Three-coloring triangle-free planar graphs in linear time (extended abstract)

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Abstract

Grötzsch's theorem states that every triangle-free planar graph is 3-colorable, and several relatively simple proofs of this fact were provided by Thomassen and other authors. It is easy to convert these proofs into quadratic-time algorithms to find a 3-coloring, but it is not clear how to find such a coloring in linear time (Kowalik used a nontrivial data structure to construct an $O(n \log n)$ algorithm).

We design a linear-time algorithm to find a 3-coloring of a given triangle-free planar graph. The algorithm avoids using any complex data structures, which makes it easy to implement. As a by-product we give a yet simpler proof of Grötzsch's theorem.

1 Introduction

The following is a classical theorem of Grötzsch [6].

THEOREM 1.1. *Every triangle-free planar graph is* 3-*colorable.*

This result has been the subject of extensive research. Thomassen [15, 16] found two short proofs and extended the result in many ways. We return to the various extensions later, but let us discuss algorithmic aspects of Theorem 1.1 first. It is easy to convert either of Thomassen's proofs into a quadratic-time algorithm to find a 3-coloring, but it is not clear how to do so in linear time. A serious problem appears very early in the algorithm. Given a facial cycle C of length four, one would like to identify a pair of diagonally opposite vertices of C and apply recursion to the smaller graph. It is easy to see that at least one pair of diagonally opposite vertices on C can be identified without creating a triangle, but how can we efficiently decide which pair? If we could test in (amortized) constant time whether given two vertices are joined by a path of length at most three, then that would take care of this issue. This can, in fact, be done, using a data structure of Kowalik and Kurowski [8] *provided* the graph does not change. In our application, however, we need to repeatedly identify vertices, and it is not clear how to maintain the data structure of Kowalik and Kurowski in overall linear time. Kowalik [7] developed a sophisticated enhancement of this data structure that supported vertex identification, but at the expense of an added $\log n$ factor. Thus he designed an $O(n \log n)$ algorithm to 3-color a triangle-free planar graph on n vertices. We improve this to a linear-time algorithm, as follows.

THEOREM 1.2. *There is a linear-time algorithm to* 3-*color an input triangle-free planar graph.*

To describe the algorithm we exhibit a specific list of five reducible configurations, called "multigrams", and show that every triangle-free planar graph contains one of those reducible configurations. Proving this is the only step that requires some effort; the rest of the algorithm is entirely straightforward, and the algorithm is very easy to implement. Given a triangle-free planar graph G we look for one of the reducible configurations in G, and upon finding one we modify G to a smaller graph G', and apply the algorithm recursively to G'. It is easy to see that every 3-coloring of G' can be converted to a 3-coloring of G in constant time. Furthermore, each reducible configuration has a vertex of degree at most three, and, conversely, given a vertex of G of degree at most three it can be checked in constant time whether it belongs to a reducible configuration. Thus at every step a reducible configuration can be found in amortized constant time by maintaining a list of candidates for such vertices. As a by-product of the proof of correctness of our algorithm we give a short proof of Theorem 1.1.

Let us briefly survey some of the related work. Since in a proof of Theorem 1.1 it is easy to eliminate faces of length four, the heart of the argument lies in proving the theorem for graphs of girth at least five. For such graphs there are several extensions of the theorem. Thomassen proved in [15] that every graph of girth at least five that admits an embedding in the projective plane or the torus is 3-colorable, and the analogous result for Klein bottle graphs was obtained in [14]. For a general surface Σ, Thomassen [17] proved the deep theorem that there are only finitely many 4-critical graphs of girth at least five that embed in Σ. (A graph is 4-critical if it is not 3-colorable, but every proper subgraph is.)

None of the results mentioned in the previous paragraph hold without the additional restriction on

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girth. Nevertheless, Gimbel and Thomassen [5] found an elegant characterization of 3-colorability of triangle-free projective-planar graphs. That result does not seem to extend to other surfaces, but two of us in joint work with Král' [3] were able to find a sufficient condition for 3-colorability of triangle-free graphs drawn on a fixed surface Σ. The condition is closely related to the sufficient condition for the existence of disjoint connecting trees in [12]. Using that condition Dvořák, Král' and Thomas were able to design a linear-time algorithm to test if a triangle-free graph on a fixed surface is 3-colorable [3].

If we allow the planar graph G to have triangles, then testing 3-colorability becomes NP-hard [4]. There is an interesting conjecture of Steinberg stating that every planar graph with no cycles of length four or five is 3-colorable, but that is still open. Every planar graph is 4-colorable by the Four-Color Theorem [1, 2, 11], and a 4-coloring can be found in quadratic time [11]. Any improvement to the running time of this algorithm would seem to require new ideas. A 5-coloring of a planar graph can be found in linear time [9].

Our terminology is standard. All *graphs* in this paper are simple and *paths* and *cycles* have no repeated vertices. By a *plane graph* we mean a graph that is drawn in the plane. On several occasions we will be identifying vertices, but when we do, we will remove the resulting parallel edges. When this will be done by the algorithm we will make sure that the only parallel edges that arise will form faces of length two. The detection and removal of such parallel edges can be done in constant time.

2 Short proof of Grötzsch's theorem

Let G be a plane graph. Somewhat nonstandardly, we call a cycle F in G *facial* if it bounds a face in a connected component of G, regardless of whether F is a face or not (another component of G might lie inside F). See Section 3 for a justification of this definition. By a *tetragram* in G we mean a sequence (v_1, v_2, v_3, v_4) of vertices of G such that they form a facial cycle in G in the order listed. We define a *hexagram* $(v_1, v_2, \ldots, v_6)$ similarly. By a *pentagram* in G we mean a sequence $(v_1, v_2, v_3, v_4, v_5)$ of vertices of G such that they form a facial cycle in G in the order listed and v_1, v_2, v_3, v_4 all have degree exactly three. We will show that every triangle-free planar graph of minimum degree at least three has a tetra-, penta- or hexagram with certain additional properties that will allow an inductive argument. But first we need the following lemma.

LEMMA 2.1. *Let G be a connected triangle-free plane graph and let f_0 be the unbounded face f_0 of G. Assume that the boundary of f_0 is a cycle C of length at most six, and that every vertex of G not on C has degree at least three. If $G \neq C$, then G has either a tetragram, or a pentagram $(v_1, v_2, v_3, v_4, v_5)$ such that $v_1, v_2, v_3, v_4 \notin V(C)$.*

Proof. We may assume that G has no separating cycle C' of length at most six, for otherwise we can apply the lemma to C' and the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C'. We define the charge of a vertex v to be $3\deg(v)-12$, the charge of the face f_0 to be $3|V(C)| + 11$ and the charge of a face $f \neq f_0$ of length l to be $3l - 12$. It follows from Euler's formula that the sum of the charges of all vertices and faces is -1.

We now redistribute the charges according to the following rules. Every vertex not on C of degree three will receive one unit of charge from each incident face, each vertex on C of degree three will receive three units from f_0, and each vertex of degree two on C will receive five units from f_0 and one unit from the other incident face. Thus the final charge of every vertex is non-negative.

We now show that the final charge of f_0 is also non-negative. Let l denote the length of C. Then f_0 has initial charge of $3l + 11$. By hypothesis at least one vertex of C has degree at least three, and hence f_0 sends a total of at most $5(l-1)+3$ units of charge, leaving it at the end with charge of at least $3l+11-5(l-1)-3 \geq 1$.

Since no charge is lost or created, there is a face $f \neq f_0$ whose final charge is negative. Since f sends at most one unit to each incident vertex, we see that f has length at most five. Furthermore, if f has length exactly five, then it sends one unit to at least four incident vertices. None of those could be a degree two vertex on C, for then f would not be sending anything to the ends of the common subpath of the boundaries of f and f_0. Thus the vertices of f form the desired tetragram or pentagram. □

Let $k = 4, 5, 6$, and let $(v_1, v_2, \ldots, v_k)$ be a tetragram, pentagram or hexagram in a triangle-free plane graph G. If $k = 4$ or $k = 6$, then we say that $(v_1, v_2, \ldots, v_k)$ is *safe* if every path in G of length at most three with ends v_1 and v_3 is a subgraph of the cycle $v_1 v_2 \cdots v_k$. For $k = 5$ we define safety as follows. For $i = 1, 2, 3, 4$ let x_i be the neighbor of v_i distinct from v_{i-1} and v_{i+1} (where $v_0 = v_5$). Then $x_i \notin \{v_1, \ldots, v_5\}$, because G is triangle-free. Assume that

- the vertices x_1, x_2, x_3, x_4 are pairwise distinct and pairwise non-adjacent, and

- there is no path in $G \setminus \{v_1, v_2, v_3, v_4\}$ of length at most three from x_2 to v_5, and
- every path in $G \setminus \{v_1, v_2, v_3, v_4\}$ of length at most three from x_3 to x_4 has length exactly two, and its completion via the path $x_3v_3v_4x_4$ results in a facial cycle of length five in G (in particular, there is at most one such path).

In those circumstances we say that the pentagram $(v_1, v_2, \ldots, v_5)$ is *safe*.

LEMMA 2.2. *Every triangle-free plane graph G of minimum degree at least three has a safe tetragram, a safe pentagram, or a safe hexagram.*

Proof. Let G be as stated. If (v_1, v_2, v_3, v_4) is a tetragram in G, then one of the tetragrams (v_1, v_2, v_3, v_4), (v_2, v_3, v_4, v_1) is safe, as G is planar and triangle-free. Thus we may assume that G has no tetragram.

Let us now define a subgraph G' of G and a facial cycle C of G' in the following way: If G has a separating cycle of length at most six, then let us select such a cycle C so that the disk it bounds is as small as possible, and let G' be the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C. If G has no separating cycle of length at most six, then let $G' := G$ and let C be a facial cycle of G of length at most five. Such a facial cycle exists, because the minimum degree of G is at least three.

From Lemma 2.1 applied to the graph G' and facial cycle C we deduce that G' has a pentagram $(v_1, v_2, v_3, v_4, v_5)$ such that $v_1, v_2, v_3, v_4 \notin V(C)$. Given the choice of C and the fact that G has no tetragram, it follows that this pentagram safe, unless there is a path x_3abx_4 in $G \setminus \{v_1, v_2, v_3, v_4\}$ forming a facial hexagon $x_4v_4v_3x_3ab$. Let us argue that $(x_4, v_4, v_3, x_3, a, b)$ is a safe hexagram. If that were not the case, then there would exist a path $x_4u_1v_3$ or $x_4u_1u_2v_3$ for some $u_1, u_2 \neq v_4$. As v_4 has degree three, the cycle $x_4u_1v_3v_4$ or $x_4u_1u_2v_3v_4$ would be separating. This contradicts the choice of C. □

Proof of Theorem 1.1. Let G be a triangle-free plane graph. We proceed by induction on $|V(G)|$. We may assume that every vertex v of G has degree at least three, for otherwise the theorem follows by induction applied to $G \setminus v$. By Lemma 2.2 there is a safe tetra-, penta-, or hexagram $(v_1, v_2, \ldots, v_k)$. If $k = 4$ or $k = 6$, then we apply induction to the graph obtained from G by identifying v_1 and v_3. It follows from the definition of safety that the new graph has no triangles, and clearly every 3-coloring of the new graph extends to a 3-coloring of G.

Thus we may assume that $(v_1, v_2, \ldots, v_5)$ is a safe pentagram in G. Let G' be obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying v_5 with x_2, and x_3 with x_4. It follows from the definition of safety that G' is triangle-free, and hence it is 3-colorable by the induction hypothesis. Any 3-coloring of G' can be extended to a 3-coloring of G: let c_1 be the color of x_1, c_2 the color of x_2 and v_5, and c_3 the color of x_3 and x_4. If $c_1 = c_2$, then we color the vertices v_4, v_3, v_2 and v_1 in this order. Note that when v_i $(i = 1, 2, 3, 4)$ is colored, it is adjacent to vertices of at most two different colors, and hence we can choose the third color for it. Similarly, if $c_2 = c_3$, then we color the vertices in the following order: v_1, v_2, v_3 and v_4. Let us now consider the case that $c_1 \neq c_2 \neq c_3$. We color v_2 with c_1, v_3 with c_2, and choose a color different from c_1 and c_2 for v_1 and a color different from c_2 and c_3 for v_4. Thus G is 3-colorable, as desired. □

Let us note that the essential ideas of the proof came from Thomassen's work [15]. For graphs of girth at least five Thomassen actually proves a stronger statement, namely that every 3-coloring of an induced facial cycle of length at most nine extends to a 3-coloring of the entire triangle-free plane graph, unless some vertex of G has three distinct neighbors on C (and those neighbors received three different colors). By restricting ourselves to Theorem 1.1 we were able to somewhat streamline the argument. Another variation of the same technique is presented in [7].

3 Graph representation

For the purpose of our algorithm, graphs will be represented by means of doubly linked adjacency lists. More precisely, the neighbors of each vertex v will be listed in the clockwise cyclic order in which they appear around v, and the two occurrences of the same edge will be linked to each other. The facial walks of the graph can be read off from this representation using the standard face tracing algorithm (Mohar and Thomassen [10], page 93). Thus all vertices and edges incident with a facial cycle of length k can be listed in time $O(k)$. This does not necessarily mean that we can list all the vertices of the face in case that G is disconnected, as we do not record which facial cycles contain the components of G. Note that such information is not necessary, as the algorithm can color the components independently; it however motivates our definition of the facial cycle.

Suppose that D is a fixed constant (in our algorithm, $D = 47$). We can perform the following operations with graphs represented in the described way in constant time:

- remove an edge
- add an edge, assuming that the edges preceding and following it in the facial walks are specified
- remove an isolated vertex
- determine the degree of a vertex v if $\deg(v) \leq D$, or prove that $\deg(v) > D$
- check whether two vertices u and v such that $\min(\deg(u), \deg(v)) \leq D$ are adjacent
- check whether the distance between two vertices u and v such that $\max(\deg(u), \deg(v)) \leq D$ is at most two
- given an edge e of a face f, either output all vertices incident with the component of the boundary of f that contains e, or establish that the length of f is strictly larger than 6
- output the subgraph consisting of vertices reachable from a vertex v_0 through a path $v_0, v_1, \ldots, v_t$ of length $t \leq D$, such that $\deg(v_i) \leq D$ for $0 \leq i < t$ (but the degree of v_t may be arbitrary).

All the transformations and queries executed in the algorithm can be expressed in terms of these simple operations.

4 The algorithm

The idea of our algorithm is to find a safe tetragram, pentagram or hexagram γ in G and use it to reduce the size of the graph as in the proof of Theorem 1.1 above. Finding γ is easy, but the difficulty lies in testing safety. To resolve this problem we prove a variant of Lemma 2.2 that will guarantee the existence of such γ with an additional property that will allow testing safety in constant time. The additional property, called security, is merely that enough vertices in and around γ have bounded degree. Unfortunately, the additional property we require necessitates the introduction of another configuration, a variation of pentagram, called "decagram". For the sake of consistency, we say that a *monogram* in a graph G is the one-vertex sequence (v) comprised of a vertex $v \in V(G)$ of degree at most two.

Now let G be a plane graph, let $k \in \{1, 4, 5, 6\}$ and let $\gamma = (v_1, v_2, \ldots, v_k)$ be a mono-, tetra-, penta-, or hexagram in G. Let C be a subgraph of G. (For the purpose of this section the reader may assume that C is the null graph, but in the next section we will need C to be a facial cycle of G.) A vertex of G is *big* if it has degree at least 48, and *small* otherwise. A vertex $v \in V(G)$ is *C-admissible* if it is small and does not belong to C; otherwise it is *C-forbidden*. A pentagram $(v_1, v_2, \ldots, v_5)$ is called a *decagram* if v_5 has degree exactly three (and hence $v_1, \ldots, v_5$ all have degree three). A *multigram* is a monogram, tetragram, pentagram, hexagram or a decagram. The vertex v_1 will be called the *pivot* of the multigram $(v_1, v_2, \ldots, v_k)$. In the following γ will be a multigram, and we will define (or recall) what it means for γ to be safe and C-secure. We will also define a smaller graph G', which will be called the *γ-reduction of G*.

If γ is a monogram, then we define it to be always *safe*, and we say that it is *C-secure* if $v_1 \notin V(C)$. We define $G' := G \setminus v_1$.

Now let γ be a tetragram. Let us recall that γ is safe if the only paths in G of length at most three with ends v_1 and v_3 are subgraphs of the facial cycle $v_1v_2v_3v_4$. We say that γ is *C-secure* if it is safe, v_1 is C-admissible and has degree exactly three, and letting x denote the neighbor of v_1 other than v_2 and v_4, the vertex x is C-admissible, and either v_3 is C-admissible, or every neighbor of x is C-admissible. We define G' to be the graph obtained from G by identifying the vertices v_1 and v_3 and deleting one edge from each of the two pairs of parallel edges that result.

Now let γ be a decagram, and for $i = 1, 2, 3, 4$ let x_i be the neighbor of v_i other than v_{i-1} or v_{i+1}, where v_0 means v_5. We say that the decagram γ is *safe* if x_1, x_3 are distinct, non-adjacent and there is no path of length two between them. We say that γ is *C-secure* if it is safe and the vertices $v_1, v_2, \ldots, v_5, x_1, x_3$ are all C-admissible. We define G' to be the graph obtained from $G \setminus \{v_1, v_2, \ldots, v_5\}$ by adding the edge x_1x_3.

Now let γ be a pentagram, and for $i = 1, 2, 3, 4$ let x_i be as in the previous paragraph. Let us recall that the safety of γ was defined prior to Lemma 2.2. We say that γ is *C-secure* if it is safe, the vertices $v_1, v_2, \ldots, v_5, x_1, x_2, x_3, x_4$ are all C-admissible, either v_5 or x_2 has no C-forbidden neighbor, and either x_3 or x_4 has no C-forbidden neighbor. We define G' as in the proof of Theorem 1.1: G' is obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying x_2 and v_5; identifying x_3 and x_4; and deleting one of the parallel edges should x_3 and x_4 have a common neighbor.

Finally, let γ be a hexagram. Let us recall that γ is safe if every path of length at most three in G between v_1 and v_3 is the path $v_1v_2v_3$. We say that γ is *C-secure* if v_1, v_3, v_6 are C-admissible, v_1 has degree exactly three, and the neighbor of v_1 other than v_2 or v_6 is C-admissible. We define G' to be the graph obtained from G be identifying the vertices v_1 and v_3 and deleting one of the parallel edges that result.

We say that a multigram γ is *secure* if it is K_0-secure, where K_0 denotes the null graph. This completes the definition of safe and secure multigrams.

Lemma 4.1. *Let G be a triangle-free plane graph, let γ be a safe multigram in G, and let G' be the γ-reduction of G. Then G' is triangle-free, and every 3-coloring of G' can be converted to a 3-coloring of G in constant time. Moreover, if γ is secure, then G' can be regarded as having been obtained from G by deleting at most 102 edges, adding at most 92 edges, and deleting at least one isolated vertex.*

Proof. The graph G' is triangle-free, because γ is safe. As in the proof of Theorem 1.1, we argue that every 3-coloring of G' can be extended to a 3-coloring of G. If γ is secure, then every time vertices u and v are identified in the construction of G', one of u, v is small. Thus the identification of u and v can be seen as a deletion of at most 47 edges and addition of at most 47 edges. The lemma follows by a more careful examination of the construction of G'. □

Let G and C be as above. We say that two small vertices $u, v \in V(G)$ are *close* if there is a path of length at most four between u and v consisting of small vertices, or if a facial cycle of length at most six contains both u and v. A vertex u is close to an edge e if both u and e belong to the same facial cycle of length at most six. Thus for every vertex v there are at most $1 + 4 \cdot 47 + 47^2 + 47^3 + 47^4$ vertices that are close to v, and for every edge e, there are at most 10 vertices that are close to e.

Lemma 4.2. *Given a triangle-free plane graph G and a vertex $v \in V(G)$, it can be decided in constant time whether G has a secure multigram with pivot v.*

Proof. This follows by inspecting the subgraph of G induced by vertices and edges that are close to v. To test safety we need to check the existence of certain paths P of bounded length with prescribed ends. However, whenever such a test is needed every vertex of P, except possibly one, is small. Thus the test can be carried out in constant time. □

Lemma 4.3. *Let G and G' be triangle-free plane graphs, such that for some pair of non-adjacent vertices $u, v \in V(G)$ the graph G' is obtained from G by adding the edge uv. Let γ be a secure multigram in exactly one of the graphs G, G'. Then the pivot of γ is close to u or v in G, or to the edge uv in G'.*

Proof. Let v_1 be the pivot of γ. The claim is obvious if $v_1 \in \{u, v\}$, and thus assume this is not the case. In particular, γ is not a monogram, and γ corresponds to a facial cycle F in G or G'. If F is not facial in G or G', then v_1 is close to the edge uv in G'. Let us now consider the case that F is facial both in G and G'. As $v_1 \notin \{u, v\}$, the degree of v_1 is three both in G and G'. Let x_1 be the neighbor of v_1 distinct from its neighbors on F. Note that x_1 is small in G.

Suppose first that γ is a tetragram or a hexagram. Observe that the removal of the edge uv from G' must decrease the degree of some of the vertices affecting the security of γ, or destroy a path affecting its safety. Also, if $\{u, v\} \cap (V(F) \cup \{x_1\}) = \emptyset$, then u or v is a small neighbor of x_1 in G that is big in G'. We conclude that v_1 is close to u or v in G.

Therefore, we may assume that $\gamma = (v_1, v_2, \ldots, v_5)$ is a decagram or a pentagram. As γ is secure in G or G', all the vertices of γ are small in G. If $\{u, v\} \cap V(F) \neq \emptyset$, then v_1 is close to u or v in G, and thus assume that this is not the case. It follows that the degree of v_i is the same in G and G', for $1 \le i \le 5$; in particular, $\deg(v_i) = 3$ for $1 \le i \le 4$. Let x_i be the neigbor of v_i not incident with F, for $1 \le i \le 4$. Similarly, we conclude that x_1 and x_3 are small in G, and if γ is a pentagram, then x_2 and x_4 are small in G. If $\{u, v\} \cap \{x_1, x_3\} \neq \emptyset$, or γ is a pentagram and $\{u, v\} \cap \{x_2, x_4\} \neq \emptyset$, then u or v is close to v_1 in G. If this is not the case, then the removal or addition of uv cannot affect the security of γ if γ is a decagram.

Finally, suppose that γ is a pentagram, and $\{u, v\} \cap \{x_1, x_2, x_3, x_4\} = \emptyset$. It follows that the neighborhoods of x_2, x_3, x_4 and v_5 are the same in G and in G'. As γ is secure in G or G', all neighbors of v_5 or x_2, and all neighbors of x_3 or x_4 are small in G. As γ is not secure both in G and G', the removal of uv

- destroys a path of length at most three between x_2 and v_5 or between x_3 and x_4, or
- removes an edge incident with the common neighbor y of x_3 and x_4, thus making the 5-cycle $x_3 v_3 v_4 x_4 y$ facial, or
- decreases the degree of a neighbor of x_2, x_3, x_4 or v_5, making it small in G.

In all the cases, u or v is a small neighbor of x_2, x_3, x_4 or v_5, and hence it is close to v_1 in G. □

The next theorem will serve as the basis for the proof of correctness of our algorithm. We defer its proof until the next section.

Theorem 4.1. *Every non-null triangle-free planar graph has a secure multigram.*

We are now ready to prove Theorem 1.2, assuming Theorem 4.1.

Algorithm 4.1. There is an algorithm with the following specifications:

Input: A triangle-free planar graph.
Output: A proper 3-coloring of G.
Running time: $O(|V(G)|)$.

Description. Using a linear-time planarity algorithm that actually outputs an embedding, such as [13] or [18], we can assume that G is a plane graph. The algorithm is recursive. Throughout the execution of the algorithm we will maintain a list L that will include the pivots of all secure multigrams in G, and possibly other vertices as well. We initialize the list L to consist of all vertices of G of degree at most three.

At a general step of the algorithm we remove a vertex v from L. There is such a vertex by Theorem 4.1 and the requirement that L include the pivots of all secure multigrams. If $v \notin V(G)$, then we go to the next iteration. Otherwise, we check if G has a secure multigram with pivot v. This can be performed in constant time by Lemma 4.2. If no such multigram exists, then we go to the next iteration. Otherwise, we let γ be one such multigram, and let G' be the γ-reduction of G. By Lemma 4.1 the graph G' is triangle-free and can be constructed in constant time by adding and deleting bounded number of edges. For every edge uv that was deleted or added during the construction of G' we add to L all vertices that are close to u or v, or to the edge uv in G or G'. By Lemma 4.3 this will guarantee that L will include pivots of all secure multigrams in G'. We apply the algorithm recursively to G', and convert the resulting 3-coloring of G' to one of G using Lemma 4.1. Since the number of vertices added to L is proportional to the number of vertices removed from G we deduce that the number of vertices added to L (counting multiplicity) is at most linear in the number of vertices of G. Thus the running time is $O(|V(G)|)$, as claimed. $\square$

Algorithm 4.1 has the following extension.

Algorithm 4.2. There is an algorithm with the following specifications:
Input: A triangle-free plane graph G, a facial cycle C in G of length at most five, and a proper 3-coloring ϕ of C.
Output: A proper 3-coloring of G whose restriction to $V(C)$ is equal to ϕ.
Running time: $O(|V(G)|)$.

Description. The description is exactly the same, except that we replace "secure" by "C-secure" and appeal to Lemma 5.1 rather than Theorem 4.1. $\square$

5 Proof of correctness

In this section we prove Theorem 4.1, thereby completing the proof of correctness of the algorithm from the previous section. The theorem will follow from the next lemma (whose proof is omitted in this extended abstract). Unfortunately, for a technical reason we need a small variation on the notion of C-secure tetragram. Let G be a triangle-free plane graph, let C be a cycle in G, let H be the subgraph of G consisting of all vertices and edges of G drawn in the closed disk bounded by C, and let $\gamma = (v_1, v_2, v_3, v_4)$ be a safe tetragram in H. We say that γ is a *C-semigram* if v_1 is C-admissible, has degree exactly three and belongs to H, the neighbor of v_1 other than v_2 and v_4 is also C-admissible, and the edge v_3v_4 belongs to C. We say that v_3 is the *tail of the C-semigram γ*. We say that a vertex $v \in V(G)$ is a *C-appendix* if either $v \in V(H) - V(C)$ and v is big, or $v \in V(C)$ and v is the tail of some K-semigram in H for some cycle K in H.

Lemma 5.1. *Let G be a connected triangle-free plane graph and let f_0 be its outer face. Assume that f_0 is bounded by a cycle of length at most six and that $|V(G) - V(C)| \geq 2$. Then either G has a C-secure multigram, or C has length exactly six and includes at least two distinct non-adjacent C-appendices.*

Proof of Theorem 4.1. Let G be a triangle-free planar graph. We may assume that G is actually drawn in the plane. If G has a vertex of degree two or less, then it has a secure monogram, and so we may assume that G has minimum degree at least three. It follows that G has a facial cycle C of length at most five. Let H be the component of G containing C. We may assume that C bounds the outer face of H. Since H has minimum degree at least three and is triangle-free it follows that $V(H) - V(C)$ has at least two vertices. By Lemma 5.1 H has a C-secure multigram; but any C-secure multigram in H is a secure multigram in G, as desired. $\square$

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