

On the Relative Strength of Split, Triangle and Quadrilateral Cuts

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Abstract

Integer programs defined by two equations with two free integer variables and nonnegative continuous variables have three types of nontrivial facets: split, triangle or quadrilateral inequalities. In this paper, we compare the strength of these three families of inequalities. In particular we study how well each family approximates the integer hull. We show that, in a well defined sense, triangle inequalities provide a good approximation of the integer hull. The same statement holds for quadrilateral inequalities. On the other hand, the approximation produced by split inequalities may be arbitrarily bad.

1 Introduction

In this paper, we consider mixed integer linear programs with two equality constraints, two free integer variables and any number of nonnegative continuous variables. We assume that the two integer variables are expressed in terms of the remaining variables as follows.

$$(1.1) \quad \begin{aligned} x &= f + \sum_{j=1}^k r^j s_j \\ x &\in \mathbb{Z}^2 \\ s &\in \mathbb{R}_+^k. \end{aligned}$$

This model is a natural relaxation of a general mixed integer linear program (MILP) and therefore it can be used to generate cutting planes for MILP. Currently, MILP solvers rely on cuts that can be generated from a single equation (such as Gomory mixed integer cuts [13], MIR cuts [16], lift-and-project cuts [3], lifted cover inequalities [8]). Model (1.1) has attracted attention recently as a way of generating new families of cuts from two equations instead of just a single one [1, 7, 10, 11, 14].

We assume $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, $k \geq 1$, and $r^j \in \mathbb{Q}^2 \setminus \{0\}$. So $s = 0$ is not a solution of (1.1). Let $R_f(r^1, \dots, r^k)$ be the convex hull of all vectors $s \in \mathbb{R}_+^k$ such that $f + \sum_{j=1}^k r^j s_j$ is integral. A classical theorem of Meyer [17] implies that $R_f(r^1, \dots, r^k)$ is a polyhedron. Andersen, Louveaux, Weismantel and Wolsey [1] showed that the

only inequalities needed to describe $R_f(r^1, \dots, r^k)$ are $s \geq 0$ (called *trivial inequalities*), split inequalities [6] and intersection cuts (Balas [2]) arising from triangles or quadrilaterals in \mathbb{R}^2 . A *lattice-free convex set* is a convex set with no integral point in its interior. Cornuéjols and Margot [7] characterized the facets of $R_f(r^1, \dots, r^k)$: The nontrivial facets are minimal inequalities related to maximal lattice-free convex sets in \mathbb{R}^2 with nonempty interior (Borozan and Cornuéjols [5]). These maximal lattice-free convex sets are splits, triangles, and quadrilaterals as proved in the following theorem of Lovász [15].

THEOREM 1.1. (Lovász [15]) *In the plane, a maximal lattice-free convex set with nonempty interior is one of the following:*

- (i) *A split $c \leq ax_1 + bx_2 \leq c + 1$ where a and b are coprime integers and c is an integer;*
- (ii) *A triangle with an integral point in the interior of each of its edges;*
- (iii) *A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.*

A nontrivial valid inequality for $R_f(r^1, \dots, r^k)$ is of the form

$$(1.2) \quad \sum_{j=1}^k \psi(r^j) s_j \geq 1$$

where $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$. A nontrivial valid inequality is *minimal* if there is no other nontrivial valid inequality $\sum_{j=1}^k \psi'(r^j) s_j \geq 1$ such that $\psi'(r^j) \leq \psi(r^j)$ for all $j = 1, \dots, k$. The following result provides a link between minimal nontrivial valid inequalities and the maximal lattice-free convex sets of Theorem 1.1.

THEOREM 1.2. (Borozan and Cornuéjols [5]) *Minimal nontrivial valid inequalities are associated with functions ψ that are nonnegative, positively homogeneous, piecewise linear and convex. Furthermore, the closure of the set*

$$(1.3) \quad B_\psi := \{x \in \mathbb{Q}^2 : \psi(x - f) \leq 1\}$$

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is a maximal lattice-free convex set with nonempty interior.

Conversely, any maximal lattice-free convex set B with f in its interior defines a minimal function $\psi_B : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ that can be used to generate a minimal nontrivial valid inequality. Indeed, define $\psi_B(0) = 0$ and $\psi_B(x - f) = 1$ for all points x on the boundary of B . Then, the homogeneity of ψ_B implies the value of $\psi_B(r)$ for any vector $r \in \mathbb{R}^2 \setminus \{0\}$: If there is a positive scalar λ such that the point $f + \lambda r$ is on the boundary of B , we get that $\psi_B(r) = 1/\lambda$. Otherwise, if there is no such λ , r is an unbounded direction of B and $\psi_B(r) = 0$.

Following Dey and Wolsey [10], the maximal lattice-free triangles can be partitioned into three types (see Figure 1):

- *Type 1 triangles*: triangles with integral vertices and exactly one integral point in the relative interior of each edge;
- *Type 2 triangles*: triangles with at least one fractional vertex v , exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;
- *Type 3 triangles*: triangles with exactly three integral points on the boundary, one integral point in the relative interior of each edge.

Figure 1 shows these three types of triangles as well as a maximal lattice-free quadrilateral and a split satisfying the properties of Theorem 1.1.

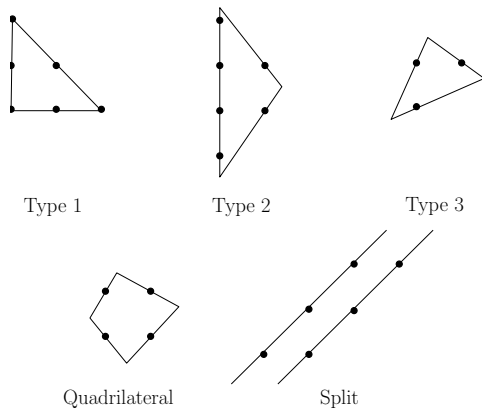


Figure 1: Maximal lattice-free convex sets with nonempty interior in \mathbb{R}^2

1.1 Motivation An unbounded maximal lattice-free set has two edges and is called a *split*. These two edges

are parallel and their direction is the *direction* of the split. Split inequalities for (1.1) are valid inequalities that can be derived by combining the two equations in (1.1) and by using the integrality of $\pi_1 x_1 + \pi_2 x_2$, where $\pi \in \mathbb{Z}^2$ defines the normal to the unbounded direction of the split. Similarly, for general MILPs, the equations can be combined into a single equality from which a split inequality is derived. Split inequalities are equivalent to Gomory mixed integer cuts [19]. Empirical evidence shows that split inequalities are effective for strengthening the linear programming relaxation of MILPs [4, 9]. Interestingly, triangle and quadrilateral inequalities cannot be derived from a single equation. They can only be derived from (1.1) without aggregating the two equations. Recent computational experiments by Espinoza [11] indicate that quadrilaterals also induce effective cutting planes in the context of solving general MILPs. In this paper, we consider the relative strength of split, triangle and quadrilateral inequalities from a theoretical point of view. We use an approach for measuring strength initiated by Goemans [12], based on the following definition and results.

Let $Q \subseteq \mathbb{R}_+^n \setminus \{0\}$ be a polyhedron of the form $Q = \{x : a^i x \geq b_i \text{ for } i = 1, \dots, m\}$ where $a^i \geq 0$ and $b_i \geq 0$ for $i = 1, \dots, m$ and let $\alpha > 0$ be a scalar. We define the polyhedron αQ as $\{x : \alpha a^i x \geq b_i \text{ for } i = 1, \dots, m\}$. Note that αQ contains Q when $\alpha \geq 1$. It will be convenient to define αQ to be \mathbb{R}_+^n when $\alpha = +\infty$.

We need the following generalization of a theorem of Goemans [12].

THEOREM 1.3. *Suppose $Q \subseteq \mathbb{R}_+^n \setminus \{0\}$ is defined as above. If convex set $P \subseteq \mathbb{R}_+^n$ is a relaxation of Q (i.e. $Q \subseteq P$), then the smallest value of $\alpha \geq 1$ such that $P \subseteq \alpha Q$ is*

$$\max_{i=1, \dots, m} \left\{ \frac{b_i}{\inf\{a^i x : x \in P\}} : b_i > 0 \right\}.$$

In other words, the only directions that need to be considered to compute α are those defined by the nontrivial facets of Q . Goemans' paper assumes that both P and Q are polyhedra, but one can easily verify that only the polyhedrality of Q is needed in the proof.

1.2 Results Let the *split closure* $S_f(r^1, \dots, r^k)$ be the intersection of all split inequalities, let the *triangle closure* $T_f(r^1, \dots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free triangles, and let the *quadrilateral closure* $Q_f(r^1, \dots, r^k)$ be the intersection of all inequalities arising from maximal lattice-free quadrilaterals. Since all the facets of $R_f(r^1, \dots, r^k)$ are induced by these three families of maximal lattice-free

convex sets, we have

$$R_f(r^1, \dots, r^k) = S_f(r^1, \dots, r^k) \cap T_f(r^1, \dots, r^k) \cap Q_f(r^1, \dots, r^k).$$

It is known that the split closure is a polyhedron (Cook, Kannan and Schrijver [6]) but such a result is not known for the triangle closure and the quadrilateral closure. Our first theorem shows that the triangle and quadrilateral closures are at least as strong as the split closure.

THEOREM 1.4. $T_f(r^1, \dots, r^k) \subseteq S_f(r^1, \dots, r^k)$ and $Q_f(r^1, \dots, r^k) \subseteq S_f(r^1, \dots, r^k)$.

We next study the strength of the triangle closure and quadrilateral closure in the sense defined in Section 1.1. We show that both the triangle closure and the quadrilateral closure are good approximations of the integer hull $R_f(r^1, \dots, r^k)$ in the sense that

THEOREM 1.5.

$$R_f(r^1, \dots, r^k) \subseteq T_f(r^1, \dots, r^k) \subseteq 2R_f(r^1, \dots, r^k) \text{ and} \\ R_f(r^1, \dots, r^k) \subseteq Q_f(r^1, \dots, r^k) \subseteq 2R_f(r^1, \dots, r^k).$$

Finally we show that the split closure may not be a good approximation of the integer hull.

THEOREM 1.6. *For any large constant α , there is a choice of f, r^1, \dots, r^k such that*

$$S_f(r^1, \dots, r^k) \not\subseteq \alpha R_f(r^1, \dots, r^k).$$

These results imply that cuts derived from two rows of the simplex tableau are provably much stronger than cutting planes currently used in MILP software, such as split cuts. In this regard, these results provide additional support for the recent interest in cuts derived from two or more rows of an integer program [1, 5, 7, 10, 11, 14].

2 Split closure vs. triangle and quadrilateral closures

We present here the proof of Theorem 1.4.

Proof. (Theorem 1.4). We show that if any point \bar{s} is cut off by a split inequality, then it is also cut off by some triangle inequality.

Consider any split inequality $\sum_{i=1}^k \psi_S(r^i) s_i \geq 1$ (see Figure 2) and denote by L_1 and L_2 its two boundary lines. Point f lies in some parallelogram of area 1 whose vertices y^1, y^2, y^3 , and y^4 are lattice points on the boundary of the split.

Assume without loss of generality that y^1 and y^2 are on L_1 . Consider the family \mathcal{T} of triangles whose edges

are supported by L_2 and by two lines passing through y^1 and y^2 and whose interior contains the segment $y^1 y^2$. See Figure 2. Note that all triangles in \mathcal{T} are of Type 2. For $T \in \mathcal{T}$ we will denote by ψ_T the minimal function associated with T .

Side 2 and the dashed lines form the triangle inequality

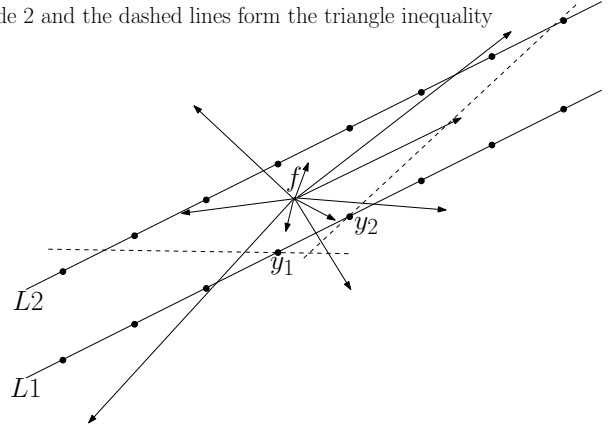


Figure 2: Approximating a Split Inequality with a Triangle Inequality

By assumption, $\sum_{i=1}^k \psi_S(r^i) \bar{s}_i < 1$. Let $\epsilon = 1 - \sum_{i=1}^k \psi_S(r^i) \bar{s}_i$.

We now make the following simple observation.

OBSERVATION 2.1. *Given a finite set X of points that lie in the interior of the split S , we can find a triangle $T \in \mathcal{T}$ as defined above, such that all points in X are in the interior of T .*

Proof. Consider the convex hull $\mathcal{C}(X)$ of X . Since all points in X are in the interior of S , so is $\mathcal{C}(X)$. This implies that the tangent lines from y^1 and y^2 to $\mathcal{C}(X)$ are not parallel to the sides of the split. If two of these four tangent lines along with side 2 of S form a triangle in \mathcal{T} with X in its interior, then we are done. If not, it is always possible to rotate two of these tangent lines so that we form a triangle T from \mathcal{T} while keeping X in its interior.

Let $s_{max} = \max\{\bar{s}_i : i = 1 \dots, k\}$ and define $\delta = \frac{\epsilon}{2 \cdot k \cdot s_{max}} > 0$. For every ray r^i define $c(r^i) = \psi_S(r^i) + \delta$. Therefore, by definition $p^i = \frac{1}{c(r^i)} \cdot r^i$ is a point strictly in the interior of S . Therefore, using Observation 2.1, there exists a triangle $T \in \mathcal{T}$ which contains all the points p^i . It follows that the coefficient $\psi_T(r^i)$ for any ray r^i is less than or equal to $c(r^i)$.

We claim that for this triangle T we have $\sum_{i=1}^k \psi_T(r^i) \bar{s}_i < 1$. Indeed,

$$\begin{aligned}
\sum_{i=1}^k \psi_T(r^i) \bar{s}_i &\leq \sum_{i=1}^k c(r^i) \bar{s}_i \\
&= \sum_{i=1}^k (\psi_S(r^i) + \delta) \bar{s}_i \\
&= \sum_{i=1}^k \psi_S(r^i) \bar{s}_i + \sum_{i=1}^k \frac{\epsilon}{2 \cdot k \cdot s_{max}} \bar{s}_i \\
&\leq \sum_{i=1}^k \psi_S(r^i) \bar{s}_i + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2} < 1
\end{aligned}$$

The first inequality follows from the definition of $c(r^i)$ and the last equality follows from the fact that $\sum_{i=1}^k \psi_S(r^i) \bar{s}_i = 1 - \epsilon$.

This shows that $T_f(r^1, \dots, r^k) \subseteq S_f(r^1, \dots, r^k)$. For the quadrilateral closure, we also use two lines passing through y^3 and y^4 on L_2 and argue similarly.

3 Tools

Cornuéjols and Margot [7] characterized the facets of $R_f(r^1, \dots, r^k)$ as follows. Let B_ψ be a maximal lattice-free split, triangle or quadrilateral with f in its interior. For any $j = 1, \dots, k$ such that $\psi(r^j) > 0$, let p^j be the intersection of the half-line $f + \lambda r^j$, $\lambda \geq 0$, with the boundary of B_ψ . The point p^j is called the *boundary point* for r^j . Let P be a set of boundary points. We say that a point $p \in P$ is *active* if it can have a positive coefficient in a convex combination of points in P generating an integral point. Note that $p \in P$ is active if and only if p is integral or there exists $q \in P$ such that the segment pq contains an integral point in its interior. We say that an active point $p \in P$ is *uniquely active* if it has a positive coefficient in *exactly one* convex combination of points in P generating an integral point.

Apply the following *Reduction Algorithm*:

- 0.) Let $P = \{p^1, \dots, p^k\}$.
- 1.) While there exists $p \in P$ such that p is active and p is a convex combination of other points in P , remove p from P . At the end of this step, P contains at most two active points on each edge of B_ψ and all points of P are distinct.
- 2.) While there exists a uniquely active $p \in P$, remove p from P .
- 3.) If P contains exactly two active points p and q (and possibly inactive points), remove both p and q from P .

The ray condition holds for a triangle or a quadrilateral if $P = \emptyset$ at termination of the Reduction Algorithm.

The ray condition holds for a split if, at termination of the Reduction Algorithm, either $P = \emptyset$, or $P = \{p_1, q_1, p_2, q_2\}$ with p_1, q_1 on one of the boundary lines and p_2, q_2 on the other and both line segments p_1q_1 and p_2q_2 contain at least two integral points.

THEOREM 3.1. (Cornuéjols and Margot [7]) *The facets of $R_f(r^1, \dots, r^k)$ are*

- (i) *split inequalities where the unbounded direction of B_ψ is r^j for some $j = 1, \dots, k$ and the line $f + \lambda r^j$ contains no integral point; or where B_ψ satisfies the ray condition,*
- (ii) *triangle inequalities where the triangle B_ψ has its corner points on three half-lines $f + \lambda r^j$ for some $j = 1, \dots, k$ and $\lambda > 0$; or where the triangle B_ψ satisfies the ray condition,*
- (iii) *quadrilateral inequalities where the corners of B_ψ are on four half-lines $f + \lambda r^j$ for some $j = 1, \dots, k$ and $\lambda > 0$, and B_ψ satisfies a certain ratio condition (the ratio condition will not be needed in this paper; the interested reader is referred to [7] for details).*

Note that the same facet may arise from different convex sets. For example quadrilaterals for which the ray condition holds define facets, but there is always also a triangle that defines the same facet, which is the reason why there is no mention of the ray condition in (iii) of Theorem 3.1.

The following technical theorem allows to simplify the discussion in the remainder of the paper. The proof of this theorem can be found in the full version of this paper [18].

THEOREM 3.2. *Let B_1, \dots, B_m be lattice-free convex sets with f in their interior. Let $R_c \subseteq \{1, \dots, k\}$ be a subset of the ray indices such that for every ray r^j with $j \notin R_c$, r^j is the convex combination of some two rays whose indices are in R_c . Define*

$$z_1 = \left\{ \begin{array}{l} \min \sum_{i=1}^k s_i \quad : \quad \sum_{i=1}^k \psi_{B_p}(r^i) s_i \geq 1, \\ \forall p \in \{1, \dots, m\}, \\ s \geq 0 \end{array} \right\}$$

and

$$z_c = \left\{ \begin{array}{l} \min \sum_{i \in R_c} s_i \quad : \quad \sum_{i \in R_c} \psi_{B_p}(r^i) s_i \geq 1, \\ \forall p \in \{1, \dots, m\}, \\ s \geq 0 \end{array} \right\}$$

Then $z_1 = z_c$.

4 Proof sketch for Theorem 1.5

In this section, we give a brief outline of the proof for Theorem 1.5.

Recall that all non trivial facet defining inequalities for $R_f(r^1, \dots, r^k)$ are of the form $a^i s \geq 1$ with $a^i \geq 0$. Therefore, Theorem 1.3 shows that to prove Theorem 1.5, we need to consider all nontrivial facet defining inequalities and optimize in the direction a^i over the triangle closure $T_f(r^1, \dots, r^k)$ and the quadrilateral closure $Q_f(r^1, \dots, r^k)$. Moreover, Theorem 1.4 shows that we can ignore the facets defined by split inequalities. Formally, consider a maximal lattice-free triangle or quadrilateral B with associated minimal function ψ that gives rise to a facet $\sum_{j=1}^k \psi(r^j) s_j \geq 1$ of $R_f(r^1, \dots, r^k)$. We want to investigate the following optimization problems:

$$(4.4) \quad \min \left\{ \sum_{j=1}^k \psi(r^j) s_j : s \in T_f(r^1, \dots, r^k) \right\}$$

and

$$(4.5) \quad \min \left\{ \sum_{j=1}^k \psi(r^j) s_j : s \in Q_f(r^1, \dots, r^k) \right\}$$

We first observe that, without loss of generality, we can make the following simplifying assumptions for problems (4.4) and (4.5).

ASSUMPTION 4.1. *Consider the objective function ψ in problems (4.4) and (4.5). For every j such that $\psi(r^j) > 0$, the ray r^j is such that the point $f + r^j$ is on the boundary of the lattice-free set B generating ψ .*

Indeed, this amounts to scaling the coefficient for the ray r^j by a constant factor in every inequality derived from all maximal lattice-free sets, including B . Therefore, this corresponds to a simultaneous scaling of variable s_j in problems (4.4) and (4.5). This does not change the optimal values of these problems. As a consequence, we can assume that the objective function of problems (4.4) or (4.5) is $\sum_{j=1}^k s_j \geq 1$.

When B_ψ is a triangle or quadrilateral and f is in its interior, define a *corner ray* to be a ray r such that $f + \lambda r$ is a corner of B_ψ for some $\lambda > 0$.

REMARK 4.1. *If $\{r^1, \dots, r^k\}$ contains the corner rays R_c of the convex set defining the objective functions of (4.4) or (4.5), then Assumption 4.1 implies that the hypotheses of Theorem 3.2 are satisfied. Therefore, when analyzing (4.4) or (4.5), we can assume that $\{r^1, \dots, r^k\} = R_c$.*

5 Integer hull vs. triangle and quadrilateral closures

In this section we present the proof of Theorem 1.5. As outlined in Section 4, to show this we need to analyze the optimization problems (4.4) and (4.5).

5.1 Approximating the integer hull by the triangle closure Theorem 1.4 shows that we can ignore the facets defined by split inequalities. We only need to consider facets of $R_f(r^1, \dots, r^k)$ derived from quadrilaterals to obtain the objective function of problem (4.4). We prove the following result.

THEOREM 5.1. *Let Q be a maximal lattice-free quadrilateral with corresponding minimal function ψ and generating a facet $\sum_{i=1}^k \psi(r^i) s_i \geq 1$ of $R_f(r^1, \dots, r^k)$. Then*

$$\min \left\{ \sum_{i=1}^k \psi(r^i) s_i : s \in T_f(r^1, \dots, r^k) \right\} \geq \frac{1}{2} .$$

Proof. The theorem holds if the facet defining inequality can also be obtained as a triangle inequality. Therefore by Theorem 3.1, we may assume that rays r^1, \dots, r^4 are the corner rays of Q (See Figure 3). We remind the reader of Remark 4.1, showing that we can assume that $k = 4$ and that the four rays are exactly the corner rays of Q .

By an affine transformation, we may further assume that the four integral points on the boundary of Q are $(0, 0), (1, 0), (1, 1), (0, 1)$. Moreover, by symmetry, we may assume that the fractional point $f = (f_1, f_2)$ satisfies $f_1 \leq \frac{1}{2}$ and $f_2 \leq \frac{1}{2}$ as rotating this region about $(\frac{1}{2}, \frac{1}{2})$ by multiples of $\frac{\pi}{2}$ covers the entire quadrilateral. Note that $f_1 < 0$ and $f_2 < 0$ are possible.

We relax Problem (4.4) by keeping only two of the triangle inequalities, defined by triangles T_1 and T_2 . T_1 has corner $f + r^4$ and edges supported by the two edges of Q incident with that corner and by the line $x = 1$. T_2 has corner $f + r^1$ and edges supported by the two edges of Q incident with that corner and by the line $y = 1$. The two triangles are depicted in dashed lines in Figure 3.

Thus, Problem (4.4) can be relaxed to the LP

$$(5.6) \quad \begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ & \sum_{i=1}^4 \psi_{T_1}(r^i) s_i \geq 1 \quad (\text{Triangle } T_1) \\ & \sum_{i=1}^4 \psi_{T_2}(r^i) s_i \geq 1 \quad (\text{Triangle } T_2) \\ & s \in \mathbb{R}_+^4. \end{aligned}$$

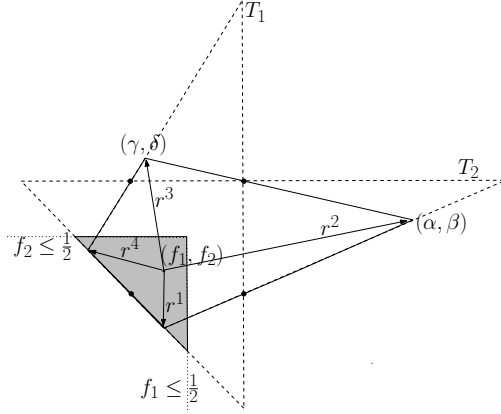


Figure 3: Approximating a quadrilateral inequality with triangle inequalities.

Computing the coefficients $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$ LP (5.6) becomes

$$(5.7) \quad \begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ & s_1 + \frac{\alpha - f_1}{1 - f_1} s_2 + s_3 + s_4 \geq 1 \quad (T_1) \\ & s_1 + s_2 + \frac{\delta - f_2}{1 - f_2} s_3 + s_4 \geq 1 \quad (T_2) \\ & s \in \mathbb{R}_+^4. \end{aligned}$$

Using the equation of the edge of Q connecting $f + r^2$ and $f + r^3$, we can find bounds on $\psi_{T_1}(r^2)$ and $\psi_{T_2}(r^3)$. The edge has equation $\frac{x}{t} + \frac{t-1}{t}y = 1$ for some $1 < t < \infty$. Therefore $\alpha \leq t$ and $\delta \leq \frac{t}{t-1}$. Using these two inequalities together with $f_1 \leq \frac{1}{2}$ and $f_2 \leq \frac{1}{2}$ we get

$$\frac{\alpha - f_1}{1 - f_1} = \frac{\alpha - 1}{1 - f_1} + 1 \leq 2(t - 1) + 1 = 2t - 1$$

and

$$\frac{\delta - f_2}{1 - f_2} \leq (2\frac{t}{t-1} - 1).$$

Using these bounds, we obtain the relaxation of LP (5.7)

$$(5.8) \quad \begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ & s_1 + (2t - 1)s_2 + s_3 + s_4 \geq 1 \quad (T_1) \\ & s_1 + s_2 + (2\frac{t}{t-1} - 1)s_3 + s_4 \geq 1 \quad (T_2) \\ & s \in \mathbb{R}_+^4. \end{aligned}$$

Set $\lambda = 2t - 1$ and $\mu = 2\frac{t}{t-1} - 1$. Then $t > 1$ implies $\lambda > 1$ and $\mu > 1$. The optimal solution of the above LP is $s_1 = s_4 = 0$, $s_2 = \frac{\mu-1}{\lambda\mu-1}$ and $s_3 = \frac{\lambda-1}{\lambda\mu-1}$ with value

$$s_1 + s_2 + s_3 + s_4 = \frac{\lambda + \mu - 2}{\lambda\mu - 1} = \frac{t^2 - 2t + 2}{t^2}.$$

To find the minimum of this expression for $t > 1$, we set its derivative to 0, and get the solution $t = 2$. Thus the minimum value of $s_1 + s_2 + s_3 + s_4$ is equal to $\frac{1}{2}$.

5.2 Approximating the integer hull by the quadrilateral closure In this section, we study Problem (4.5). Theorem 1.4 shows that we can ignore the facets defined by split inequalities. Moreover, if the facet is derived from a Type 3 triangle satisfying the ray condition, the same facet can be derived from a maximal lattice-free quadrilateral (This fact is quite easy to demonstrate and the proof appears in the full version of this paper[18]). So we only need to consider facets from Type 1, Type 2 and Type 3 triangles with corner rays. Define another relaxation $\bar{T}_f(r^1, \dots, r^k)$ as the convex set defined by the intersection of the inequalities derived only from Type 1 and Type 2 triangles. It can also be shown that

CLAIM 5.1. $Q_f(r^1, \dots, r^k) \subseteq \bar{T}_f(r^1, \dots, r^k)$.

The proof of this claim is very similar to the proof of Theorem 1.4 and can be found in the full version of the paper [18].

Hence (4.5) can be relaxed to

$$(5.9) \quad \min \left\{ \sum_{i=1}^k \psi(r^i) s_i : s \in \bar{T}_f(r^1, \dots, r^k) \right\}.$$

This implies that we need to consider only facets derived from triangles of Type 3 with corner rays for the objective function. We can show the following theorem.

THEOREM 5.2. Let T be a triangle of Type 3 with corresponding minimal function ψ and generating a

facet $\sum_{i=1}^k \psi(r^i) s_i \geq 1$ of $R_f(r^1, \dots, r^k)$. Assume that

the three corner rays are among $\{r^1, \dots, r^k\}$. Then,

$$\min \{ \psi(r^i) s_i : s \in \bar{T}_f(r^1, \dots, r^k) \} \geq \frac{1}{2}.$$

Theorem 5.2 is proved in a similar way as Theorem 5.1, using two Type 2 triangles. The complete proof is not included here and can be found in the full version [18]. This theorem implies directly the following corollary.

COROLLARY 5.1. $R_f(r^1, \dots, r^k) \subseteq 2Q_f(r^1, \dots, r^k)$.

6 The split closure vs. a single triangle or quadrilateral inequality

In this section, we prove Theorem 1.6. This is done by showing that there exist examples of integer programs (1.1) where the gap between the split closure $S_f(r^1, \dots, r^k)$ and a single triangle or quadrilateral inequality can be arbitrarily large. We give such examples for facets derived from triangles of Type 2 and Type 3, and from quadrilaterals. In our examples, the rays will be joining f to the corners of the triangle or quadrilateral. Hence the facet obtained from it will be of the form $\sum_{j=1}^k s_j \geq 1$. Consider the following LP

$$(6.10) \quad z_{SPLIT} = \left\{ \min \sum_{j=1}^k s_j : s \in S_f(r^1, \dots, r^k) \right\}.$$

We first present the following two technical results which we need for the proof. The proofs are omitted from this extended abstract and can be found in the full version of this paper[18].

Refer to Figure 4 for an illustration of the following lemma.

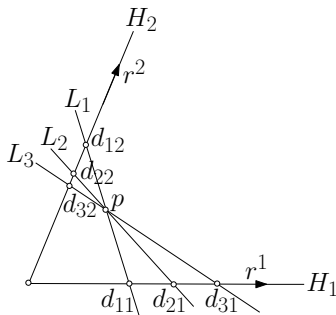


Figure 4: Illustration for Lemma 6.1

LEMMA 6.1. Let r^1 and r^2 be two rays that are not multiples of each others and let H_1 and H_2 be the half-lines generated by nonnegative multiples of r^1 and r^2 respectively. Let $p := k_1 r^1 + k_2 r^2$ with $k_1, k_2 > 0$. Let L_1, L_2 , and L_3 be three distinct lines going through p such that each of the lines intersect both H_1 and H_2 at points other than the origin. Let d_{ij} be the distance from the origin to the intersection of line L_i with the half-line H_j for $i = 1, 2, 3$ and $j = 1, 2$. Assume that $d_{11} < d_{21} < d_{31}$. Then there exists $0 < \lambda < 1$ such that

$$\begin{aligned} \frac{1}{d_{21}} &= \lambda \frac{1}{d_{11}} + (1 - \lambda) \frac{1}{d_{31}}, \\ \frac{1}{d_{22}} &= \lambda \frac{1}{d_{12}} + (1 - \lambda) \frac{1}{d_{32}}. \end{aligned}$$

COROLLARY 6.1. In the situation of Lemma 6.1, let L_4 be a line parallel to r^1 going through p . Let d_{42} be the distance between the origin and the intersection of H_2 with L_4 . Then there exists $0 < \lambda < 1$ such that

$$\frac{1}{d_{21}} = \lambda \frac{1}{d_{11}} \quad \text{and} \quad \frac{1}{d_{22}} = \lambda \frac{1}{d_{12}} + (1 - \lambda) \frac{1}{d_{42}}.$$

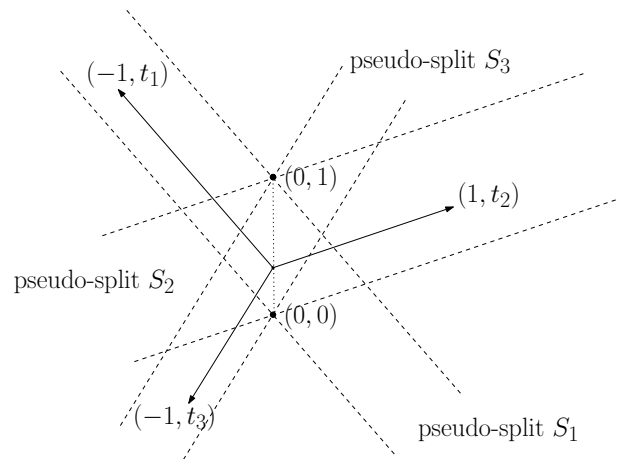


Figure 5: Dominating the Split closure with pseudo-splits

6.1 A polyhedron contained in the split closure Our examples for proving Theorem 1.6 have the property that the point f lies in the relative interior of a segment joining two integral points y^1, y^2 at distance 1.

To obtain an upper bound on the value z_{SPLIT} of the split closure, we define some inequalities which dominate the split closure (6.10). A *pseudo-split* is the convex set between two distinct parallel lines passing through y^1 and y^2 respectively. The direction of the lines, called *direction* of the pseudo-split, is a parameter. Figure 5 illustrates three pseudo-splits in the directions of three rays. The *pseudo-split inequality* is derived from a pseudo-split exactly in the same way as from any maximal lattice-free convex set. Note that pseudo-splits are in general not lattice-free and hence do not generate valid inequalities for $R_f(r^1, \dots, r^k)$. However, we can dominate any split inequality cutting f by an inequality derived from these convex sets. Indeed, consider any split S containing the fractional point f in its interior. Since f lies on the segment $y^1 y^2$, both boundary lines of S pass through the segment $y^1 y^2$. The pseudo-split with direction identical to the direction of S generates an inequality that dominates the split inequality derived from S , as the coefficient for any ray is smaller in the pseudo-split inequality.

The next lemma states that we can dominate the split closure by using only the inequalities generated by the pseudo-splits with direction parallel to the rays r^1, \dots, r^k assuming mild conditions on the rays and f . We say that vectors in a given set are *not pairwise collinear* if no two of them are multiple of each other.

LEMMA 6.2. *Assume that none of the rays r^1, \dots, r^k has a zero first component and that at least three of them are not pairwise collinear. Assume also that $f = (0, f_2)$ with $0 < f_2 < 1$. Let $y^1 = (0, 1)$ and $y^2 = (0, 0)$, these two points being used to construct pseudo-splits. Let S_1, \dots, S_k be the pseudo-splits in the directions of rays r^1, \dots, r^k and denote the corresponding minimal functions by $\psi_{S_1}, \dots, \psi_{S_k}$. Let S be any split with f in its interior and let S' be the corresponding pseudo-split. Then the inequality $\sum_{j=1}^k \psi_{S'}(r^j) s_j \geq 1$ corresponding to S' is dominated by a convex combination of the inequalities $\sum_{j=1}^k \psi_{S_i}(r^j) s_j$, $i = 1, \dots, k$. Therefore, the split inequality corresponding to S is dominated by a convex combination of the inequalities corresponding to $\psi_{S_1}, \dots, \psi_{S_k}$.*

Proof. As a convention, the direction of a pseudo-split forms an angle with the x_1 -axis in the range of $]-\frac{\pi}{2}, \frac{\pi}{2}[$. Without loss of generality, assume that the slope of the directions of the pseudo-splits corresponding to the rays r^1, \dots, r^k are monotonically non increasing. We can assume that the direction of S' is different than the direction of any of the rays in $\{r^1, \dots, r^k\}$ as otherwise the result trivially holds.

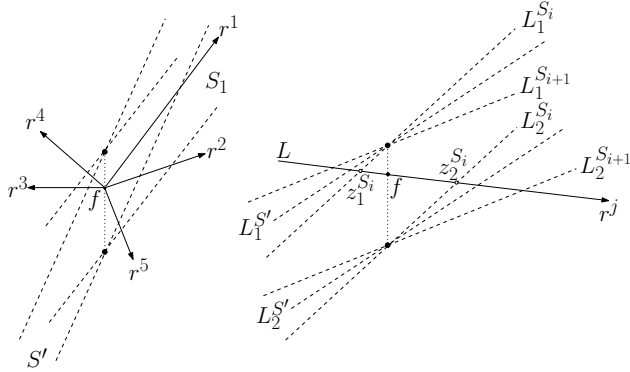


Figure 6: Bounding the split closure with a finite number of pseudo-splits

First note that, if S' has a direction with slope greater than the slope of r^1 , then the inequality generated by S' is dominated by the one generated by S_1 . Indeed, any ray r^j having a slope smaller than r^1 has its boundary point for S' closer to f than the one for S_1 . It follows that $\psi_{S'}(r^j) \geq \psi_{S_1}(r^j)$. See Figure 6.

A similar reasoning holds for the case where S' has a direction with slope smaller than the slope of r^k .

Thus we only have to consider the case where the slope of the direction of S' is strictly between the slopes of the directions of S_i and S_{i+1} , for some $1 \leq i \leq k-1$. We claim the following.

OBSERVATION 6.1. *There exists a $0 < \lambda < 1$ such that $\psi_{S'}(r) = \lambda \psi_{S_i}(r) + (1 - \lambda) \psi_{S_{i+1}}(r)$ for every ray $r \in \{r^1, \dots, r^k\}$.*

Proof. For each pseudo-split $S \in \{S', S_i, S_{i+1}\}$, we denote by L_1^S its boundary line passing through $(0, 1)$ and by L_2^S its boundary line passing through $(0, 0)$.

Consider first any ray r^j with $j < i$ and let L^{r^j} be the half-line $f + \mu r^j$, $\mu \geq 0$. We have that L^{r^j} has a slope greater than the slope of the direction of S_i and thus L^{r^j} intersects the boundaries of S', S_i and S_{i+1} on $L_1^{S'}$, $L_1^{S_{i+1}}$ and $L_1^{S_i}$. By Lemma 6.1, there exists a $0 < \lambda_1 < 1$ such that, for all $r \in \{r^1, \dots, r^{i-1}\}$

$$(6.11) \quad \psi_{S'}(r) = \lambda_1 \psi_{S_i}(r) + (1 - \lambda_1) \psi_{S_{i+1}}(r).$$

By Corollary 6.1, equation (6.11) also holds for $r = r^i$.

Using a similar reasoning for the rays $\{r^{i+1}, \dots, r^k\}$ and the boundary lines $L_2^{S'}$, $L_2^{S_{i+1}}$ and $L_2^{S_i}$, there exists a $0 < \lambda_2 < 1$ such that, for all $r \in \{r^{i+1}, \dots, r^k\}$

$$(6.12) \quad \psi_{S'}(r) = \lambda_2 \psi_{S_i}(r) + (1 - \lambda_2) \psi_{S_{i+1}}(r).$$

It remains to show that $\lambda_1 = \lambda_2$. Consider any ray r^j that is not collinear with r^i or r^{i+1} and let L be the line passing through f with direction r^j . For each $S \in \{S', S_i, S_{i+1}\}$, let z_1^S (resp. z_2^S) be the intersection of L with L_1^S (resp. L_2^S) and let d_1^S (resp. d_2^S) be the distance from f to z_1^S (resp. z_2^S). See Figure 6. By Lemma 6.1

$$(6.13) \quad \frac{1}{d_1^{S'}} = \lambda_1 \frac{1}{d_1^{S_i}} + (1 - \lambda_1) \frac{1}{d_1^{S_{i+1}}},$$

$$(6.14) \quad \frac{1}{d_2^{S'}} = \lambda_2 \frac{1}{d_2^{S_i}} + (1 - \lambda_2) \frac{1}{d_2^{S_{i+1}}}.$$

The length of segments $fz_1^S, fz_2^S, fz_1^{S'}$ and $fz_2^{S'}$ are respectively $1 - f_2, d_1^S, f_2$ and d_2^S . Observe that the triangles $fz_1^S y^1$ and $fz_2^S y^2$ are homothetic with homothetic ratio $t = \frac{1-f_2}{f_2}$. It follows that $\frac{d_1^S}{d_2^S} = t$. Substituting d_1^S by $t \cdot d_2^S$ in (6.13) yields $\lambda_1 = \lambda_2$.

This observation proves the lemma.

Using the above lemma, we can bound the split closure for three rays, assuming that none of the rays has

a zero first component and that they are not pairwise collinear. The latter condition is always verified if we assume that nonnegative combinations of the three rays generate \mathbb{R}^2 , the situation we will consider in the remainder of this section. Without loss of generality, we make the following assumptions. The rays are $r^1 = \mu_1(-1, t_1)$, $r^2 = \mu_2(1, t_2)$ and $r^3 = \mu_3(-1, t_3)$, where t_i 's are rational numbers in the range $[-\infty, \infty]$, with $t_1 > t_3$ and μ_i 's are scaling factors with $\mu_i > 0$. Any configuration of three rays satisfying the above assumptions either fits this description or is a reflection of it about the segment $(0, 0), (0, 1)$. In addition, we must have $-t_1 < t_2 < -t_3$. See Figure 5 for an illustration.

THEOREM 6.1. *Assume that $f = (0, f_2)$ with $0 < f_2 < 1$. Consider rays $r^1 = \mu_1(-1, t_1)$, $r^2 = \mu_2(1, t_2)$ and $r^3 = \mu_3(-1, t_3)$, where t_i 's are rational numbers with $-t_1 < t_2 < -t_3$ and $\mu_i > 0$. Then*

$$z_{SPLIT} \leq \frac{1}{t_1 - t_3} \left(\frac{1 - f_2}{\mu_1} + \frac{f_2}{\mu_3} \right).$$

Proof. Let $y^1 = (0, 1)$ and $y^2 = (0, 0)$, these two points being used to construct pseudo-splits. By Lemma 6.2, we know that the three pseudo-splits S_1, S_2, S_3 corresponding to the directions of r^1, r^2, r^3 dominate the entire split closure. More formally, the following LP is a strengthening of (6.10) in this example of three rays.

$$(6.15) \quad \begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ & \psi_{S_1}(r^1)s_1 + \psi_{S_1}(r^2)s_2 + \psi_{S_1}(r^3)s_3 \geq 1 \\ & \psi_{S_2}(r^1)s_1 + \psi_{S_2}(r^2)s_2 + \psi_{S_2}(r^3)s_3 \geq 1 \\ & \psi_{S_3}(r^1)s_1 + \psi_{S_3}(r^2)s_2 + \psi_{S_3}(r^3)s_3 \geq 1 \\ & s \in \mathbb{R}_+^3. \end{aligned}$$

It is fairly straightforward to compute the coefficients in the above inequalities. Analyzing the resulting LP, we get the bound stated in the theorem. For the detailed computations, the reader is referred to the full version[18].

6.2 Triangles of Type 2 that do much better than the split closure Consider the Type 2 triangle T with the following three edges. The line parallel to the x_2 -axis and passing through $(-1, 0)$ supports one of the edges, and the other two edges are supported by lines passing through $(0, 1)$ and $(0, 0)$ respectively. See left part of Figure 7. Note that in this example, the rays are of the form $r^1 = (-1, t_1), r^2 = \mu(1, t_2), r^3 = (-1, t_3)$. In the notation of Theorem 6.1, $\mu_1 = \mu_3 = 1$.

THEOREM 6.2. *Given any $\alpha > 1$, there exists a Type 2 triangle T as shown in Figure 7 such that for any point*

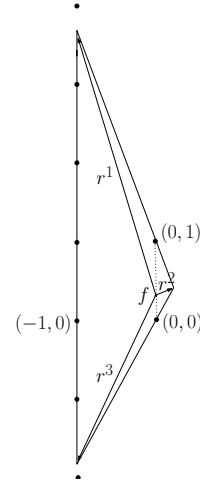


Figure 7: Facets from Type 2 triangles with large gap versus the split closure

f in the relative interior of the segment joining $(0, 0)$ to $(0, 1)$, LP (6.10) has value $z_{SPLIT} \leq \frac{1}{\alpha}$.

Proof. Let $M = \lceil \alpha \rceil$. When the fractional point f is on the segment connecting $(0, 0)$ and $(0, 1)$, consider the triangle T with M integral points in the interior of the vertical edge (the triangle in Figure 7). This implies $t_1 - t_3 \geq M$. Therefore, from the result of Theorem 6.1, $\mu_1 = \mu_3 = 1$ implies that $z_{SPLIT} \leq \frac{1}{t_1 - t_3} \leq \frac{1}{\alpha}$.

In this example, for any large constant α , optimizing over the split closure in the direction of the facet defined by these Type 2 triangles yields at most $\frac{1}{\alpha}$. This implies Theorem 1.6.

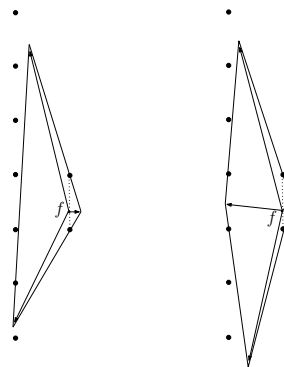


Figure 8: Facets from Type 3 triangles and quadrilaterals on which the split closure does poorly

6.3 Triangles of Type 3 and quadrilaterals Using a construction similar to the one in the previous section, we can find a family of examples where z_{SPLIT} tends to 0. We tilt the vertical edge of the triangle in Figure 7 around an integral point. See Figure 8. The integral point about which the edge is rotated is chosen to be the lowest (minimum x_2 -value) on the vertical side.

Similarly, quadrilaterals can be constructed by breaking the vertical side in Figure 7 into two sides of the quadrilateral. See Figure 8. By very similar arguments as the previous section, we can show that z_{SPLIT} tends to 0.

7 Conclusion

In this paper we gave examples of integer programs with two equality constraints, two free integer variables and nonnegative integer variables where the fraction of the integrality gap closed by the split closure is arbitrarily small. On the other hand we showed that the triangle closure always closes at least half of the integrality gap. Similarly, the quadrilateral closure always closes at least half of the integrality gap.

8 Acknowledgments

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