstract we prove only a special case of this theorem, NP-hardness of $\text{EMBED}_{2\to 4}$, whose proof contains most of the ideas. For the proof of the remaining cases, as well as for the other omitted proofs, we refer to the full version of this paper [23] (available on-line).

Let us briefly mention where the dimension restriction $k \ge (2d-2)/3$ comes from. There is a certain necessary condition for embeddability of a simplicial complex into $\mathbb{R}^d$, called the *deleted product obstruction.* A celebrated theorem of Haefliger and Weber, which is a far-reaching generalization of the ideas of Van Kampen mentioned above, asserts that this condition is also *sufficient* provided that $k \le \frac{2}{3}d - 1$ (these k are said to lie in the *metastable range*). The condition on k in Theorem 1.2 is exactly that k must be outside of the metastable range

There are examples showing that the restriction to the metastable range in the Haefliger–Weber theorem is indeed necessary, in the sense that whenever $d \ge 3$ and $d \ge k > (2d-3)/3$, there are k-dimensional complexes that cannot be embedded into $\mathbb{R}^d$ but the deleted product obstruction fails to detect this. We use constructions of this kind, namely, examples due to Segal and Spież [40], Freedman, Krushkal, and Teichner [12], and Segal, Skopenkov, and Spież [39], as the main ingredient in our proof of Theorem 1.2.

Discussion. The current complexity status of $\text{EMBED}_{k\to d}$ is summarized in Table 1. In our opinion, the most interesting currently open cases are $(k, d) = (2, 3)$ and $(3, 3)$.

A variation on the proof of our undecidability result (Theorem 1.1) shows that both $\text{EMBED}_{2\to 3}$ and $\text{EMBED}_{3\to 3}$ are at least as hard as the problem of recognizing the 3-sphere (that is, given a simplicial complex, decide whether it is homeomorphic to S^3). The latter problem is in NP [19, 37], but no hardness result seems to be known.

For all we know, $\text{EMBED}_{k\to d}$ might turn out to be undecidable in all cases except for those listed above as tractable, i.e., $d \le 2$ or $d = 2k \ge 6$.

Related work. Among the most important computational problems in topology are the *homeomorphism problem* for manifolds, and the *equivalence problem* for knots. The first one asks if two given manifolds M_1 and M_2 (given as simplicial complexes, say) are homeomorphic. The second one asks if two given knots, i.e., PL embeddings $f, g\colon S^1 \to \mathbb{R}^3$, are equivalent, i.e., if there is a PL homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f = h \circ g$. An important special case of the latter in the *knot triviality* problem: Is a given knot equivalent to the trivial knot (i.e., the standard geometric circle placed in $\mathbb{R}^3$)?

There is a vast amount of literature on compu-

$k =$						$d =$							
	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P	+	+	+	+	+	+	+	+	+	+	+	+
2	P	**?**	NPh	+	+	+	+	+	+	+	+	+	+
3		**?**	NPh	NPh	P	+	+	+	+	+	+	+	+
4			NPh	UND	NPh	NPh	P	+	+	+	+	+	+
5				UND	UND	NPh	NPh	**?**	P	+	+	+	+
6					UND	UND	NPh	NPh	NPh	**?**	P	+	+
7						UND	UND	NPh	NPh	NPh	**?**	**?**	P

Table 1: The complexity of $\text{EMBED}_{k\to d}$ (P = polynomial-time solvable, UND = algorithmically undecidable, NPh = NP-hard, + = always embeddable, **?** = no result known).

tational problems for 3-manifolds and knots. For instance, it is algorithmically decidable whether a given 3-manifold is homeomorphic to S^3 [36, 44], or whether a given polygonal knot in $\mathbb{R}^3$ is trivial [14]. Indeed, both problems have recently shown to lie in NP [19, 37], [16]. The knot equivalence problem is also algorithmically decidable [14, 18, 24], but nothing seems to be known about its complexity status. We refer the reader to the above-mentioned sources and to [1] for further results, background and references.

In higher dimensions, all of these problems are undecidable. Markov [21] showed that the homeomorphism problem for d-manifolds is algorithmically undecidable for every $d \geq 4$. For $d \geq 5$, this was strengthened by Novikov to the undecidability of recognizing S^d (or any other fixed d-manifold), as was mentioned above. Nabutovsky and Weinberger [27] showed that for $d \geq 5$, it is algorithmically undecidable whether a given PL embedding $f\colon S^{d-2} \to \mathbb{R}^d$ is equivalent to the standard embedding (placing S^{d-2} as the "equator" of the unit sphere S^{d-1}, say). For further undecidability results, see, e.g., [28] and the survey by Soare [43].

Another direction of algorithmic research in topology is the computability of *homotopy groups*. While the fundamental group $\pi_1(X)$ is well-known to be uncomputable [21], all higher homotopy groups of a given finite simply connected simplicial (or CW) complex are computable (Brown [7]). More recently, there appeared several works (Schön [38], Smith [42], and Rubio, Sergeraert, Dousson, and Romero, e.g. [34]) aiming at making methods of algebraic topology, such as spectral sequences, "constructive"; the last of these has also resulted in an impressive software called KENZO.

A different line of research relevant to the embedding problem concerns linkless embeddings of graphs. Most notably, results of Robertson, Seymour, and Thomas [33] on linkless embeddings provide an interesting *sufficient* condition for embeddability of a 2-dimensional complex in $\mathbb{R}^3$: If the 1-skeleton of a 2-dimensional complex K is linklessly embeddable in $\mathbb{R}^3$ (which can be tested in polynomial time), then K embeds in $\mathbb{R}^3$ (see Section 5).

2 Preliminaries on PL topology

Here we review definitions and facts related to piecewise linear (PL) embeddings. For more information on PL topology, and for facts mentioned below without proofs, we refer to Rourke and Sanderson [35], Bryant [9], or Buoncristiano [10].

Linear and PL mappings of simplicial complexes. A *linear* mapping of a simplicial complex K into $\mathbb{R}^d$ is a mapping $f\colon |K| \to \mathbb{R}^d$ that is linear on each simplex. More explicitly, each point $x \in |K|$ is a convex combination $t_0v_0+t_1v_1+\cdots+t_sv_s$, where $\{v_0, v_1, \ldots, v_s\}$ is the vertex set of some simplex $\sigma \in K$ and $t_0, \ldots, t_s$ are nonnegative reals adding up to 1. Then we have $f(x) = t_0f(v_0) + t_1f(v_1) + \cdots + t_sf(v_s)$.

A *PL mapping* of K into $\mathbb{R}^d$ is a linear mapping of some subdivision K' of K into $\mathbb{R}^d$.

Embeddings. A general *topological embedding* of K into $\mathbb{R}^d$ is any continuous mapping $f\colon |K| \to \mathbb{R}^d$ that is a homeomorphism of $|K|$ with $f(|K|)$. Since we only consider finite simplicial complexes, this is equivalent to requiring that f be injective.

By contrast, for a *PL embedding* we require additionally that f be PL, and for a *linear embedding* we are even more restrictive and insist that f be (simplexwise) linear.

PL embeddings versus linear embeddings. In contrast to planarity of graphs, linear and PL embeddability do not always coincide in higher dimensions (see [5, 4] for examples with $k = 2$, $d = 3$ and [6] for higher-dimensional examples). On the algorithmic side, the problem of *linear* embeddability of a given finite simplicial complex into $\mathbb{R}^d$ is at least algorithmically decidable, and for k and d fixed, it even belongs to

PSPACE (since the problem can easily be formulated as the solvability over the reals of a system of polynomial inequalities with integer coefficients, which lies in PSPACE [31]).

PL structures. Two simplicial complexes K and L are *PL homeomorphic* if there are a subdivision K' of K and a subdivision L' of L such that K' and L' are isomorphic.

Let Δ^d denote the simplicial complex consisting of all faces of a d-dimensional simplex (including the simplex itself), and let $\partial\Delta^d$ consist of all faces of Δ^d of dimension at most $d-1$. Thus, $|\Delta^d|$ is topologically B^d, the d-dimensional ball, and $|\partial\Delta^d|$ is topologically S^{d-1}.

A d-dimensional *PL ball* is a simplicial complex PL homeomorphic to Δ^d, and a d-dimensional *PL sphere* is a simplicial complex PL homeomorphic to $\partial\Delta^{d+1}$. Let us mention that a simplicial complex K is PL embeddable in $\mathbb{R}^d$ iff it is PL homeomorphic to a subcomplex of a d-dimensional PL ball (and similarly, K is PL embeddable in $|\partial\Delta^{d+1}|$ iff it is PL homeomorphic to a subcomplex of a d-dimensional PL sphere).

One of the great surprises in higher-dimensional topology was the discovery that simplicial complexes with homeomorphic polyhedra need not be PL homeomorphic (the failure of the "Hauptvermutung"). In particular, there exist *non-PL spheres*, i.e., simplicial complexes homeomorphic to a sphere that fail to be PL spheres. More precisely, every simplicial complex homeomorphic to S^1, S^2, S^3, and S^4 is a PL sphere,[2] but there are examples of non-PL spheres of dimensions 5 and higher.

A weak PL Schoenflies theorem. The well-known Jordan curve theorem states that if S^1 is embedded (topologically) in $\mathbb{R}^2$, the complement of the image has exactly two components. Equivalently, but slightly more conveniently, if S^1 is embedded in S^2, the complement has two components. The *Schoenflies theorem* asserts that in the latter setting, the closure of each of the components is homeomorphic to the disk B^2.

While the Jordan curve theorem generalizes to an arbitrary dimension (if S^{d-1} is topologically embedded in S^d, the complement has exactly two components), the Schoenflies theorem does not. There are embeddings $h\colon S^2 \to S^3$ such that the closure of one of the components of $S^3 \setminus h(S^2)$ is not a ball; a well known example is the *Alexander horned sphere*.

Thus, one needs to put some additional conditions on the embedding to make a "higher-dimensional Schoenflies theorem" work. We will need the following version, in which we assume a $(d-1)$-dimensional PL sphere sitting in a d-dimensional PL sphere.

Theorem 2.1. (Weak PL Schoenflies Theorem) *Let f be a PL embedding of $\partial\Delta^d$ into $\partial\Delta^{d+1}$. Then the complement $|\partial\Delta^{d+1}| \setminus f(|\partial\Delta^d|)$ has two components, whose closures are topological d-balls.*

For a proof of this theorem, see, e.g., [29] or [13]. A simple, inductive proof is to appear in the upcoming revised edition of the book [10] by Buoncristiano and Rourke.

PL embeddings versus topological embeddings. Let us say that *TOP and PL embeddability coincide for* (k,d) if every finite simplicial complex of dimension at most k that can be topologically embedded in $\mathbb{R}^d$ can also be PL embedded in $\mathbb{R}^d$.

On the positive side, it is known that TOP and PL embeddability coincide for (k,d) whenever $d-k \geq 3$ [8], and also for $(k,d) = (2,3)$. The latter follows from Theorem 5 of Bing [2], which shows that the image of a topological embedding of a 2-dimensional complex in $\mathbb{R}^3$ is homeomorphic to a polyhedron PL embedded in $\mathbb{R}^3$, and from a result of Papakyriakopoulos [30] ("Hauptvermutung" for 2-dimensional polyhedra) that any two 2-dimensional polyhedra that are homeomorphic are also PL homeomorphic.

However, TOP and PL embeddability do *not* always coincide: There is an example of a 4-dimensional complex (namely, the suspension of the Poincaré homology 3-sphere) that embeds topologically, but not PL, into $\mathbb{R}^5$. For this example we are indebted to Colin Rourke (private communication); unfortunately, his proof, although short, uses too advanced concepts to be reproduced here. It would be interesting to clarify in general for what (k,d) TOP and PL embeddability coincide.

Genericity. First let us consider a linear mapping f of a simplicial complex K into $\mathbb{R}^d$. We say that f is *generic* if $f(V(K))$ is a set of distinct points in $\mathbb{R}^d$ in general position. A PL mapping of K into $\mathbb{R}^d$ is generic if the corresponding linear mapping of the subdivision K' of K is generic.

A PL embedding can always be made generic (by an arbitrarily small perturbation).

Linking and linking numbers. Let k,ℓ be integers, and let $f\colon S^k \to \mathbb{R}^{k+\ell+1}$ and $g\colon S^\ell \to \mathbb{R}^{k+\ell+1}$ be PL embeddings with $f(S^k) \cap g(S^\ell) = \emptyset$ (so here we regard S^k and S^ℓ as PL spheres). We will need two notions capturing how the images of f and g are "linked" (the basic example is $k = \ell = 1$, where we deal with two disjoint simple closed curves in $\mathbb{R}^3$). For our purposes, we may assume that f and g are mutually generic (i.e. $f\dot\cup g$, regarded as a PL embedding of the disjoint union

[2] The proof for S^4 relies on the recent solution of the Poincaré conjecture by Perelman.

$S^k \dot{\cup} S^\ell$ into $\mathbb{R}^{k+\ell+1}$, is generic).

Consider an arbitrary extension of f to a PL mapping $\bar{f}\colon B^{k+1} \to \mathbb{R}^{k+\ell+1}$ so that $\bar{f}$ and g are still mutually generic. We say that the images $f(S^k)$ and $g(S^\ell)$ have an *odd* or *even linking number* depending on whether $\bar{f}(B^{k+1})$ and $g(S^\ell)$ intersect in an odd or even number of points (the number of intersections is always finite, by genericity, and it turns out that its parity does not depend on the choice of $\bar{f}$). If $\bar{f}$ can be chosen in such a way that $\bar{f}(B^{k+1}) \cap g(S^\ell) = \emptyset$, we say that $f(S^k)$ and $g(S^\ell)$ are *unlinked*.

3 Undecidability: Proof of Theorem 1.1

We begin with a statement of Novikov's result mentioned in the introduction (undecidability of S^d recognition for $d \geq 5$) in a form convenient for our purposes.

Theorem 3.1. (Novikov) *Fix $d \geq 5$. There is an effectively constructible sequence of simplicial complexes Σ_i, $i \in \mathbb{N}$, with the following properties:*

(1) *Each $|\Sigma_i|$ is a homology d-sphere.*

(2) *For each i, either Σ_i is a* PL *d-sphere, or the fundamental group of Σ_i is nontrivial (in particular, Σ_i is not homeomorphic to the d-sphere).*

(3) *There is no algorithm that decides for every given Σ_i which of the two cases holds.*

We refer to the appendix in [26] for a detailed proof. For the proof of Theorem 1.1, the undecidability of $\mathrm{EMBED}_{(d-1)\to d}$ for $d \geq 5$ is an immediate consequence of Theorem 3.1 and the following lemma, whose proof is omitted in this extended abstract.

Lemma 3.1. *Let $d \geq 2$. Suppose that Σ is a homology d-sphere, and let K be its $(d-1)$-skeleton. If Σ is a* PL *sphere, then K* PL *embeds into $\mathbb{R}^d$. Conversely, if K* PL *embeds into $\mathbb{R}^d$, then Σ is (not necessarily PL) homeomorphic to S^d.*

4 Hardness of embedding 2-dimensional complexes in $\mathbb{R}^4$

We will reduce the problem 3-SAT to $\mathrm{EMBED}_{2\to4}$. Given a 3-CNF formula φ, we construct a 2-dimensional simplicial complex K that is PL embeddable in $\mathbb{R}^4$ exactly if φ is satisfiable.

First we define two particular 2-dimensional simplicial complexes G (the *clause gadget*) and X (the *conflict gadget*). They are closely related to the main example of Freedman et al. [12]: X is taken over exactly, and G is a variation on a construction in [12] (which, in turn, is similar in some respects to an example of Segal and Spież [40], with some of the ideas going back to Van Kampen [45]).

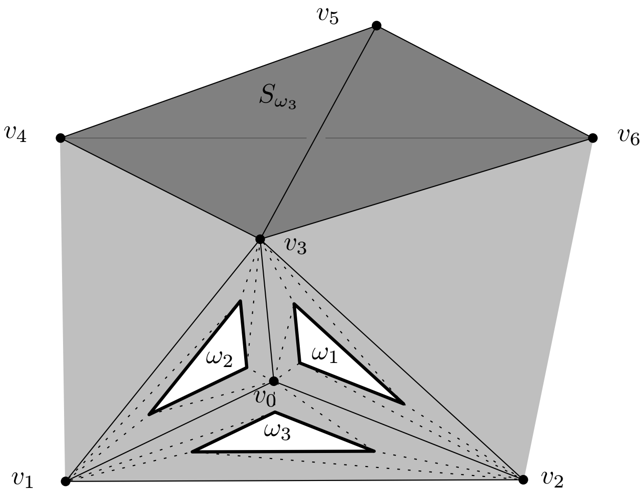


Figure 1: The clause gadget G T, its openings, and one of the complementary spheres.

4.1 The clause gadget To construct G, we begin with a 6-dimensional simplex on the vertex set $\{v_0, v_1, \ldots, v_6\}$, and we let F be the 2-skeleton of this simplex (F for "full" skeleton). Then we make a hole in the interior of the three triangles (2-simplices) $v_0v_1v_2$, $v_0v_1v_3$, and $v_0v_2v_3$. That is, we subdivide each of the triangles and from each of these subdivisions we remove a small triangle in the middle, as is indicated in Fig. 1. This yields the simplicial complex G.

Let $\omega_1, \omega_2, \omega_3$ be the three small triangles we have removed (where ω_1 comes from the triangle $v_0v_2v_3$ etc.). We call them the *openings* of G and we let $O_G := \{\omega_1, \omega_2, \omega_3\}$ be the set of openings. Thus, $G \cup O_G$ is a subdivision of the full 2-skeleton F.

If we remove from F the vertices v_0, v_1, v_2 and all simplices containing them, we obtain the boundary of the 3-simplex $\{v_3, v_4, v_5, v_6\}$. Topologically it is an S^2, we call it the *complementary sphere* of the opening ω_3, and we denote it by S_{ω_3}. The complementary spheres of the openings ω_1 and ω_2 are defined analogously. The following lemma is a variation on results in Van Kampen [45] and we omit its proof in this extended abstract:

Lemma 4.1.

(i) *For every generic PL embedding f of G into $\mathbb{R}^4$ there is at least one opening $\omega \in O_G$ such that the images of the boundary $\partial\omega$ and of the complementary sphere S_ω have odd linking number.*

(ii) *For every opening $\omega \in O_G$ there exists an embedding of G into $\mathbb{R}^4$ in which only $\partial\omega$ is linked with its complementary sphere. More precisely, there exists a generic linear mapping of the full 2-skeleton F*

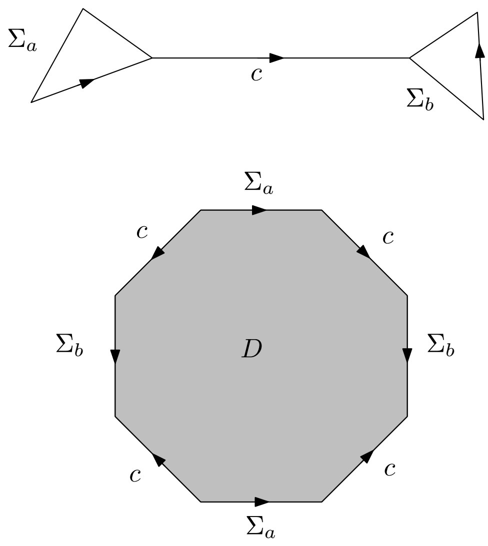


Figure 2: Attaching a disk to the polygonal line E.

into $\mathbb{R}^4$ whose restriction to $|G \cup O_G \setminus \{\omega\}|$ is an embedding.

4.2 The conflict gadget To construct X, we start with the 1-dimensional simplicial complex E shown in Fig. 2 up, consisting of two triangular loops Σ_a and Σ_b and an edge c connecting them. We also fix an orientation of Σ_a, Σ_b, and c (marked by arrows). Then we take a disk D and we attach its boundary to E as indicated in Fig. 2 down; the disk is triangulated sufficiently finely so that the result of the attachment is still a simplicial complex. This is the complex X.

We observe that topologically, X is a "squeezed torus" (the reader may want to recall the usual construction of a torus by gluing the opposite sides of a square; this well-known construction would be obtained from the attachment as above if the edge c were contracted to a point). Fig. 3 shows such a squeezed torus embedded in $\mathbb{R}^3$ (with the loops Σ_a and Σ_b drawn circular rather than triangular).

LEMMA 4.2.

(i) [12, Lemma 7] *Let S_a and S_b be PL 2-spheres. Then there is no PL embedding f of $S_a \dot\cup S_b \dot\cup X$ (disjoint union) into $\mathbb{R}^4$ such that the 1-sphere $f(\Sigma_i)$ and the 2-sphere $f(S_i)$ have odd linking number, $i \in \{a, b\}$, but $f(\Sigma_i)$ and $f(S_j)$ are unlinked, $i, j \in \{a, b\}$, $i \neq j$.*

(ii) *Let f be a generic linear embedding of E in $\mathbb{R}^3$ (not $\mathbb{R}^4$ this time) and let $\delta > 0$. Then there is a PL embedding $\overline{f}$ of X in $\mathbb{R}^3$ extending f whose image is contained in the set $N = N(f, \delta) := N(T_a, \delta) \cup N(f(\Sigma_b), \delta) \cup N(f(c), \delta)$, where T_a is the triangle bounded by the loop $f(\Sigma_a)$ and $N(A, \delta)$ denotes the δ-neighborhood of a set A (in $\mathbb{R}^3$ in our case).[3] (Symmetrically, and this is the main point of the construction, we can also embed X into $N(f(\Sigma_a), \delta) \cup N(T_b, \delta) \cup N(f(c), \delta)$, thus leaving a hole on the other side.)*

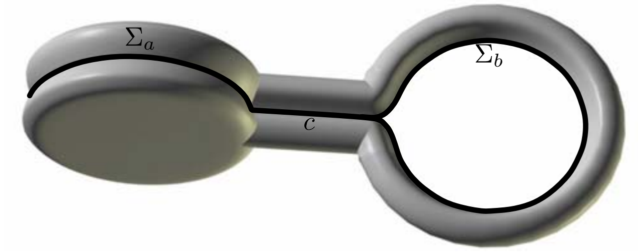


Figure 3: A 3-dimensional embedding of the conflict gadget.

For a proof of part (i) we refer to [12], and for part (ii) to Fig. 3.

4.3 The reduction Let the given 3-CNF formula be $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where each C_i is a clause with three literals (each literal is either a variable or its negation). For each C_i, we take a copy of the clause gadget G and we denote it by G_i (the G_i have pairwise disjoint vertex sets). We fix a one-to-one correspondence between the literals of C_i and the openings of G_i, letting $\omega(\lambda)$ be the opening corresponding to a literal λ.

Let us say that a literal λ in a clause C_i *is in conflict* with a literal μ in a clause C_j if both λ and μ involve the same variable x but one of them is x and the other the negation $\overline{x}$. For convenience we assume, without loss of generality, that two literals from the same clause are never in conflict.

Let Ξ consist of all (unordered) pairs $\{\omega(\lambda), \omega(\mu)\}$ of openings corresponding to pairs $\{\lambda, \mu\}$ of conflicting literals in φ. For every pair $\{\omega, \psi\} \in \Xi$ we take a fresh copy $X_{\omega\psi}$ of the conflict gadget X. We identify the loop Σ_a in $X_{\omega\psi}$ with the boundary $\partial\omega$ and the loop Σ_b with $\partial\psi$ (the rest of $X_{\omega\psi}$ is disjoint from the clause gadgets and the other conflict gadgets).

The simplicial complex K assigned to the formula

[3]Formally $N(A, \delta) = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, A) \leq \delta\}$, where $\mathrm{dist}(x, A)$ is the Euclidean distance of x from the set A.

φ is

$$K := \left(\bigcup_{i=1}^{m} G_i\right) \cup \left(\bigcup_{\{\omega,\psi\}\in\Xi} X_{\omega\psi}\right).$$

It remains to show that K is PL embeddable in $\mathbb{R}^4$ exactly if φ is satisfiable.

Nonembeddability for unsatisfiable formulas. This is a straightforward consequence of Lemma 4.1(i) and Lemma 4.2(i).

Indeed, if f is a PL embedding of K into $\mathbb{R}^4$, which we may assume to be generic, there is an opening in each clause gadget G_i such that $f(\partial\omega_i)$ has odd linking number with the complementary sphere $f(S_{\omega_i})$; let us call it a *linked opening* of G_i. Since φ is not satisfiable, whenever we choose one literal from each clause, there are two of the chosen literals in conflict. Thus, there are two linked openings $\omega \in O_{G_i}$ and $\psi \in O_{G_j}$ that are connected by a conflict gadget $X_{\omega\psi}$.

Then the supposed PL embedding f provides us an embedding as in Lemma 4.2(i) with $S_a = S_\omega$, $S_b = S_\psi$, and $X = X_{\omega\psi}$. Concerning the assumptions in the lemma, we already know that $f(S_\omega)$ and $f(\partial\omega)$ have odd linking number, and so do $f(S_\psi)$ and $f(\partial\psi)$. It remains to observe that $f(\partial\omega)$ cannot be linked with $f(S_\psi)$ (and vice versa), since G_i contains a disk bounded by $\partial\omega$: For example (refer to Fig. 1), $\partial\omega_3$ is the boundary of the disk consisting of the triangles $v_0v_1v_4$, $v_0v_2v_4$, $v_1v_2v_4$ and the triangles in the subdivision of $v_0v_1v_2$ different from ω_3. So the lemma applies and K is not embeddable.

Embedding for satisfiable formulas. Given a satisfying assignment for φ, we choose a *witness literal* λ_i for each clause C_i that is true under the given assignment (and we will refer to the remaining two literals of C_i as *non-witness* ones). No two witness literals can be in conflict.

We describe an embedding of K into $\mathbb{R}^4$ corresponding to this choice of witness literals.

Let us choose distinct points $p_1, \dots, p_m \in \mathbb{R}^4$. For each $i = 1, 2, \dots, m$, we let f_i be a generic linear embedding of the clause gadget G_i into a small neighborhood of p_i (and far from the other p_j) as in Lemma 4.1(ii), where the role of ω in the lemma is played by the witness opening of G_i (i.e., the one corresponding to to the witness literal of C_i). In particular, the interiors of the triangles bounded by $f_i(\partial\omega')$ and by $f_i(\partial\omega'')$ are disjoint from $f_i(G_i)$, where ω' and ω'' are the non-witness openings of G_i.

Taking all the f_i together defines an embedding f of the union of the clause gadgets, and it remains to embed the conflict gadgets.

To this end, we will assign to each conflict gadget $X_{\omega\psi}$ a "private" set $P_{\omega\psi} \subset \mathbb{R}^4$ homeomorphic to the 3-dimensional set N from Lemma 4.1(ii), and we will embed $X_{\omega\psi}$ into $P_{\omega\psi}$. Each $P_{\omega\psi}$ will be disjoint from all other $P_{\omega'\psi'}$ and also from all the images $f(G_i)$, *except* that $P_{\omega\psi}$ has to contain the loops $f(\partial\omega)$ and $f(\partial\psi)$ where the conflict gadget $X_{\omega\psi}$ should be attached. In order to fit enough almost-disjoint homeomorphic copies of N into the space, we will "fold" them suitably.

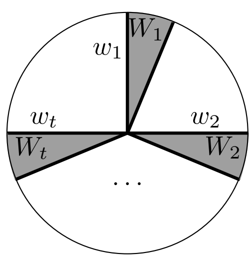
We know that for every pair $\{\omega, \psi\}$ of openings connected by a conflict gadget, at least one of ω and ψ is non-witness. Let us choose the notation so that ω is non-witness and thus unlinked in the embedding f.

We will build $P_{\omega\psi}$ from three pieces: a set $Q^+_{\omega\psi}$ that plays the role of $N(T_a, \delta)$ in Lemma 4.1(ii), a set $Q_{\psi\omega}$ that plays the role of $N(f(\Sigma_b), \delta)$, and a "connecting ribbon" in the role of $N(f(c), \delta)$.

Now let ω be an opening of some G_i, witness or non-witness. Let t be the number of openings ψ that are connected to ω by a conflict gadget. The sets $Q_{\omega\psi}$ and $Q^+_{\omega\psi}$ we want to construct are indexed by these ψ, but with some abuse of notation, we will now regard them as indexed by an index j running from 1 to t, i.e., as $Q_{\omega 1}$ through $Q_{\omega t}$ (and similarly for $Q^+_{\omega\psi}$).

For concise notation let us write $\Sigma = f(\partial\omega)$ and let T be the triangle in $\mathbb{R}^4$ having Σ as the boundary. Let $\varepsilon > 0$ be a parameter and let $T^\varepsilon := \{x \in T : \mathrm{dist}(x, \partial T) \le \varepsilon\}$ be the part of T at most ε away from the boundary of T. Since the subdivided triangle in G_i containing ω in its interior is embedded linearly by f, there is an $\varepsilon > 0$ such that if we start at a point $x \in T^\varepsilon$ and go distance at most ε in a direction orthogonal to T, we do not hit $f(G_i)$. Moreover, if ω is non-witness and thus all of T is free of $f(G_i)$, we can take any $x \in T$ with the same result. Fig. 4 tries to illustrate this in dimension one lower, where we have a segment T in $\mathbb{R}^3$ instead of a triangle T in $\mathbb{R}^4$. Thus, there are a set $Q_\omega \subset \mathbb{R}^4$ with $Q_\omega \cap f(G_i) = \Sigma$ and a homeomorphism (actually, a linear isomorphism) $h\colon Q_\omega \to T^\varepsilon \times B^2$ with $h(T^\varepsilon) = T^\varepsilon \times \{0\}$, where 0 is the center of the disk B^2. Similarly, if ω is non-witness, there are Q^+_ω and $h^+\colon Q^+_\omega \to T \times B^2$ with $h^+(T) = T \times \{0\}$.

Let $W_1, \dots, W_t \subset B^2$ be disjoint wedges as in the drawing below, and let w_j consist of the two radii bounding W_j.



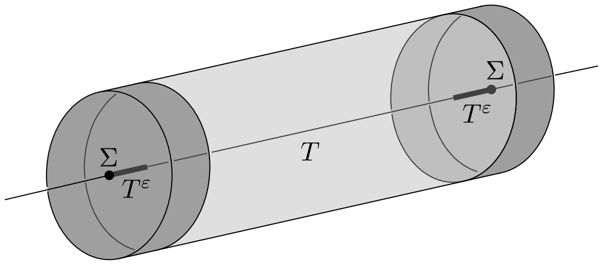


Figure 4: A free region around the triangle T; illustration in $\mathbb{R}^3$ instead of $\mathbb{R}^4$.

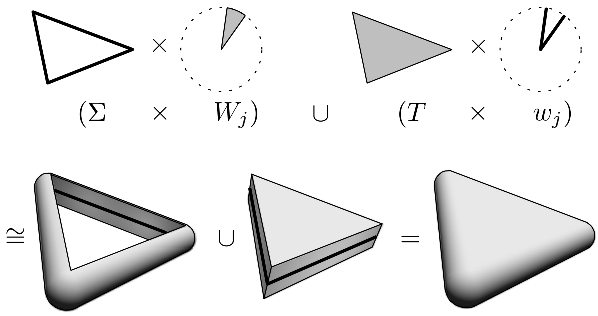


Figure 5: Folding a 3-dimensional neighborhood in $\mathbb{R}^4$.

We set

$$Q_{\omega j} := h^{-1}((\Sigma \times W_j) \cup (T^\varepsilon \times w_j)),$$

$$Q^+_{\omega j} := (h^+)^{-1}((\Sigma \times W_j) \cup (T \times w_j)).$$

As Fig. 5 tries to illustrate, $Q^+_{\omega j}$ is homeomorphic to a 3-dimensional neighborhood of T (by a homeomorphism sending T to T), and $Q_{\omega j}$ is similarly homeomorphic to a 3-dimensional neighborhood of Σ. Thus, the sets $Q_{\omega j}$ and $Q^+_{\omega j}$ can indeed play the roles of $N(f(\Sigma_b), \delta)$ and $N(T_a, \delta)$, respectively, in Lemma 4.1(ii).

It remains to construct the "connecting ribbons": For every conflict gadget $X_{\omega\psi}$, we want to connect a vertex of $f(\partial\omega)$ to a vertex of $f(\partial\psi)$ by a narrow 3-dimensional "ribbon" (it need not be straight since we are looking only for PL homeomorphic copies of N).

We observe that each of the sets $Q_{\omega j}$ and $Q^+_{\omega j}$ can be deformation-retracted to the corresponding loop $f(\partial\omega)$ or to the corresponding triangle, respectively. It follows that the complement of the union U of all the $Q_{\omega j}$, $Q^+_{\omega j}$, and $f(G_i)$ is path-connected (formally, this follows from Alexander duality, since this union is homotopy equivalent to a 2-dimensional space). Since all the considered embeddings are piecewise linear, any two points on the boundary of U can be connected by a PL path within $\mathbb{R}^4 \setminus U$.

Thus, the 3-dimensional "ribbon" connecting $f(\partial\omega)$ to $f(\partial\psi)$ can first go within the appropriate $Q_{\omega j}$ to a point on the boundary, then continue along a path connecting this boundary point to a boundary point of $Q_{\psi j'}$, and then reach $f(\partial\psi)$ within $Q_{\psi j'}$.

In this way, we have allocated the desired "private" sets $P_{\omega\psi}$ for all conflict gadgets $X_{\omega\psi}$, and hence K can be PL embedded in $\mathbb{R}^4$ as claimed. This finishes the proof of Theorem 1.2. □

5 Linkless embeddings

A PL embedding f of a graph G into $\mathbb{R}^3$ is called *linkless* if the images of any two vertex-disjoint cycles in G are unlinked, i.e., each of them bounds a PL disk that is disjoint from the other.

Robertson, Seymour, and Thomas [32, 33] showed, establishing a conjecture of Sachs, that a finite graph G is linklessly embeddable in $\mathbb{R}^3$ if and only if G d oes not contain one of the seven graphs in the so-called *Petersen family* as a minor. Moreover, they show (confirming a conjecture by Böhme [3]) that every linklessly embeddable graph G has even a *panelled* embedding (also called a *flat* embedding in some sources) into $\mathbb{R}^3$, i.e., a PL embedding such that for every cycle C in G there exists a PL disk D in $\mathbb{R}^3$ whose boundary equals $f(C)$ and that is otherwise disjoint from $f(G)$. It follows from the forbidden minor criterion that linkless embeddability, as well as panelled embeddability, can be tested in polynomial time (although the algorithm does not find an embedding).

The following lemma can be used to relate panelled embeddability to embeddability of 2-dimensional complexes into $\mathbb{R}^3$.

LEMMA 5.1. (BÖHME [3]) *Let f be a panelled embedding of G into $\mathbb{R}^3$, and let $C_1, \ldots, C_m$ be a family of cycles in G any two of which are either disjoint or intersect in a path. Then there exist* PL *disks $D_1, \ldots, D_m$ in $\mathbb{R}^3$ such that $\partial D_i = f(C_i)$ and the interiors of the D_i are pairwise disjoint and disjoint from $f(G)$.*

COROLLARY 5.1. *Let K be a 2-dimensional simplicial complex whose 1-skeleton does not have a minor from the Petersen family (and thus is linklessly embeddable). Then K embeds in $\mathbb{R}^3$.*

Proof. If G is the 1-skeleton of K, then the boundaries of the triangles in K form a family of cycles as in Lemma 5.1. Hence a panelled embedding of G can be extended to an embedding of K.

We note that the general problem EMBED$_{2\to 3}$ can be rephrased as a *partially panelled* embedding problem for graphs, whose input is a graph G and a family of triangles $C_1, C_2, \ldots, C_m$ in G, and the question is whether G admits a PL embedding in which each C_i can be panelled. This in itself does not tell us anything new about the computational complexity of the problem, of course.

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A A decision algorithm for $\mathrm{EMBED}_{2\to 2}$ (sketch)

Given a 2-dimensional simplicial complex K, we want to test whether it is embeddable in $\mathbb{R}^2$. To this end, we can use a characterization of 2-dimensional simplicial complexes embeddable in $\mathbb{R}^2$ due to Halin and Jung [15]. They give a list of seven small simplicial complexes, denoted by K_{I} through K_{VII} and shown in Fig. 6, such that a 2-dimensional simplicial complex K is embeddable in $\mathbb{R}^2$ iff it does not contain a subdivision of some of K_{I}–K_{VIII} as a subcomplex.

An inspection of K_{I}–K_{VIII} reveals that they are of three basic types:

(a) K_{III} is homeomorphic to an S^2.

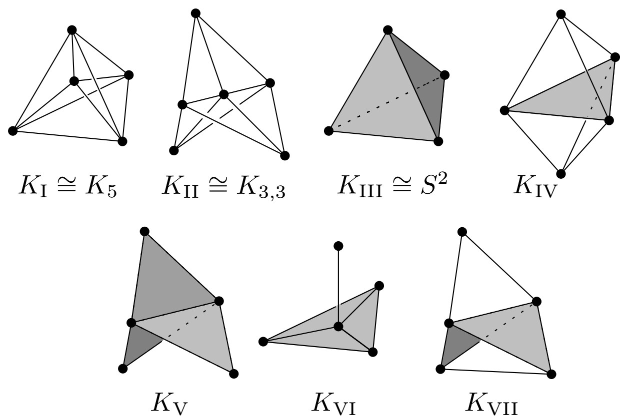
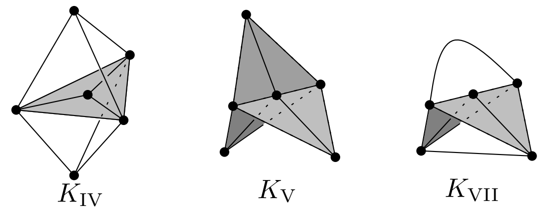


Figure 6: The forbidden subcomplexes K_{I}–K_{VII}.

(b) K_{VI} is a "disk with a stick", i.e. a subdivision of a triangle with an edge attached to a vertex in the middle.

(c) Each of the remaining five types contain a subgraph isomorphic to $K_{3,3}$ or K_5 in the 1-skeleton or in the 1-skeleton of their first barycentric subdivision (see the drawing below for the latter cases).



(A similar characterization of 2-dimensional complexes embeddable in S^2 can be found in Mardešić and Segal [20]; it is somewhat less convenient for our purposes.)

Thus, the following algorithm decides the embeddability of a given 2-dimensional complex K into $\mathbb{R}^2$: Test if the second homology (with $\mathbb{Z}_2$ coefficients) vanishes (this excludes (a)),[4] test if the 1-skeleton of the first barycentric subdivision of K is a planar graph (this takes care of (c)), and test if the link of each vertex of K is either acyclic or consists of a single cycle (this deals with (b)). By the above, K is embeddable iff it passes these three tests. It is clear that the tests can be done in polynomial time. We believe that with some additional effort even a linear-time algorithm could be obtained.

This gives only a decision algorithm and does not actually construct the embedding.

[4] Testing the second homology also excludes other cases besides a sphere; for example, a torus does not pass this test. But since every complex embeddable in $\mathbb{R}^2$ has vanishing second homology, this test does not exclude any embeddable complexes.