The Rank of Diluted Random Graphs

Charles Bordenave[*] Marc Lelarge[†]

Abstract

We investigate the rank of the adjacency matrix of large diluted random graphs: for a sequence of graphs converging locally to a tree, we give new formulas for the asymptotic of the multiplicity of the eigenvalue 0. In particular, the result depends only on the limiting tree structure, showing that the normalized rank is 'continuous at infinity'. Our work also gives a new formula for the mass at zero of the spectral measure of a Galton-Watson tree. Our techniques of proofs borrow ideas from analysis of algorithms, random matrix theory, statistical physics and analysis of Schrödinger operators on trees.

1 Introduction

The aim of this paper is to study some spectral properties of the adjacency matrix of random graphs. To motivate our results, we first show consequences of our work for G_n, the Erdős-Rényi graph with parameter $\lambda > 0$. Let $G_n = (V, E)$ be an Erdős-Rényi graph on the vertex set $V = [n] = \{1, \cdots, n\}$ and edge set E: for each pair of distinct vertices i, j in $[n]$, $(ij) \in E$ with probability λ/n independently of everything else. The adjacency matrix A_n of G_n is the $n \times n$ symmetric matrix defined by $(A_n)_{ij} = \mathbf{1}((ij) \in E)$. Let $\lambda_1(A_n) \geq \cdots \geq \lambda_n(A_n)$ denote the eigenvalues of A_n (with multiplicities) and

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A_n)}$$

denote the spectral measure of A_n. The convergence of the spectral measure μ_n as n goes to infinity to a measure μ was first rigorously proved by Khorunzhy, Shcherbina and Vengerovsky [17], for an alternative proof see [8].

THEOREM 1.1. *(i) For all $\lambda > 0$, there exists a deterministic symmetric measure μ such that, almost surely, for the weak convergence of probability measures,*

$$\lim_n \mu_n = \mu.$$

(ii) Almost surely,

$$\lim_n \mu_n(\{0\}) = \mu(\{0\}).$$

[*]Institut de Mathématiques - Université de Toulouse & CNRS
[†]INRIA-École Normale Supérieure

Note that the first statement only implies a.s.-$\limsup_n \mu_n(\{0\}) \leq \mu(\{0\})$. With "almost surely" replaced by "in probability", the first statement is contained in the above mentioned references.

Our main concern will be the rank of the adjacency matrix A_n:

$$\operatorname{rank}(A_n) = n - \dim\ker(A_n) = n - n\mu_n(\{0\}).$$

THEOREM 1.2. *For all $\lambda > 0$, almost surely,*

$$(1.1)\quad \lim_{n\to\infty} \frac{\operatorname{rank}(A_n)}{n} = 1 - \mu(\{0\}) = 2 - q - e^{-\lambda q} - \lambda q e^{-\lambda q}$$

where $0 < q < 1$ is the smallest solution of the equation $q = \exp(-\lambda \exp(-\lambda q))$.

In the sparse case, where λ tends to infinity as n tends to infinity as $\lambda_n = a \log n$ with $a > 0$, Costello, Tao and Vu [11] and Costello and Vu [10] study the rank of A_n. Their results imply that for $a > 1$, with high probability $\dim\ker(A_n) = 0$ while for $0 < a < 1$, $\dim\ker(A_n)$ is of order of magnitude n^{1-a}. Our theorem answers one of their open questions in [10].

The formula (1.1) already appeared in the remarkable paper of Karp and Sipser [13] as the asymptotic size of the number of vertices left unmatched by a maximum matching of G_n. To be more precise, it is easy to see that the function $G \mapsto \dim\ker(G)$ is additive on disjoint components and invariant under 'leaf removal', i.e. if x is a leaf of G and y its unique neighbor, $V' = V\backslash\{x, y\}$ and G' is the induced graph on V', then $\dim\ker(G') = \dim\ker(G)$. Karp and Sipser [13] study the iterative leaf removal algorithm on G_n which gives a final graph with isolated vertices and a core consisting of vertices with degree ≥ 2. They showed that the number of isolated vertices in the graph after leaf removal is approximately $(2 - q - e^{-\lambda q} - \lambda q e^{-\lambda q})n$. Moreover for $\lambda \leq e$, the size of the core is $o(n)$ so that (1.1) follows (by additivity of dim ker) as observed by Bauer and Golinelli [5]. However for $\lambda > e$, the size of the core is not negligible and the same argument only leads to the following statement: $\liminf_n \frac{\dim\ker(A_n)}{n} \geq q + e^{-\lambda q} + \lambda q e^{-\lambda q} - 1$. Bauer and Golinelli [5] conjectured the formula for all λ, which is equivalent to saying that asymptotically the

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

dimension of the kernel of the core is zero. The proof of this conjecture follows from our work.

In the rest of this paper, we will not restrict to Erdős-Rényi graphs. We consider sequences of random graphs converging locally to a random tree as the number of vertices goes to infinity. We will state a precise definition of the local convergence in Section 4 which was introduced by Benjamini and Schramm [6], Aldous and Steele [3]. The limiting tree $\mathcal{T}$ is characterized by a distribution F_* on $\mathbb{N}$ (with finite mean) and is a rooted Galton-Watson Tree (GWT) with *degree* distribution F_* [2]: the root $\varnothing$ has offspring distribution F_* and all other genitors have offspring distribution F, where for all $k \geq 1$, $F(k-1) = kF_*(k)/\sum_\ell \ell F_*(\ell)$. For Erdős-Rényi graphs, the limiting tree is a GWT with Poisson degree distribution with parameter λ.

The adjacency operator A of a GWT $\mathcal{T} = (V, E)$ is a symmetric linear operator over the set of functions of $L^2(V)$ with a finite support, defined for u, v in V by $\langle Ae_u, e_v\rangle = \mathbf{1}((uv) \in E)$. If F_* has exponential moments, then we show that A has a self-adjoint extension also denoted by A. By the spectral theorem, there is a spectral measure $\mu_\varnothing$ for A with vector $e_\varnothing$ associated to the root $\varnothing$ of the tree $\mathcal{T}$, so that for any bounded continuous function of A,

$$\langle f(A)e_\varnothing, e_\varnothing\rangle = \int f(x)d\mu_\varnothing(x).$$

Our first main result computes the value of $\mathbb{E}\mu_\varnothing(\{0\})$ for rooted GWT $\mathcal{T}$.

THEOREM 1.3. *Let $\mathcal{T}$ be a GWT with degree distribution F_* such that F_* has a finite exponential moment $\int e^{sx}F_*(dx)$ for some $s > 0$. Let φ be the generating function of F and φ_* be the generating function of F_*. We define $\pi_0 \in [0, 1]$ as the smallest solution of the equation $x = 1 - \varphi(1 - \varphi(x))$. We set $\pi_\infty = \varphi(\pi_0)$. If $\varphi'(\pi_0)\varphi'(1 - \pi_\infty) < 1$, then we have: $\mathbb{E}\mu_\varnothing(\{0\}) = \varphi_*(\pi_0) + \varphi_*(1-\pi_\infty) - 1 + \varphi_*'(1)\pi_\infty(1-\pi_0)$.*

The condition $\varphi'(\pi_0)\varphi'(1 - \pi_\infty) < 1$ is required for an application of the implicit function theorem. This theorem has already been used in the nearby context of random Schrödinger operators on the Bethe lattice by Klein [18]. At this stage, we may just notice that the fixed point π_0 is locally stable if $\frac{d}{dx}(1 - \varphi(1 - \varphi(x)))_{|x=\pi_0} = \varphi'(\pi_0)\varphi'(1 - \pi_\infty) < 1$. For regular trees, there is an exact expression for $\mu_\varnothing$ which is absolutely continuous. Hence, in this case, the value of $\mu_\varnothing(\{0\})$ is just zero. In the case where F_* is a Poisson distribution with parameter λ, the right hand side of the formula is compatible with (1.1). For finite trees, we will prove that $\mu_\varnothing(\{0\})$ is the probability that the root of the tree is left unmatched in a realization of a leaf removal algorithm producing a maximal matching. In Zdeborová and Mézard [23, Equation (38)], the right hand side of the formula already appeared in the context of the maximal matchings. To the best of our knowledge, this question has not been addressed for general GWT. Bauer and Gollineli [4] have however computed explicitly the asymptotic rank of the uniform spanning tree of n vertices. Also Bhamidi, Evans and Sen [7] have recently analyzed the convergence of the spectrum of the adjacency matrix of growing random trees.

Our second main result (Theorem 4.1) states that for a sequence of graphs G_n converging locally to a GWT, we have $\lim_n n^{-1}\text{rank}(A_n) = 1 - \mathbb{E}\mu_\varnothing(\{0\})$. By a truncation argument, we will also replace (see [9]) the finite exponential moment condition on F_* by a first moment condition $\int xF_*(dx) < \infty$.

In this paper, we have performed a detailed analysis of the atom at 0 of the limiting spectral measure μ. It is however a small achievement for the global understanding of this measure. For example, for Erdős-Rényi graphs, the atomic part of μ is dense in $\mathbb{R}$ and nothing is known on the mass of atoms apart 0. There is also a conjecture on the absolutely continuous part μ_{ac} of the measure μ. We say that μ has *extended states* (resp. no extended state) at $E \in \mathbb{R}$ if the partition function $x \mapsto \mu_{ac}(-\infty, x)$ is differentiable at $x = E$ and its derivative is positive (resp. null). This notion of extended states was introduced in mathematical physics for studying the spectrum of random Schrödinger operators, for a recent treatment on the subject see for example Aizenman, Sims and Warzel [1]. For Erdős-Rényi graphs, Bauer and Gollineli conjectured that if $0 < \lambda \leq e$, μ has no extended state at $E = 0$, whereas if $\lambda > e$, μ has extended states at $E = 0$. More generally, one might conjecture that if $0 < \lambda \leq e$, $\mu_{ac} = 0$. Finally, the existence of a singular continuous part in μ is apparently unkown.

The remainder of the paper is organized as follows, in Section 2, we analyze the adjacency operator of a GWT. In Section 3, we study $\mu_\varnothing(\{0\})$ and prove Theorem 1.3. In Section 4, we prove finally the convergence of the spectrum of finite graphs and prove Theorems 1.1 and 1.2. We also include a section on statistical physics of matchings in trees in Section 5.

2 Galton-Watson Trees and their adjacency operators

2.1 Self-Adjointness. Let F be a distribution on $\mathbb{N}$. A GWT with *offspring* distribution F is a random rooted tree obtained by a standard Galton-Watson branching process with offspring distribution F. More-

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

over if F_* is a distribution on $\mathbb{N}$ with finite first moment, a GWT with *degree* distribution F_* is a random rooted tree obtained by a Galton-Watson branching process where the root has offspring distribution F_* and all other genitors have offspring distribution F, where for all $k \geq 1$,

$$F(k-1) = kF_*(k) / \sum_{\ell} \ell F_*(\ell). \tag{2.2}$$

In this paper, we will make the following assumption on the distribution F_* (we refer to [9] to see how one can relax it for some of the results presented here):

(A) For some $s > 0$, $\sum_k \exp(sk) F_*(k) < \infty$.

We consider $\mathcal{T} = (V, E)$ a GWT with degree distribution F_* as a rooted graph. This graph is locally finite (i.e. the degree of each vertex in V is finite) and we denote $\mathbf{i} = i_1 i_2 \dots i_d \in V$ a dth-generation individual, the i_dth child of its parent $i_1 i_2 \dots i_{d-1}$ and we denote by $|\mathbf{i}| = d$ its generation. Label the root $\emptyset$ and let $|\emptyset| = 0$. Assume first that the cardinal of V the vertex set of $\mathcal{T}$, is infinite. Then the breadth-first search in the tree defines a bijection ϕ from V to $\mathbb{N}$, with $\phi(\emptyset) = 0$. If the cardinal of V is finite, say equal to n, then ϕ is a bijection from V to $\{0, \cdots, n-1\}$ and we set $\phi(k) = k$ for $k \geq n$.

Let $e_k = \{\delta_{ik} : i \in \mathbb{N}\}$ be the specified complete orthonormal system of $L^2(\mathbb{N})$. The adjacency operator A of $\mathcal{T}$ is a linear operator over $L^2(\mathbb{N})$, which is defined on the basis vector e_k as follows:

$$\langle Ae_k, e_j \rangle = \mathbf{1}((\phi^{-1}(k), \phi^{-1}(j)) \in E).$$

For $\mathbf{i} \in V$, we will sometimes use the notation $e_{\mathbf{i}}$ for the vector $e_{\phi(\mathbf{i})}$ if no confusion is possible, so that $\langle Ae_{\mathbf{k}}, e_{\mathbf{j}} \rangle = \mathbf{1}((\mathbf{k}, \mathbf{j}) \in E)$. Since $\mathcal{T}$ is locally finite, Ae_k is an element of $L^2(\mathbb{N})$ and A can be extended by linearity to a dense subspace H_0 of $L^2(\mathbb{N})$, the set of functions of $L^2(\mathbb{N})$ with a finite support. Let A_0 be the corresponding operator with domain $D(A_0) = H_0$. The operator A_0 is symmetric on H_0 and thus closable (Section VIII.2 in [21]). We will denote the closure of A_0 by the symbol A. The operator A is by definition a closed symmetric transformation: the coordinates of $y = Ax$ are

$$y_i = \sum_j \mathbf{1}((\phi^{-1}(i), \phi^{-1}(j)) \in E) x_j, \quad i \in \mathbb{N},$$

The next proposition (proved in [9]) gives a sufficient condition for A to be self-adjoint.

Proposition 2.1. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. If (A) holds then with probability one A is self-adjoint.*

2.2 Spectral measure. As above let $\mathcal{T}$ be a GWT with degree distribution F_* such that (A) holds. Since its adjacency operator A is a.s. self-adjoint, by the spectral theorem, there is a.s. a spectral measure $\mu_\emptyset$ for A with vector $e_\emptyset \in L^2(\mathbb{N})$ associated to the root $\emptyset$. That is, so that for any bounded function of A

$$\langle f(A) e_\emptyset, e_\emptyset \rangle = \int f(x) d\mu_\emptyset(x). \tag{2.3}$$

Since $e_\emptyset \in D(A)$, the domain of A, (2.3) extends to polynomially bounded function and we have

$$\begin{aligned} \gamma_n = \langle A^n e_\emptyset, e_\emptyset \rangle &= \int x^n d\mu_\emptyset(x) \\ &= \#\{\text{paths of length } n \text{ from } \emptyset \text{ to } \emptyset \text{ in } \mathcal{T}\}. \end{aligned} \tag{2.4}$$

The measure $\mu_\emptyset$ is a random probability measure on $\mathbb{R}$ and we now characterize its law. Let $\mathcal{H}$ be the set of holomorphic functions f from $\mathbb{C}_+$ to $\mathbb{C}_+$ such that $|f(z)| \leq (\Im z)^{-1}$. We denote by $\mathcal{P}(\mathcal{H})$ the set of probability measures on $\mathcal{H}$. The Cauchy-Stieltjes transform of a probability measure on $\mathbb{R}$ is the function in $\mathcal{H}$ given by

$$m_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}.$$

For any $\mathbf{i} \in \mathcal{T}$, let $\mathcal{T}_{\mathbf{i}}$ be the subtree of $\mathcal{T}$ containing $\mathbf{i}$ and restricted to $\mathbf{i}\mathbb{N} = \{\mathbf{j}, |\mathbf{j}| \geq |\mathbf{i}|, j_1 = i_1, \dots, j_{|\mathbf{i}|} = i_{|\mathbf{i}|}\}$. If $i \neq \emptyset$, then $\mathcal{T}_{\mathbf{i}}$ is a GWT with offspring distribution F. The adjacency operator of $\mathcal{T}_{\mathbf{i}}$ denoted $A_{\mathbf{i}}$, is the projection of the adjacency operator of $\mathcal{T}$ on $\mathbf{i}\mathbb{N}$. Assumption (A) implies that $A_{\mathbf{i}}$ is self-adjoint with probability one. As above, we define $\mu_{\mathbf{i}}$ the spectral measure for $A_{\mathbf{i}}$ with vector $e_{\mathbf{i}}$ and $m_{\mathbf{i}} := m_{\mu_{\mathbf{i}}}$ the associated Cauchy-Stieltjes transform. Each $\mu_{\mathbf{i}}$ is a random probability measure and hence each $m_{\mathbf{i}}$ is a random element in $\mathcal{H}$. The recursive structure of the tree implies a simple recursion for Cauchy-Stieltjes transforms $m_{\mathbf{i}}$. We show in the proof of the forthcoming Proposition 2.2 that for any $\mathbf{i} \in \mathcal{T}$:

$$m_{\mathbf{i}}(z) = -\left(z + \sum_{\mathbf{j} \in D(\mathbf{i})} m_{\mathbf{j}}(z) \right)^{-1}, \tag{2.5}$$

where $D(\mathbf{i}) = \{\mathbf{j} \in \mathbf{i}\mathbb{N}, |\mathbf{j}| = |\mathbf{i}| + 1\}$. The recursion (2.5) is valid for any locally finite tree with self-adjoint adjacency operator. In the particular case of GWT with offspring distribution F, the $(m_{\mathbf{j}})_{\mathbf{j} \in D(\mathbf{i})}$ are i.i.d. with distribution $m_{\mathbf{i}}$ and it allows us to characterize the law of the random spectral measures $\mu_{\mathbf{i}}$. Namely, the law of its Cauchy-Stieltjes transform is given as the unique solution of the Recursive Distributional

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

Equation (RDE) given below (2.6). Then, the recursion (2.5) applied at the root, allows us to characterize the laws of the random $m_\varnothing$ as follows:

PROPOSITION 2.2. *Let $\mathcal{T}$ be a GWT with degree distribution F_* . We assume that (A) holds.*

(i) There exists a unique probability measure $Q \in \mathcal{P}(\mathcal{H})$ such that for all $z \in \mathbb{C}_+$,

$$Y(z) \stackrel{d}{=} -\left(z + \sum_{i=1}^{N} Y_i(z)\right)^{-1}, \tag{2.6}$$

where N has distribution F and Y, Y_i are iid copies with law Q and independent of N.

(ii) If $|\mathbf{i}| \geq 1$, the law of $m_\mathbf{i} \in \mathcal{H}$ is Q and

$$m_\varnothing(z) \stackrel{d}{=} -\left(z + \sum_{i=1}^{N_*} Y_i(z)\right)^{-1}, \tag{2.7}$$

where N_* has distribution F_* and Y_i are iid copies with law Q and independent of N_*.

Proof. Point (i) is proved in [8] under a weaker assumption on F_*. Point (ii) follows from a classical operator version of Schur complement formula (see e.g. Proposition 2.1 in Klein [18] for a similar argument). We define the operator U on $L^2(\mathbb{N})$ by its matrix elements:

$$\langle e_\varnothing, U e_\mathbf{i}\rangle = \langle e_\mathbf{i}, U e_\varnothing\rangle = \langle e_\varnothing, A e_\mathbf{i}\rangle = 1,$$

for all $\mathbf{i}$ with $|\mathbf{i}| = 1$ and $\langle e_\mathbf{j}, U e_\mathbf{k}\rangle = 0$ otherwise. We now have the following decomposition:

$$A = U + \bigoplus_{|\mathbf{i}|=1} A_\mathbf{i},$$

where $A_\mathbf{i}$, is the projection of the adjacency operator of $\mathcal{T}$ on $\mathbf{i}\mathbb{N}$. Clearly U and $\tilde{A} = \bigoplus_{|\mathbf{i}|=1} A_\mathbf{i}$ are self-adjoint operators and we write $R(z) = (A - zI)^{-1}$ and $\tilde{R}(z) = (\tilde{A} - zI)^{-1}$ for the associated resolvents with $z \in \mathbb{C}_+$. The resolvent identity gives:

$$\tilde{R}(z) U R(z) = \tilde{R}(z) - R(z). \tag{2.8}$$

First note that $\langle e_\varnothing, \tilde{R}(z) e_\varnothing\rangle = -z^{-1}$ and $\langle e_\varnothing, \tilde{R}(z) e_\mathbf{k}\rangle = \langle e_\mathbf{k}, \tilde{R}(z) e_\varnothing\rangle = 0$ for all $|\mathbf{k}| \geq 1$. For any $\mathbf{k}$, we have:

$$\begin{aligned}
\langle e_\mathbf{k}, \tilde{R}(z) U R(z) e_\varnothing\rangle &= \sum_{|\mathbf{i}|=1} \langle e_\mathbf{k}, \tilde{R}(z) e_\varnothing\rangle \langle e_\mathbf{i}, R(z) e_\varnothing\rangle \\
&\quad + \langle e_\mathbf{k}, \tilde{R}(z) e_\mathbf{i}\rangle \langle e_\varnothing, R(z) e_\varnothing\rangle \\
&= \langle e_\mathbf{k}, \tilde{R}(z) e_\varnothing\rangle \sum_{|\mathbf{i}|=1} \langle e_\mathbf{i}, R(z) e_\varnothing\rangle \\
&\quad + \langle e_\varnothing, R(z) e_\varnothing\rangle \sum_{|\mathbf{i}|=1} \langle e_\mathbf{k}, \tilde{R}(z) e_\mathbf{i}\rangle.
\end{aligned}$$

In particular we get from (2.8),

$$\langle e_\mathbf{k}, \tilde{R}(z) U R(z) e_\varnothing\rangle = \langle e_\mathbf{k}, \tilde{R}(z) e_\varnothing\rangle - \langle e_\mathbf{k}, R(z) e_\varnothing\rangle.$$

First consider $\mathbf{k}$ with $|\mathbf{k}| = 1$, then we get

$$\begin{aligned}
-\langle e_\mathbf{k}, R(z) e_\varnothing\rangle &= \langle e_\varnothing, R(z) e_\varnothing\rangle \sum_{|\mathbf{i}|=1} \langle e_\mathbf{k}, \tilde{R}(z) e_\mathbf{i}\rangle \\
&= \langle e_\varnothing, R(z) e_\varnothing\rangle \langle e_\mathbf{k}, \tilde{R}(z) e_\mathbf{k}\rangle \\
&= m_\varnothing(z) m_\mathbf{k}(z)
\end{aligned}$$

Now consider $\mathbf{k} = \varnothing$, then we get:

$$\begin{aligned}
-z^{-1} - m_\varnothing(z) &= (-z)^{-1} \sum_{|\mathbf{i}|=1} \langle e_\mathbf{i}, R(z) e_\varnothing\rangle \\
&= z^{-1} m_\varnothing(z) \sum_{|\mathbf{i}|=1} m_\mathbf{i}(z)
\end{aligned}$$

Hence we proved that for any locally finite tree with self-adjoint adjacency operator:

$$m_\varnothing(z) = -\left(z + \sum_{|\mathbf{i}|=1} m_\mathbf{i}(z)\right)^{-1}.$$

Point (ii) follows then easily. □

3 Computation of $\mathbb{E}\mu_\varnothing(\{0\})$

3.1 Recursion Equation. First note that for any symmetric measure μ, we have for $t > 0$,

$$m_\mu(it) = \int_\mathbb{R} \frac{x}{x^2 + t^2} + \frac{it}{x^2 + t^2} d\mu(x) = i \int_\mathbb{R} \frac{t}{x^2 + t^2} d\mu(x).$$

Now, consider a rooted tree $\mathcal{T}$ with self-adjoint adjacency operator. By (2.4), we see that $\mu_\varnothing$, the spectral measure with the root vector $e_\varnothing$, is symmetric. In what follows we denote $h_\varnothing(t) = \Im(m_{\mu_\varnothing}(it))$. Then we have for $t > 0$,

$$t h_\varnothing(t) = \int_\mathbb{R} \frac{t^2 d\mu_\varnothing(x)}{x^2 + t^2},$$

so that by the dominated convergence theorem, we have

$$\mu_\varnothing(\{0\}) = \lim_{t \to 0} t h_\varnothing(t) \in [0, 1]. \tag{3.9}$$

For any $\mathbf{i} \in \mathcal{T}$, $\mu_\mathbf{i}$ is also symmetric and we define $h_\mathbf{i}(t) = \Im(m_\mathbf{i}(it)) = -i m_\mathbf{i}(it)$. Then (2.5) gives:

$$h_\mathbf{i}(t) = \left(t + \sum_{\mathbf{j} \in D(\mathbf{i})} h_\mathbf{j}(t)\right)^{-1}. \tag{3.10}$$

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

Since we are interested in the limit of $th_{\emptyset}(t)$, we iterate once Equation (3.10) to obtain an equation with $th_{\mathbf{i}}(t)$ on each side:

$$th_{\mathbf{i}}(t) = \left(1 + \sum_{\mathbf{j}\in D(\mathbf{i})} \left(t^2 + \sum_{\mathbf{k}\in D(\mathbf{j})} th_{\mathbf{k}}(t)\right)^{-1}\right)^{-1}. \quad (3.11)$$

As in (3.9), we know that $\mu_{\mathbf{i}}(\{0\}) = \lim_{t\to 0} th_{\mathbf{i}}(t) \in [0,1]$. With the convention $1/0 = \infty$ and $1/\infty = 0$, the map $x \mapsto 1/x$ is continuous on $[0,\infty]$, so that we have

$$\mu_{\mathbf{i}}(\{0\}) = \left(1 + \sum_{\mathbf{j}\in D(\mathbf{i})} \left(\sum_{\mathbf{k}\in D(\mathbf{j})} \mu_{\mathbf{k}}(\{0\})\right)^{-1}\right)^{-1}. \quad (3.12)$$

First consider the case where $\mathcal{T}$ is a finite tree. Then for any leaf $\mathbf{i}$ of the tree, we have $h_{\mathbf{i}}(t) = t^{-1}$ so that $\mu_{\mathbf{i}}(\{0\}) = 1$. Then one can use the recursion (3.12) to compute $\mu_{\emptyset}(\{0\})$ 'bottom up'. Now consider the general case, where $\mathcal{T}$ might be infinite. We define $\mathcal{T}^k = \mathcal{T} \cap B_{\mathcal{T}}(\emptyset, k)$ as the truncated tree of depth k from the root, i.e. $B_{\mathcal{T}}(\emptyset, k) = \{\mathbf{i} \in \mathcal{T},\ |\mathbf{i}| \le k\}$. Since $\mu_{\mathbf{i}}(\{0\}) \in [0,1]$, thanks to the monotonicity of the recursion (3.12), we get:

LEMMA 3.1. *Let $\mathcal{T}$ be a locally finite rooted tree with self-adjoint adjacency operator. We have for any k:*

$$\mu_{\emptyset}^{\mathcal{T}^{2k+1}}(\{0\}) \le \mu_{\emptyset}(\{0\}) \le \mu_{\emptyset}^{\mathcal{T}^{2k}}(\{0\}),$$

where $\mu_{\emptyset}^{\mathcal{T}^k}$ is the spectral measure with vector $e_{\emptyset}$ of the adjacency operator of the finite truncated tree $\mathcal{T}^k$.

If $\mathcal{T}$ has leaves, it is natural to try to extend the 'bottom up' procedure described above for finite trees to infinite trees. We follow this idea in the next Section 3.2 by defining a leaf removal process similar to the one of Karp and Sipser [13]. When $\mathcal{T}$ is finite, this leaf removal process remove all the vertices and leave an empty graph. However, when $\mathcal{T}$ is infinite, this leaf removal process can leave a non-empty tree. This remaining tree is infinite with no leaf. We will compute $\mu_{\emptyset}(\{0\})$ for trees with no leaf in Section 3.3. In the last Section 3.4, we put our results together and give the general formula for $\mathbb{E}\mu_{\emptyset}(\{0\})$.

3.2 The leaf removal Markov process. Let $G = (V,E)$ be a finite graph (not necessarily a tree). The leaf removal (LR) is a Markov process that describes a pruning of G. Denote by $\mathcal{G}_V$ the set of graphs on V, $\mathcal{M}_V$ the set of pairs of V and $\mathcal{U}_V$ the subsets of V. The state space of the Markov process is $\mathcal{G}_V \times \mathcal{M}_V \times \mathcal{U}_V$. Let $(G,M,U) \in \mathcal{G}_V \times \mathcal{M}_V \times \mathcal{U}_V$. If $v \in V$ has degree one in G, then its only neighbor is say u. We denote by N_u the set of neighbors of u that are of degree one. In the LR process, at rate 1 the following transition occurs:

$$(G,M,U) \to (G\backslash(N_u \cup \{u\}), M \cup \{(u,v)\}, U \cup N_u\backslash\{v\}), \quad (3.13)$$

where as usual $G\backslash(N_u \cup \{u\})$ is the graph on $[n]\backslash(N_u \cup \{u\})$ induced by G. All other transitions have rate 0. Now we consider the initial condition $(G\backslash V_0, \emptyset, V_0)$, where V_0 is the set of vertices of degree 0 in G, i.e. isolated points. Let $(G(t), M(t), U(t))$ denote the state of the Markov process at time t. Since the size of $G(t)$ is non-increasing, the LR process reaches a.s. an absorbing state (C,M,U) in finite time. C is called the *core* of G, all vertices have a degree at least 2 in C. U is called the set of *uncovered* vertices of G and M is a *matching* of G (i.e. a set of edges with distinct adjacent vertices). By recursion on the size of G it is straightforward to check that $|U|$, $|M|$, and C do not depend on the order of the leaf removal. As a consequence, we get:

LEMMA 3.2. *For a finite graph G, $|U|$, $|M|$, and C are not random.*

For a finite graph G, let P_G be the probability measure associated to the LR process with initial condition $(G\backslash V_0, \emptyset, V_0)$. A consequence of the above lemma is that for any $v \in V$, $\mathrm{P}_G(v \in M \cup U) = 1 - \mathrm{P}_G(v \in C) \in \{0,1\}$.

Now, for $k \in \mathbb{N}$ and a locally finite rooted graph G with root $\emptyset$, we define a new process, the Leaf Removal process at depth k, denoted by LR_k. With the above notation, if the graph distance from $\emptyset$ to v is less or equal to k, in the LR_k process, at rate 1 the transition (3.13) occurs. All other transitions have rate 0. Since G is locally finite, the process LR_k is properly defined. Note that by construction LR_k is the restriction of LR to G_k the subgraph of G with vertices at distance at most k from the root (and with possibly some inactive vertices at distance k from the root). Again, the LR_k process reaches a.s. an absorbing state (C_k, M_k, U_k) in finite time. Note that any vertex v at distance from the root $\emptyset$ larger than $k+1$ is in C_k.

For a fixed (possibly infinite) locally finite rooted graph G with root $\emptyset$, we couple the LR_k processes for all $k \in \mathbb{N}$ as follows: to each vertex $v \in V$, we attach an independent exponential random variable $\xi(v)$ of parameter 1. If v is at distance less than k from the root $\emptyset$, define $\tau_k(v)$ as the first time that v has degree one in the LR_k process. Note that $\tau_k(v) = \infty$ if vertex v is removed ($v \in M_k$) or becomes of degree 0 ($v \in U_k$) after a leaf removal. Let $\tau_k(v) + \xi(v)$ be the

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

time of activation of vertex v. At any time t, the next transition (3.13) to occur is the removal of the leaf with the least activation time and this transition occurs at its activation time. With this coupling, we see that the event $\{\emptyset \in C_k\}$ is a non-increasing event in k and has probability 0 or 1. We define

$$\begin{aligned} P_G(\emptyset \in C) &:= \lim_k \mathrm{P}_{G_k}(\emptyset \in C_k) \\ &= \inf_k \mathrm{P}_{G_k}(\emptyset \in C_k) \in \{0,1\}. \end{aligned}$$

We have thus defined the core $C(G)$ of any locally finite graph G. We now consider $\mathcal{T}$ a locally finite rooted tree with adjacency operator A which is self-adjoint (as in Section 2.1) and show how the spectral measure $\mu_\emptyset$ associated to the root $\emptyset$ is related to the LR process. We also define $\mathcal{T}^k = \mathcal{T} \cap B_{\mathcal{T}}(\emptyset, k)$ as the truncated tree of depth k from the root (as in previous section).

LEMMA 3.3. *Let $\mathcal{T}$ be a locally finite rooted tree with self-adjoint adjacency operator. If $\emptyset \notin C$ then there exists a random a.s.-finite $K_{\mathcal{T}}$ such that for any integers $k \geq \ell \geq K_{\mathcal{T}}$, we have*

$$\mu_\emptyset(\{0\}) = \mathrm{P}_{\mathcal{T}^k}(\emptyset \in U_\ell) =: \mathrm{P}_{\mathcal{T}}(\emptyset \in U).$$

Proof. If $\mathcal{T}$ is a finite tree, then C is empty and the matching obtained by the LR process is maximal. More precisely, the matching M obtained by the LR process is uniformly distributed over the set of maximal matching of $\mathcal{T}$. Then it follows from Section 5 see (5.22) that

$$\mathrm{P}_{\mathcal{T}}(\emptyset \in U) = \mu_\emptyset(\{0\}).$$

Now we deal with the general case. First, we define the weighted distance between $\emptyset$ and $\mathbf{i} = i_1 \cdots i_k$ as $w_{\mathbf{i}} = \xi(\mathbf{i}) + \xi(i_1 \dots i_{k-2}) + \dots + \xi(\emptyset)$ if $|\mathbf{i}|$ is even and $w_{\mathbf{i}} = \xi(\mathbf{i}) + \xi(i_1 \dots i_{k-2}) + \dots + \xi(i_1)$ if $|\mathbf{i}|$ is odd.

Since $\mathrm{P}_{\mathcal{T}}(\emptyset \notin C) = 1$, there exists an integer k_0 and a finite sequence of transitions of the Markov process LR_{k_0} such that if these transitions occur, one of the event $\{\emptyset \in U_{k_0}\}$ or $\{\emptyset \in M_{k_0}\}$ occurs. For concreteness, let $\mathbf{i}_1, \cdots, \mathbf{i}_p$ be the vertices which trigger these transitions. We define $T_0 = \sum_{\ell=1}^p \xi(\mathbf{i}_\ell)$.

Then for any $k \geq k_0$, at time $t \geq T_0$, the root $\emptyset$ either belong to $U_k(t)$ or to $M_k(t)$. Since $\mathcal{T}$ is locally finite, there exists a finite $K \geq k_0$ such that for all $v \notin B_{\mathcal{T}}(\emptyset, K)$, $w_{\mathbf{v}} > T_0$. Then, for all $k \geq k_0$, the event $\{\emptyset \in U_k\}$ is measurable with respect to the filtration $\{\xi_{\mathbf{v}},\ \mathbf{v} \in B_{\mathcal{T}}(\emptyset, K)\}$. In summary, we have for any $k \geq K$, $\mathbf{1}(\emptyset \in U_k) = \mathbf{1}(\emptyset \in U_K)$. Note that with our coupling for all $k \geq \ell > K$, $\mathbf{1}(\emptyset \in U_K) = \mathbf{1}(\emptyset \in U_\ell(\mathcal{T}^k))$, where $U_\ell(\mathcal{T}^k)$ is the set of uncovered vertices when the LR_ℓ is applied to $\mathcal{T}^k$. For any $k \geq \ell > K$, we have $\mathrm{P}_{\mathcal{T}^k}(\emptyset \in U_\ell(\mathcal{T}^k)) = \mathrm{P}_{\mathcal{T}^k}(\emptyset \in U_k(\mathcal{T}^k)) = \mathrm{P}_{\mathcal{T}^k}(\emptyset \in U) = \mu_\emptyset^{\mathcal{T}^k}(\{0\})$, where the last equality follows from the first part of the proof. Thus, by Lemma 3.1, we have $\mu_\emptyset(\{0\}) = \mathrm{P}_{\mathcal{T}^k}(\emptyset \in U)$ for all $k > K$. ☐

In conclusion, we proved that for any locally finite rooted tree $\mathcal{T} = (V, E)$ with self-adjoint adjacency operator, we have:

$$\mu_\emptyset(\{0\}) = \mathbf{1}(\emptyset \notin C)\mathrm{P}_{\mathcal{T}}(\emptyset \in U) + \mathbf{1}(\emptyset \in C)\mu_\emptyset(\{0\}).$$

In the proof of Theorem 1.3, we will not explicitly use the above decomposition. However it will be useful to prove the convergence of the rank of finite graphs in Section 4. In the next section, we deal with the case where the tree has no leaf which corresponds to $C = V$. We defer the explicit computation of $\mu_\emptyset(\{0\})$ and $\mathrm{P}_{\mathcal{T}}(\emptyset \in U)$ to Section 3.4.

3.3 Proof of Theorem 1.3 : tree with no leaf.

In this section, we consider $\mathcal{T}$ a GWT with degree distribution F_* such that assumption (A) holds. We define F by (2.2), and N and N_* will denote random variables with distribution F and F_*. The generating function of N is

$$\varphi(x) = \mathbb{E}[x^N] = \sum_{k=0}^{\infty} F(k) x^k.$$

We assume that the tree $\mathcal{T}$ has no leaf and an additional condition on the distribution, namely,

(C) $F_*(0) = F_*(1) = 0$ and $\mathbb{P}(N = 1)\mathbb{E}[N] < 1 \Leftrightarrow \varphi'(0)\varphi'(1) < 1$.

PROPOSITION 3.1. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. If (A) and (C) hold then,with probability one, $\mu_\emptyset(\{0\}) = 0$.*

Proof. First consider the particular case of the 3-regular tree, corresponding to $F_*(3) = F(2) = 1$. In this case, the spectral measure $\mu_\emptyset$ is well-known and is due to McKay [19] and Kesten [16]. We can recover it easily from Proposition 2.2 (see [8]): for any $|\mathbf{i}| \geq 1$, $h_{\mathbf{i}}(t) = \frac{-t+\sqrt{t^2+4}}{4}$ and $\mu_\emptyset(dx) = \frac{3}{2\pi} \frac{\sqrt{8-x^2}}{9-x^2} dx$. In particular, this measure is absolutely continuous with respect to the Lebesgue measure and $\mu_\emptyset(\{0\}) = \lim_{t\downarrow 0} t h_\emptyset(t) = 0$. We also have for any $|\mathbf{i}| \geq 1$, $\lim_{t\downarrow 0} t h_{\mathbf{i}}(t) = \mu_{\mathbf{i}}(\{0\}) = 0$ (recall that $\mu_{\mathbf{i}}$ is the spectral measure of the adjacency operator of $\mathcal{T}_{\mathbf{i}}$ with vector $e_{\mathbf{i}}$).

We now consider a GWT $\mathcal{T}$ satisfying assumptions (A) and (C). For any $|\mathbf{i}| \geq 1$, $h_{\mathbf{i}}$ have the same distribution and we define a random variable $h \stackrel{d}{=} h_{\mathbf{i}}$. We shall prove that in this case also, we have with probability one,

$$\lim_{t\downarrow 0} t h(t) = 0. \tag{3.14}$$

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

Thanks to (3.11), we know that the law of $th(t) \in [0,1]$ satisfies the following RDE:

$$th(t) \stackrel{d}{=} \left(1 + \sum_{i=1}^{N}\left(t^2 + \sum_{j=1}^{N_i} th_{i,j}(t)\right)^{-1}\right)^{-1}, \tag{3.15}$$

where $h_{i,j}$ and N_i are iid copies of h and N. Then we obtain for the root:

$$th_{\varnothing}(t) = \left(1 + \sum_{i=1}^{N_*}\left(t^2 + \sum_{j=1}^{N_i} th_{i,j}(t)\right)^{-1}\right)^{-1},$$

where $h_{i,j}$ and N_i are iid copies of h and N. If (3.14) holds, then we deduce that with probability one, that $\mu_{\varnothing}(\{0\}) = \lim_{t\downarrow 0} th_{\varnothing}(t) = 0$ and Proposition 3.1 follows.

In order to prove (3.14), we will use an interpolation argument on the distribution of N. For $q \in [0,1]$, we define the distribution $F_q = qF + (1-q)\delta_2$. φ_q is the associated generating function and N_q is a random variable with distribution F_q. Hence we have $N_0 = 2$ and $N_1 \stackrel{d}{=} N$. Note that Assumption (C) implies that $\varphi'(1) = \mathbb{E}[N] > 1$, and the function $q \mapsto q(2-2q+qm)$ being increasing on $[0,1]$ for all $m \geq 1$, we get for any $q \in [0,1]$,

$$\varphi_q'(0)\varphi_q'(1) \leq F(1)\mathbb{E}N < 1. \tag{3.16}$$

We define $h_q(t)$ as the imaginary part of $Y_q(it)$, where Y_q is a solution of the RDE (2.6) in Proposition 2.2 with N having distribution F_q. We define $L_{q,t}(x) = \mathbb{E}\left[\exp(-xth_q(t))\right]$. We have the following lemma (proved in [9]):

LEMMA 3.4. *For any $t > 0$, the function $q \mapsto L_{q,t}$ is continuous for the point-wise convergence.*

Let $\mathcal{P}_\infty$ and $\mathcal{P}_1$ be the sets of probability measures on $[0,\infty]$ and $[0,1]$ respectively. Note that for any $t \geq 0$, the law of $th(t)$ belongs to $\mathcal{P}_1$. For each $t \geq 0$, we can see the RDE (3.15) as a fixed point equation on the space $\mathcal{L}_1$ of Laplace transform of elements of $\mathcal{P}_1$. We define $\mathcal{L}_\infty$ as the Laplace transforms of elements $\mu \in \mathcal{P}_\infty$ defined as follows: $f(x) = \int_0^\infty e^{-ux}\mu(du) + \mu(\{\infty\})\mathbf{1}_{x=0}$. If X is a random variable with law $\mu \in \mathcal{P}_\infty$, we denote $f(x) = \mathbb{E}[e^{-xX}]$. In order to write the fixed point equation in term of Laplace transforms, we define formally the following operators:

$$\begin{aligned}(B(q,t)f)(x) &= e^{-xt}\mathbb{E}[f(x)^{N_q}] = e^{-xt}\varphi_q(f(x)),\\ (Tf)(x) &= f(0) - \sqrt{x}\int_0^\infty \frac{J_1(2\sqrt{xt})}{\sqrt{t}} f(t)dt,\end{aligned}$$

where $J_1(x) = \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{1+2k}}{k!(k+1)!}$ is the Bessel function of integral order 1. We note that if $f \in \mathcal{L}_\infty$ is the Laplace transform of $\mu \in \mathcal{P}_\infty$ and X_i's are independent random variables with law μ, then we have for $t \geq 0$

$$(B(q,t)f)(x) = \mathbb{E}\left[e^{-x(t+\sum_{i=1}^{N_q} X_i)}\right]$$

Recall the formula, for $x \geq 0, w > 0$:

$$e^{-xw^{-1}} = 1 - \sqrt{x}\int_0^\infty \frac{J_1(2\sqrt{tx})}{\sqrt{t}} e^{-tw}dt. \tag{3.17}$$

We can extend (3.17) to $w = 0$ by continuity with the convention $\frac{1}{0} = \infty$ since $\lim_{A\to\infty}\sqrt{x}\int_0^A \frac{J_1(2\sqrt{tx})}{\sqrt{t}}dt = 1$.

In particular, if $f \in \mathcal{L}_\infty$ is the Laplace transform of the random variable X with law $\mu \in \mathcal{P}_\infty$, then we have:

$$(Tf)(x) = \mathbb{E}\left[e^{-xX^{-1}}\right].$$

Hence we see that $(Tf)(x)$ is the Laplace transform of the law of X^{-1} (with the convention $\frac{1}{0} = \infty$) which belongs to $\mathcal{P}_\infty$. We now define for $t \geq 0$, $TB(q,t^2)f$, and it follows that for any $t > 0$

$$(TB(q,t^2)f)(x) = \mathbb{E}\left[e^{-x(t^2+\sum_{i=1}^{N_q} X_i)^{-1}}\right].$$

We may thus define $G : [0,1] \times [0,\infty) \times \mathcal{L}_1 \to \mathcal{L}_1$ as

$$G(q,t,L) = TB(q,1)TB(q,t^2)L. \tag{3.18}$$

Then for any $q \in [0,1]$, $t \geq 0$ the function $L_{q,t}(x) = \mathbb{E}\left[\exp(-xth_q(t))\right]$ is in $\mathcal{L}_1$ and is a solution of the fixed point equation $G(q,t,L) = L$.

We are now going to enlarge the domain of the function G on a suitable Banach space. Let $\mathcal{M}_1$ and $\mathcal{M}_\infty$ be the set of finite signed measures on $[0,1]$ and $[0,\infty]$ respectively. By a finite signed measure μ we mean that $\mu_+([0,\infty]) + |\mu_-([0,\infty])| < \infty$, where μ_+ and μ_- are the positive and negative parts of μ. We also define the real vector space of Laplace transforms of elements of $\mathcal{M}_1$:

$$\mathcal{K}_1 = \left\{f,\ f(x) = \int e^{-ux}\mu(du) \text{ for some } \mu \text{ in } \mathcal{M}_1\right\}.$$

In order to define $\mathcal{K}_\infty$, some care is required to define the Laplace transform f of a finite signed measure μ in $[0,\infty]$: we set $f(0) = \mu([0,\infty])$ and for all $x > 0$, $f(x) = \int e^{-ux}\mu_{|[0,\infty)}(du)$. Then $\mathcal{K}_\infty$ is defined as the real vector space of Laplace transforms of elements of $\mathcal{M}_\infty$. We endow $\mathcal{K}_\infty$ with the topology of point-wise convergence. Note that for all $f \in \mathcal{K}_1$, the measure μ is uniquely defined. We endow $\mathcal{K}_1$ with the Wasserstein norm: for $f(x) = \int e^{-ux}\mu(du)$,

$$\|f\|_1 = \sup\left(\int_0^1 h(x)d\mu(x) : h(0) = 1, \|h\|_{Lip} \leq 1\right),$$

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

where $\|h\|_{Lip} = \sup_{x\neq y\in[0,\infty)} |h(x) - h(y)|/|x - y|$. For details on Wasserstein norm, refer e.g. to Chapter 7 in [22]. It follows from the Riesz representation theorem that, with this norm, $\mathcal{K}_1$ is a Banach space. We recall also that in $\mathcal{L}_1 \subset \mathcal{K}_1$, the pointwise convergence is metrizable with the Wasserstein norm. More precisely, for $(L_n)_{n\in\mathbb{N}}$, L in $\mathcal{L}_1$, if for all $x \in \mathbb{R}_+$, $\lim_n L_n(x) = L(x)$ then $\lim_{n\to\infty} \|L_n - L\|_1 = 0$. Indeed, recall that point-wise convergence of L_n to L is equivalent to weak convergence of μ_n to μ, where μ_n and μ have Laplace transforms L_n and L respectively. Then, if $\lim_{n\to\infty} \|L_n - L\|_1$ was not 0 there would exist a sequence of functions from $[0,1] \to \mathbb{R}$, $(h_n)_n$, with $h_n(0) = 1$, $\|h_n\|_{Lip} \leq 1$, such that $\int_0^1 h_n d(\mu_n - \mu)$ does not converges to 0. From Arzelà-Ascoli theorem, there would exist a continuous function h such that $\liminf \int_0^1 h d(\mu_n - \mu) > 0$. This would lead to a contradiction.

We may now extend the domain of our operators T, B and G.

Lemma 3.5. *(i) T is a continuous $\mathcal{K}_\infty \to \mathcal{K}_\infty$ linear function and $T^2 = I$.*

(ii) B is a continuous $[0,1] \times [0,\infty) \times \mathcal{K}_\infty \to \mathcal{K}_\infty$ function.

(iii) G is a continuous $[0,1]\times[0,\infty)\times\mathcal{K}_1 \to \mathcal{K}_1$ function.

Proof. The proof follows easily from the definition. In order to check that the image of B is in $\mathcal{K}_\infty$, recall that the product of Laplace transforms is the Laplace transform of a convolution. □

Lemma 3.6. *Let $(q,t,f) \in [0,1] \times [0,\infty) \times \mathcal{K}_\infty$.*

(i) The Gâteaux derivative of B at (q,t,f) with respect to f in the direction g is given by

$$(B_f(q,t)(f)g)(x) = e^{-xt}\mathbb{E}\left[N_q f(x)^{N_q-1}\right] g(x)$$

(ii) For $f \in \mathcal{K}_1$, the Fréchet derivative of G at (q,t,f) with respect to f is given by:

$$G_f(q,t,f) = TB_f(q,1)(TB(q,t^2)f)TB_f(q,t^2)(f)$$

Proof. Point (i) is checked easily. Point (ii) follows taking the derivative of a composition. □

Now, we compute $G_f(q,0,1)$ and we have:

$$\begin{aligned}
(B_f(q,0)(1)g)(x) &= \mathbb{E}[N]g(x)\\
(TB_f(q,0)(1)g)(x) &= \mathbb{E}[N](Tg)(x)\\
(B(q,0)1)(x) &= 1\\
(TB(q,0)1)(x) &= \mathbf{1}_{x=0}\\
(B_f(q,1)(\mathbf{1}_{x=0})g)(x) &= e^{-x}\mathbb{P}(N=1)g(x)\\
(TB_f(q,1)(\mathbf{1}_{x=0})g)(x) &= \mathbb{P}(N=1)(TEg)(x),
\end{aligned}$$

where $Eg(x) = e^{-x}g(x)$. Hence we get

$$G_f(q,0,1)(g)(x) = \mathbb{P}(N_q = 1)\mathbb{E}[N_q](TETg)(x). \tag{3.19}$$

Lemma 3.7. *The map $G - I : [0,1]\times[0,\infty)\times\mathcal{K}_1 \to \mathcal{K}_1$ is continuous. $G - I$ is continuously Fréchet differentiable with respect to f, the partial derivative being*

$$G_f(q,t,f) = \mathbb{P}(N_q = 1)\mathbb{E}[N_q]TB_f(q,1)(TB(q,t^2)f)TB_f(q,t^2)(f).$$

Moreover, if $\mathbb{P}(N = 1)\mathbb{E}[N] < 1$, then $G_f(q,0,1) - I$ is an automorphism of $\mathcal{K}_1$.

Proof. The proof follows from previous Lemmas except the last point. Let $\alpha_q = \mathbb{P}(N_q = 1)\mathbb{E}[N_q] < 1$, then $\alpha_q E - I$ is invertible on $\mathcal{K}_\infty$ and

$$\begin{aligned}
((\alpha_q E - I)^{-1} f)(x) &= (\alpha_q e^{-x} - 1)^{-1} f(x)\\
&= -\sum_{n\geq 0} \alpha_q^n e^{-nx} f(x).
\end{aligned}$$

Hence the inverse of $G_f(q,0,1) - I$ given by (3.19) is $T(\alpha_q E - I)^{-1}T$. □

For any $q \in [0,1]$, since $T\mathbf{1}_{x=0} = 1$, we have $G(q,0,1) = 1$. Then by the implicit function theorem we have for any $q_0 \in [0,1]$:

Lemma 3.8. *There exists a ball $V = \{(q,t), (q-q_0)^2 + t^2 \leq r\}$ and $\delta > 0$, such that there exists a unique map $u : V \to \{f \in \mathcal{K}_1, \|f - 1\|_1 < \delta\}$ such that $G(q,t,u(q,t)) = u(q,t)$. The map $(q,t) \in V \to u(q,t)$ is continuous.*

This statement of the implicit function theorem is slightly stronger than the one given in Theorem 2.7.2 in [20], but it is the result actually proved in [20].

Note that the unicity in Lemma 3.8 implies that $u(q,0) = 1$ for $q - q_0 \leq \sqrt{r}$.

We may now conclude the proof of Proposition 3.1. Recall that for all $(q,t) \in [0,1] \times [0,\infty)$, $G(q,t,L_{q,t}) = L_{q,t}$ and $L_{0,0}(x) = 1$ where $L_{q,0} = \lim_{t\downarrow 0} L_{q,t}$ is the Laplace transform of the random variable $\lim_{t\downarrow 0} th_q(t)$. Suppose that there exists $q > 0$ such that $L_{q,0}(x) \neq 1$ and define $q_0 = \sup\{q,\ L_{q,0}(x) = 1\} \geq 0$. Then, by Lemma 3.8 and 3.4, we can find $\epsilon > 0$ such that for $q \in (q_0 - \epsilon, q_0 + \epsilon)$ and $t \in (0,\epsilon)$, we have $L_{q,t} = u(q,t)$. Moreover by continuity, we have $L_{q,0}(x) = \lim_{t\downarrow 0} u(q,t) = 1$. This leads to a contradiction. Hence for all $q \in [0,1]$, $L_{q,0}(x) = 1$ and for $q = 1$ (3.14) follows. □

3.4 Proof of Theorem 1.3 : tree with leaves.

In this paragraph we study $\mu_\varnothing(\{0\})$ for a general GWT.

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

We will prove that if $\emptyset \in C$, then we have $\mu_\emptyset(\{0\}) = 0$ (Corollary 3.1). We already proved this statement in the previous paragraph for trees without leaf and the general proof will be similar. More generally, by using a labeling of the vertices closely related to the leaf removal algorithm, we will be able to compute $\mathbb{E}\mu_\emptyset(\{0\})$.

As above, $\mathcal{T}$ is a GWT with degree distribution F_* such that assumption (A) holds, φ is the generating function of F and φ_* is the generating function of F_*. We define $\pi_0 \in [0,1]$ as the smallest solution of the equation $x = 1 - \varphi(1-\varphi(x))$. We set $\pi_\infty = \varphi(\pi_0)$ and $\pi_c = 1 - \pi_0 - \pi_\infty$. The real $x_* \in [0,1]$ such that $x_* + \varphi(x_*) = 1$ is solution of the above equation, hence $0 \le \pi_0 \le x_*$ and $\pi_c = 1 - \pi_0 - \varphi(\pi_0) \ge 0$.

We consider a labeling procedure of the vertices of a locally finite rooted tree $\mathcal{T}$ as Karp and Sipser did in [13]. Initially all vertices have label or type c and we repeat the following procedure which changes the value of the labels until nothing new is generated. We put a label ∞ on all vertices with all its offsprings of type 0 (possibly vacuously defined) and a label 0 to the vertices having at least one offspring of type ∞.

Lemma 3.9. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. If $\mathbf{i} \neq \emptyset$ then $\mathbb{P}(\ \mathbf{i}$ is of type $0\ ) = \pi_0$, $\mathbb{P}(\ \mathbf{i}$ is of type $\infty\ ) = \pi_\infty$ and $\mathbb{P}(\ \mathbf{i}$ is of type $c\ ) = \pi_c$.*

Proof. For all integer n and $\tau \in \{0,\infty\}$, define $\pi_\tau^{(n)}$ as the probability that $\mathbf{i}$ is of type τ after n steps of the labeling procedure. The probabilities $\pi_0^{(n)}$ and $\pi_\infty^{(n)}$ are non-decreasing in n so that it suffices to prove that

$$\lim_{n\to\infty} \pi_0^{(n)} = \pi_0 \quad \text{and} \quad \lim_{n\to\infty} \pi_\infty^{(n)} = \pi_\infty.$$

By construction, $\mathbf{i}$ is of type 0 after n steps of the labeling procedure if one its offsprings is of type ∞ after $n-1$ steps. If N denote the number of offsprings of $\mathbf{i}$, we have

$$\pi_0^{(n)} = \mathbb{E}\left[1 - (1-\pi_\infty^{(n-1)})^N\right] = 1 - \varphi(1-\pi_\infty^{(n-1)}).$$

Similarly,

$$\pi_\infty^{(n)} = \varphi(\pi_0^{(n-1)}).$$

For all $n \ge 2$ we get $\pi_0^{(n)} = 1 - \varphi(1-\varphi(\pi_0^{(n-2)}))$. Since $\pi_0^{(0)} = 0$ and $x \mapsto 1-\varphi(1-\varphi(x))$ is non-decreasing, $\pi_0^{(n)}$ converges to the smallest fixed point of the equation $x = 1-\varphi(1-\varphi(x))$. □

The next lemma (proved in [9]) is a consequence of the recursive structure of GWT's.

Lemma 3.10. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. Let $\mathbf{i} \neq \emptyset$, $N \stackrel{d}{=} F$ be the total number of offsprings of $\mathbf{i}$, and N_0, N_c, N_∞ be the number of offsprings of $\mathbf{i}$ of type 0, c, ∞. Let L_i be an iid sequence of random variables on $\{0,c,\infty\}$ independent of N such that $\mathbb{P}(L_i = \tau) = \pi_\tau$, $\tau \in \{0,c,\infty\}$.*

(i) $(N_0, N_c, N_\infty) \stackrel{d}{=} \left(\sum_{i=1}^N \mathbf{1}_{L_i=0}, \sum_{i=1}^N \mathbf{1}_{L_i=c}, \sum_{i=1}^N \mathbf{1}_{L_i=\infty}\right)$.

(ii) Conditioned on $\{\mathbf{i}$ is of type $0\}$, (i) holds by conditioning on $\{\exists 1 \le i \le N:\ L_i = \infty\}$.

(iii) Conditioned on $\{\mathbf{i}$ is of type $\infty\}$, (i) holds by conditioning on $\{N = 0$ or $\forall 1 \le i \le N:\ L_i = 0\}$.

(iv) Conditioned on $\{\mathbf{i}$ is of type $c\}$, (i) holds by conditioning on $\{\exists 1 \le i \le N:\ L_i = c$ and $\forall 1 \le i \le N:\ L_i \neq \infty\}$.

Let $\mathbf{i} \neq \emptyset$. From Equation (3.10), conditioned on $\{\mathbf{i}$ is of type $\tau\}$, we obtain $h_{\mathbf{i}} \stackrel{d}{=} h^\tau$ with

$$h^\tau(t) \stackrel{d}{=} \left(t + \sum_{\theta \in \{0,c,\infty\}} \sum_{i=1}^N \mathbf{1}_{L_i=\theta} h_i^\theta(t)\right)^{-1}, \tag{3.20}$$

where $(h_i^\theta)_{\theta, i\ge 1}$ is an independent sequence, independent of $((L_i)_{i\ge1}, N)$, and h_i^θ has law h^θ. $((L_i)_{i\ge1}, N)$ is as in Lemma 3.10, conditioned on $\{\exists 1 \le i \le N:\ L_i = \infty\}$ if $\tau = 0$, on $\{N = 0$ or $\forall 1 \le i \le N:\ L_i = 0\}$ if $\tau = \infty$, or on $\{\exists 1 \le i \le N:\ L_i = c$ and $\forall 1 \le i \le N:\ L_i \neq \infty\}$ if $\tau = c$. The RDE (3.20) dispatched over the types is significantly more complicated than the original recursion (3.10). The reason we have introduced the types lies in the following proposition (proved in [9]) where we show that the behavior of $h_{\mathbf{i}}(t)$ as $t \downarrow 0$ is determined by the type. It implies in particular that if $\mathbf{i}$ is of type 0 then $h_{\mathbf{i}}$ converges to 0, whereas if $\mathbf{i}$ is of type ∞ then $h_{\mathbf{i}}$ converges to ∞ (hence the name chosen for the types).

Proposition 3.2. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. Assume that (A) holds and $\varphi'(\pi_0)\varphi'(1-\pi_\infty) < 1$. With probability one, $0 < \lim_{t\downarrow 0} \frac{h^0(t)}{t} < \infty$, $0 < \lim_{t\downarrow 0} t h^\infty(t) \le 1$, $\lim_{t\downarrow 0} \frac{h^c(t)}{t} = \infty$, $\lim_{t\downarrow 0} t h^c(t) = 0$.*

The next lemma proved in [9] decomposes $t\mathbb{E}h_\emptyset(t)$ into two computable terms by an anti-symmetrization trick.

Lemma 3.11. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. For each $\mathbf{i} \in \mathcal{T}$, let $\varepsilon_{\mathbf{i}} = \mathbf{1}(\mathbf{i}$ is of type $0) -$*

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

$\mathbf{1}(\mathbf{i}$ *is of type* $\infty)$. *For all* $t > 0$, *we have*

$$\mathbb{E}th_\varnothing(t) = \mathbb{E}\left[\frac{t+\sum_{i=1}^{N_*} h_i(t)\varepsilon_i}{t+\sum_{i=1}^{N_*} h_i(t)}\right] - \frac{\mathbb{E}[N_*]}{2}\mathbb{E}\left[\frac{h_1(t)h_2(t)}{1+h_1(t)h_2(t)}(\varepsilon_1+\varepsilon_2)\right].$$

Now, we use the dominated convergence theorem and Proposition 3.2 to get

$$\begin{aligned}\lim_{t\downarrow 0}\mathbb{E}\frac{t+\sum_{i=1}^{N_*} h_i(t)\varepsilon_i}{t+\sum_{i=1}^{N_*} h_i(t)} = \\ & F_*(0) - \mathbb{P}(\exists 1 \le i \le N_* : L_i = \infty) \\ & +\mathbb{P}(N_* \ge 1, \forall 1 \le i \le N_* : L_i = 0) \\ = \quad & F_*(0) - \sum_{k=1}^{\infty} F_*(k)(1-(1-\pi_\infty)^k) \\ & +\sum_{k=1}^{\infty} F_*(k)\pi_0^k \\ = \quad & \varphi_*(\pi_0) + \varphi_*(1-\pi_\infty) - 1.\end{aligned}$$

Similarly, $h_1 h_2/(1+h_1h_2) \le 1$, we may use the dominated convergence and obtain

$$\begin{aligned}-\lim_{t\downarrow 0}\mathbb{E}\frac{h_1h_2}{1+h_1h_2}(\varepsilon_1+\varepsilon_2) = \\ & 2\mathbb{P}(L_1=\infty, L_2=\infty) + \mathbb{P}(L_1=\infty, L_2=c) \\ & +\mathbb{P}(L_1=c, L_2=\infty) \\ = \quad & 2\pi_\infty^2 + 2\pi_\infty\pi_c \\ = \quad & 2\pi_\infty(1-\pi_0).\end{aligned}$$

By Lemma 3.11, we obtain

$$\lim_{t\downarrow 0}\mathbb{E}th_\varnothing(t) = \varphi_*(\pi_0)+\varphi_*(1-\pi_\infty)-1+\mathbb{E}[N_*]\pi_\infty(1-\pi_0).$$

The proof of Theorem 1.3 is now complete.

We conclude this section with a corollary of Proposition 3.2 on the leaf removal process introduced in §3.2 needed for the proof of the convergence of the rank in Section 4.

COROLLARY 3.1. *Let* $\mathcal{T}$ *be a GWT with degree distribution* F_*. *Assume that* (*A*) *holds and* $\varphi'(\pi_0)\varphi'(1-\pi_\infty) < 1$. *With probability one,* $\mu_\varnothing(\{0\})\mathbf{1}(\varnothing \in C) = 0$ *and* $\mu_\varnothing(\{0\}) = \mathrm{P}_\mathcal{T}(\varnothing \in U)$.

We start with a simple lemma that relates the types and the leaf removal process.

LEMMA 3.12. *Let* $\mathcal{T}$ *be a rooted locally finite tree. If* $\mathbf{i} \in C$ *then* $\mathbf{i}$ *is of type* c.

Proof. From Lemma 3.3, we shall prove that if $\mathbf{i}$ is of type 0 or ∞ then $\mathbf{i}$ is not in C. We prove this statement by induction on the number of steps of the labeling procedure. Assume that $\mathbf{i}$ is of type 0 or ∞ after n steps of the labeling procedure. If $n=1$, then $\mathbf{i}$ is of type ∞, it is a leaf and not in C. If $n=2$, $\mathbf{i}$ is of type 0, one of its offsprings, say $\mathbf{j}$, is a leaf of type ∞. As explained in §3.2, the core C is not random and we may consider a particular realization of the leaf removal process. Consider a realization, where the transition triggered by $\mathbf{j}$ occurs. Then $\{\mathbf{ij}\} \in M$ and in particular $\mathbf{i} \notin C$. The general induction step is similar. □

Proof of Corollary 3.1. Assume that $\varnothing \in C$ then at least one of its the neighboring vertices, say $\mathbf{i}$, is in C, we may apply Lemma 3.12 to $\mathbf{i}$. Then by Proposition 3.2, with probability one, $\lim_{t\downarrow 0} h_\mathbf{i}(t)/t = \infty$. Then from (3.10), we get, a.s. $\limsup_{t\downarrow 0} th_\varnothing(t) \le \limsup_{t\downarrow 0}\frac{t}{h_\mathbf{i}(t)} = 0$. The conclusion follows from (3.9). □

4 Convergence of the spectral measure

4.1 Local weak convergence In this paragraph, we define formally the notion of local weak convergence introduced by Benjamini and Schramm [6], Aldous and Steele [3], see also Aldous and Lyons [2]. We follow their framework. Let $G=(V,E)$ be a locally finite graph. We also assume that G is simple (at most one edge between two vertices) and has no self loop. A path from u to v of length n, $\pi = (u_0, u_1, \cdots, u_n)$, is a sequence of vertices in V with $u_0 = u$, $u_n = v$ and $(u_k, u_{k+1}) \in E$, $k = 0 \dots n-1$. If $G=(V,E)$ is connected, we define $d_G(u,v)$, the distance on V associated to G, the infimum of the length of the paths from u to v. The ball of radius t and center u is naturally

$$B_G(u,t) := \{v \in V : d_G(u,v) < t\}.$$

A rooted graph $(G,\varnothing)$ is a graph G with a distinguished vertex $\varnothing$ of V, called the root. A rooted isomorphism of rooted graphs is an isomorphism of the graph that takes the root of one to the root of the other. If G is a graph $[G,\varnothing]$ will denote the class of rooted graphs that are rooted graph isomorphic to $(G,\varnothing)$. Let $\mathcal{G}_*$ denote the set of all $[G,\varnothing]$, with G ranging over connected locally finite graphs. With the terminology of combinatorics, $\mathcal{G}_*$ is simply the set of rooted unlabeled connected locally finite graphs. For a rooted graph $(G,\varnothing)$ and any $t>0$, let $(G,\varnothing)[t]$ denote the graph whose vertex set is $B_G(\varnothing,t)$ and whose edge set consists of the edges of G that have both vertices in $B_G(\varnothing,t)$.

We define a metric on $\mathcal{G}_*$ by letting the distance between $[G_1,\varnothing_1]$ and $[G_2,\varnothing_2]$ be $1/(1+T)$, where $T = \sup\{t > 0$, there exists a rooted isomorphism from $(G_1,\varnothing_1)[t]$ to $(G_2,\varnothing_1)[t]\}$. $\mathcal{G}_*$ is separable and complete in this

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

metric, see §2 in [2]. For probability measures ρ, ρ_n on $\mathcal{G}_*$, we write $\rho_n \to \rho$ when ρ_n converges weakly to ρ with respect to this metric. Here $\mathcal{G}_*$ is given the Borel σ–algebra and weak convergence is defined as usual by $\rho_n(f) \to \rho(f)$ for all bounded continuous functions $f : \mathcal{G}_* \to \mathbb{R}$. This is called the *local weak* convergence.

As mentioned in the Introduction, there are three important examples of graphs on $[n]$ that converge weakly to Galton-Watson trees. The Erdős-Rényi graph on n vertices with parameter λ rooted at $1 \in [n]$ converges to the GWT with degree distribution $Poi(\lambda)$. For $k \geq 3$, the uniform k-regular graph on n vertices rooted at $1 \in [n]$ converges weakly with the infinite k-regular tree as local limit. More generally, the random graph with asymptotic degree distribution F_* rooted at $1 \in [n]$ converges to the GWT with degree distribution F_* provided than $\int xF_*(dx) < \infty$. Note that in the above examples, the vertices are exchangeable and the choice of the root 1 is arbitrary. The same result holds with any root chosen independently from the graph.

Let G_n denote a graph on $[n]$ with adjacency matrix A_n. The resolvent of the adjacency matrix of G_n is denoted by $R_{G_n}(z) = (A_n - zI)^{-1}$.

Proposition 4.1. *Let $\mathcal{T}$ be a GWT with degree distribution F_*. If (A) holds and $[G_n, 1]$ converges weakly to $[\mathcal{T}, \varnothing]$ then for all $z \in \mathbb{C}_+$, $R_{G_n}(z)_{11}$ converges weakly to $\langle R_{\mathcal{T}}(z)e_\varnothing, e_\varnothing\rangle = m_{\mu_\varnothing}(z)$, where $\mu_\varnothing$ is the spectral measure at the root of $\mathcal{T}$.*

Proof. As in Section 2.1, we may assume that $V = \mathbb{N}$ is the vertex set of $\mathcal{T}$, we define $A = A(\mathcal{T})$ the adjacency operator of $\mathcal{T}$ and extend A_n on all $\mathbb{N}$ by setting $\langle Ae_v, e_u\rangle = A_{u,v}$ if $1 \leq u, v \leq n$ and 0 otherwise. We also define $\mathcal{D}$ as the set of vectors with finite support in $L^2(\mathbb{N})$, it is dense in $L^2(\mathbb{N})$. By the Skorokhod Representation Theorem, we can assume that $[G_n, 1]$ and $[\mathcal{T}, \varnothing]$ are defined on a common probability space and that the convergence holds almost surely. Then, there exists a (random) sequence σ_n of injective mappings from $[n]$ to $\mathbb{N}$ and $t_n \in \mathbb{R}$ tending to ∞ such that σ_n is a rooted isomorphism from $(G_n, 1)[t_n]$ to $(\mathcal{T}, \varnothing)[t_n]$. Hence, extending in an arbitrary bijective way σ_n to $\mathbb{N}$, we deduce that for all $\phi \in \mathcal{D}$, $\sigma_n A_n \sigma_n^{-1}\phi$ converges to $A\phi$. Since A_n and A are self adjoint operators, we can apply Theorem VIII.25(a) in Reed and Simon [21] and deduce that $\sigma_n A_n \sigma_n^{-1}$ converges in the strong resolvent sense to A. This concludes the proof. □

Let G_n be a random graph on $[n]$, let $U(G)$ denote the distribution on $\mathcal{G}^*$ obtained by choosing a uniform random vertex $\varnothing \in [n]$ of G as root. Note that there are two levels of randomness here, in the uniformly random vertex $\varnothing$ and in the randomness of the graph. We also define $U_2(G_n)$ as the distribution on $\mathcal{G}^* \times \mathcal{G}^*$ of the pair of rooted graphs $((G_n, \varnothing_1), (G_n, \varnothing_2))$ where $(\varnothing_1, \varnothing_2)$ is a uniform random pair of vertices of G. Finally let μ_n be the spectral measure of A_n the adjacency matrix of G_n.

Corollary 4.1. *Assume that $U(G_n)$ converges weakly to $[\mathcal{T}, \varnothing]$, a GWT with degree distribution F_* and that (A) holds. Then $\lim_n \mathbb{E}\mu_n = \mathbb{E}\mu_\varnothing$, where $\mu_\varnothing$ is the spectral measure at the root of $\mathcal{T}$. Moreover, if $U_2(G_n)$ converges weakly to $([\mathcal{T}_1, \varnothing], [\mathcal{T}_2, \varnothing])$, two independent copies of $[\mathcal{T}, \varnothing]$, then in probability, $\lim_n \mu_n = \mathbb{E}\mu_\varnothing$.*

The assumption on $U_2(G_n)$ is easily checked in the above mentioned cases : the Erdős-Rényi graph and the random graph with asymptotic degree distribution F_*. This corollary implies that the study of the limiting spectral measure of random tree-like graphs boils down to the study of the spectral measure of a GWT at the root which is described by RDE (2.6) and (2.7). Note however that this result does not give the full statement of Theorem 1.1(i), the almost sure convergence will be considered later. In [8], we proved a stronger version of the above corollary which states that $\lim_n \mu_n$ converges to the measure $\mathbb{E}\mu_\varnothing$ with a second moment condition on F_*.

Proof of Corollary 4.1. By definition the Stieltjes transform of μ_n is equal to, for all $z \in \mathbb{C}_+$,

$$\begin{aligned} m_{\mu_n}(z) &= \frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda_i - z} = \frac{1}{n}\mathrm{tr}(A_n - zI)^{-1} \\ &= \frac{1}{n}\sum_{i=1}^n R_{G_n}(z)_{ii} = \mathbf{E}R_{G_n}(z)_{\varnothing\varnothing}, \end{aligned} \tag{4.21}$$

where the expectation $\mathbf{E}$ is with respect to the uniformly chosen root $\varnothing$. By Proposition 4.1, $R_{G_n}(z)_{\varnothing\varnothing}$ converges weakly to $m_{\mu_\varnothing}(z)$. Recall that $|R_{G_n}(z)_{\varnothing\varnothing}| \leq (\Im z)^{-1}$, thus taking expectation (with respect to the graph randomness), we get for all $z \in \mathbb{C}_+$,

$$\lim_n \mathbb{E}m_{\mu_n}(z) = \mathbb{E}m_{\mu_\varnothing}(z).$$

By linearity, $\mathbb{E}m_\mu = m_{\mathbb{E}\mu}$, it follows that

$$\lim_n \mathbb{E}\mu_n = \mathbb{E}\mu_\varnothing.$$

Assume now that moreover, $U_2(G_n)$ converges weakly to $([\mathcal{T}_1, \varnothing], [\mathcal{T}_2, \varnothing])$. It suffices to check that for all $z \in \mathbb{C}^+$, $m_{\mu_n}(z) - \mathbb{E}m_{\mu_n}(z)$ converges in $L^2(\mathbb{P})$ to 0. Let $\overline{R}_{G_n}(z)_{ii} = R_{G_n}(z)_{ii} - \mathbb{E}R_{G_n}(z)_{ii}$, from Equation (4.21), we have

$$\mathbb{E}\left(m_{\mu_n}(z) - \mathbb{E}m_{\mu_n}(z)\right)^2 =$$

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

$$\frac{1}{n^2}\mathbb{E}\left(\sum_{i\neq j}\overline{R}_{G_n}(z)_{ii}\overline{R}_{G_n}(z)_{jj}+\sum_{i=1}^{n}\overline{R}_{G_n}(z)_{ii}^2\right)$$
$$= \mathbb{E}\left[\overline{R}_{G_n}(z)_{\varnothing_1\varnothing_1}\overline{R}_{G_n}(z)_{\varnothing_2\varnothing_2}\right],$$

where $(\varnothing_1, \varnothing_2)$ is a uniform pair of vertices. Now since $U_2(G_n)$ converges weakly to $([\mathcal{T}_1, \varnothing], [\mathcal{T}_2, \varnothing])$, we deduce that $(\overline{R}_{G_n}(z)_{\varnothing_1\varnothing_1}, \overline{R}_{G_n}(z)_{\varnothing_2\varnothing_2})$ converges weakly to two independent copies of $\langle R_{\mathcal{T}}(z)e_\varnothing, e_\varnothing\rangle - \mathbb{E}\langle R_{\mathcal{T}}(z)e_\varnothing, e_\varnothing\rangle$. It follows that $\lim_n \mathbb{E}\left[\overline{R}_{G_n}(z)_{\varnothing_1\varnothing_1}\overline{R}_{G_n}(z)_{\varnothing_2\varnothing_2}\right] = 0$. □

4.2 Main result : convergence of the rank. We are now in position to state the main result of this paper. We consider a sequence of graphs G_n on $[n]$ which converges to $[\mathcal{T}, \varnothing]$, a GWT with degree distribution F_*. We assume that $\sum_k kF_*(k) < \infty$, let F be defined by (2.2). We define the generating functions,

$$\varphi_*(x) = \sum_k F_*(k)x^k \quad , \quad \varphi(x) = \sum_k F(k)x^k.$$

Let π_0 be the smallest solution of $x = 1 - \varphi(1 - \varphi(x))$. We set $\pi_\infty = \varphi(\pi_0)$.

THEOREM 4.1. *Assume that $U_2(G_n)$ converges weakly to $([\mathcal{T}_1, \varnothing], [\mathcal{T}_2, \varnothing])$, two independent copies of $[\mathcal{T}, \varnothing]$, a GWT with degree distribution F_* and $\varphi'_*(1) < \infty$. Then if $\varphi'(\pi_0)\varphi'(1 - \pi_\infty) < 1$, in probability,*

$$\lim_{n\to\infty}\frac{\text{rank}A_n}{n} = 2 - \varphi_*(\pi_0) - \varphi_*(1-\pi_\infty) - \varphi'_*(1)\pi_\infty(1-\pi_0).$$

Note that if $U(G_n)$ converges weakly to $[\mathcal{T}, \varnothing]$, then the statement holds in expectation, not in probability. The proof of Theorem 4.1 is divided into two parts, we first prove it under the additional assumption (A) and then we use a truncation argument to extend it (see [9]). The first part of the proof follows from Theorem 1.3 and the following proposition.

PROPOSITION 4.2. *We assume that $U_2(G_n)$ converges weakly to $([\mathcal{T}_1, \varnothing], [\mathcal{T}_2, \varnothing])$, two independent copies of $[\mathcal{T}, \varnothing]$ a GWT with degree distribution F_*. Assume moreover that Assumption (A) holds and $\varphi'(\pi_0)\varphi'(1 - \pi_\infty) < 1$. Then in probability,*

$$\lim_n \frac{\text{rank}(A_n)}{n} = 1 - \mathbb{E}\mu_\varnothing(\{0\}).$$

Proof. Recall that $\frac{\text{rank}(A_n)}{n} = 1 - \mu_n(\{0\})$. From Corollary 4.1, we have in probability,

$$\limsup_n \mu_n(\{0\}) \le \mathbb{E}\mu_\varnothing(\{0\}).$$

It is thus sufficient to prove that

$$\liminf_n \mathbb{E}\mu_n(\{0\}) \ge \mathbb{E}\mu_\varnothing(\{0\}).$$

We now relate the leaf removal process with some spectral properties of A, the adjacency matrix of G. The following identity was used in [5] and a proof can be found in [12]:

LEMMA 4.1. *If G is a finite graph, let $A(C)$ denote the adjacency matrix of the core C of G, then* $\dim\ker(A) = |U| + \dim\ker(A(C))$, *where $|U|$ is the number of uncovered vertices in the leaf-removal process defined in Section 3.2.*

If the vertex set of G_n is $[n]$ and its spectral measure is μ_n, we have

$$\mu_n(\{0\}) \ge \frac{|U|}{n} = \text{P}_{G_n}(\varnothing \in U),$$

where the first inequality follows from Lemma 4.1 and the last equality follows from the fact that under P_{G_n}, the set of uncovered vertices U is uniformly distributed among all such sets with minimal cardinality.

Let $k \ge 1$, now by Skorokhod representation theorem, we can assume that there exists N such that for $n \ge N$, there exists an isomorphism from $G_n \cap B(\varnothing, k+1)$ to $\mathcal{T} \cap B(\varnothing, k+1) = \mathcal{T}^{k+1}$. Let $K_{\mathcal{T}}$ be the a.s. finite variable in Lemma 3.3, note that $K_{\mathcal{T}}$ is doubly stochastic since $\mathcal{T}$ itself is random. $\text{P}_{G_n}(\varnothing \in U_k)$ is measurable with respect to $G_n \cap B(\varnothing, k+1)$. Thus for $n \ge N$:

$$\begin{aligned}\mu_n(\{0\}) &\ge \text{P}_{G_n}(\varnothing \in U_k)\mathbf{1}(K_{\mathcal{T}} \le k)\\ &= \text{P}_{\mathcal{T}^{k+1}}(\varnothing \in U_k)\mathbf{1}(K_{\mathcal{T}} \le k)\end{aligned}$$

Hence, by Lemma 3.3

$$\begin{aligned}\liminf_n \mathbb{E}\mu_n(\{0\}) &\ge \mathbb{E}\left[\text{P}_{\mathcal{T}}(\varnothing \in U)\mathbf{1}(K_{\mathcal{T}} \le k)\right]\\ &= \mathbb{E}\left[\mu_\varnothing(\{0\})\mathbf{1}(K_{\mathcal{T}} \le k)\mathbf{1}(\varnothing \notin C)\right].\end{aligned}$$

The above inequality holds for all k. However, as k goes to infinity, $\mu_\varnothing(\{0\})\mathbf{1}(K_{\mathcal{T}} \le k)\mathbf{1}(\varnothing \notin C)$ converges a.s. to $\mu_\varnothing(\{0\})\mathbf{1}(\varnothing \notin C)$. Hence, by the dominated convergence Theorem:

$$\begin{aligned}\liminf_n \mathbb{E}\mu_n(\{0\}) &\ge\\ &\lim_k \mathbb{E}\left[\mu_\varnothing(\{0\})\mathbf{1}(K_{\mathcal{T}} \le k)\mathbf{1}(\varnothing \notin C)\right]\\ &= \mathbb{E}\left[\mu_\varnothing(\{0\})\mathbf{1}(\varnothing \notin C)\right].\end{aligned}$$

Finally, by Corollary 3.1, a.s. $\mu_\varnothing(\{0\})\mathbf{1}(\varnothing \in C) = 0$ and we deduce

$$\liminf_n \mathbb{E}\mu_n(\{0\}) \ge \mathbb{E}\mu_\varnothing(\{0\}).$$

□

We proved so far that Theorem 4.1 holds under the additional assumption (A). We remove this assumption in [9] by a truncation argument.

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

4.3 Proof of Theorem 1.1 and 1.2 If F_* is a Poisson distribution then we simply have $F = F_*$, $\varphi = \varphi_*$. The generating function of F is

$$\varphi(x) = \exp(\lambda(x-1)).$$

As usual, we define π_0 as the smallest solution of the equation $x = 1 - \exp(-\lambda(\exp(-\lambda(1-x))$. So that, with the notation of Theorem 1.2,

$$\pi_\infty = q \quad \text{and} \quad \pi_0 = 1 - \exp(-\lambda q).$$

If $0 \leq \lambda \leq e$ then $q = \exp(-\lambda q)$ is solution of the Lambert equation, and $\pi_c = 0$. If $\lambda > e$ then $\pi_c > 0$. Since $\varphi_* = \varphi$ and $\pi_\infty = \varphi(\pi_0)$, $\pi_0 = 1 - \varphi(1-\pi_\infty)$, we have the identity $\varphi_*(\pi_0) + \varphi_*(1-\pi_\infty) - 1 + \varphi_*'(1)\pi_\infty(1-\pi_0) = q - 1 + e^{-\lambda q} + \lambda q e^{-\lambda q}$.

This is the required expression for $\mu(\{0\})$ in Theorem 1.2. It easy to check the assumption of Theorem 4.1, indeed

$$\varphi'(\pi_0)\varphi'(1-\pi_\infty) = \lambda e^{\lambda(\pi_0 - 1)} \lambda e^{-\lambda \pi_\infty} = \lambda^2 q e^{-\lambda q} \leq \lambda e^{-1}.$$

(recall that $xe^{-x} \leq e^{-1}$). Then if $\lambda > e$, we get $\varphi'(\pi_0)\varphi'(1-\pi_\infty) < 1$. Otherwise $\lambda \leq e$ and $\pi_\infty = 1 - \pi_0$, thus $\varphi'(\pi_0)\varphi'(1-\pi_\infty) = (\varphi'(\pi_0))^2 < 1$ easily holds . We can thus apply Theorem 4.1 to the sequence of Erdős Rényi graphs $(G_n)_{n\in\mathbb{N}}$.

To complete the proof Theorems 1.1 and 1.2, it remains to improve the convergence in probability into an almost sure convergence. This is done in [9].

5 Statistical physics of matchings

Let $G = (V, E)$ be a finite graph. For each $v \in V$, we denote E_v the subset of edges in E incident to v. Recall that a matching is a set of edges with distinct adjacent vertices. A matching is denoted by $\sigma = (\sigma_e) \in \{0,1\}^E$. Let $\mathcal{M}_G$ be the set of possible matching on G. Following Zdeborová and Mézard in [23], we define the energy of a matching as the number of vertices uncovered by the matching:

$$\mathcal{E}(\sigma) = |U(\sigma)| = \sum_{v\in V}(1 - \sum_{e\in E_v} \sigma_e).$$

On $\mathcal{M}_G$, we introduce a random matching S with Boltzmann probability measure

$$\mathbb{P}_G(S = \sigma) = \frac{1}{Z_G} e^{-\beta \mathcal{E}(\sigma)},$$

where β is the inverse temperature, $Z_G = \sum_{\sigma\in\mathcal{M}_G} e^{-\beta\mathcal{E}(\sigma)}$ is called the partition function and $\log Z_G$ is the free energy of the system. As β goes to infinity, the measure $\mathbb{P}_G$ converges to the uniform measure on the maximal matchings of G.

Kasteleyn [14] has developed a beautiful theory for perfect matchings on planar bipartite graphs. In this paragraph, we adapt his ideas to compute the Boltzmann measure on trees at any temperature. Let A be the adjacency matrix of a finite tree $G = (V, E)$, and for $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$,

$$R(z) = (A - zI)^{-1}$$

its resolvent. Recall that the spectrum of A is symmetric since T is a tree. We also have for all $z \in \mathbb{C}_+$ and $u \in V$, $R(-\overline{z})_{uu} = -\overline{R}(z)_{uu}$ and $R(ix)_{uu} \in i\mathbb{R}_+$ for all $x > 0$.

Proposition 5.1. *If G is a finite simple tree, then for all $\beta \in \mathbb{R}$,*

$$Z_G = i^{|V|} \det(A - ie^{-\beta} I) = \frac{1}{\det -iR(ie^{-\beta})}.$$

Proof. If S_V is the set of permutations on V, we have

$$\det(A - zI) = \sum_{\sigma\in S_V} (-1)^{|\sigma|} \prod_{i\in V} (A - zI)_{i\sigma(i)}$$
$$= \sum_{\sigma\in S_V} (-1)^{|\sigma|} \prod_{i\in V} (\mathbf{1}(\{\sigma(i), i\} \in E) - z\mathbf{1}(\sigma(i) = i)).$$

We may decompose a permutation into disjoint cycles. A permutation will give a non null term in the above sum if all its cycles are along the edges of the graph. Since G is a tree, such cycles have length one or two. Note that we may identify a matching σ with a permutation on V which is an involution : if $e = \{u, v\} \in \sigma$, we write $\sigma(u) = v$ and $\sigma(v) = u$. It follows that

$$\det(A - zI) = \sum_{\sigma\in\mathcal{M}_G} (-1)^{|\sigma|} (-z)^{|U(\sigma)|}.$$

Note that $|V| - |U(\sigma)|$ is even and $(-1)^{|\sigma|} = (-1)^{\frac{|V|-|U(\sigma)|}{2}} = (-i)^{|V|-|U(\sigma)|}$, we get

$$\det(A - ie^{-\beta}I) = \sum_{\sigma\in\mathcal{M}_G} (-i)^{|V|} e^{-\beta|U(\sigma)|} = (-i)^{|V|} Z_G.$$

□

On planar bipartite graphs, Kenyon in [15, Theorem 6] has shown that the uniform measure on perfect matchings is a determinantal process on the edges. Here, similarly, the same phenomena appears as a consequence of Proposition 5.1. If m is the subset of edges in E, let V_m denote the set of adjacent vertices of m.

Proposition 5.2. *Let $G = (V, E)$ be a simple finite tree and $m = \{u_1v_1, \cdots, u_kv_k\} \in \mathcal{M}_G$ then for all $\beta \in \mathbb{R}$,*

$$\mathbb{P}_G(m \subset S) = \det\left(-iR(ie^{-\beta})_{uv}\right)_{u,v\in V_m}.$$

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.

Moreover, if $V_0 \subset V$ *and* $V_0 \cap V_m = \emptyset$*,*

$$\mathbb{P}_G(m \subset S; V_0 \subset U(S)) = \\ e^{-\beta|V_0|} \det\left(-iR(ie^{-\beta})_{uv}\right)_{u,v\in V_m\cup V_0}.$$

Proof. The first statement follows from the second statement by setting $V_0 = \emptyset$. Let G_1 be the subgraph of G spanned by the vertices $V_1 = V\backslash(V_0 \cup V_m)$. Note that if $m \subset S$ and $V_0 \subset U(S)$ then $S = m \cup \sigma$ where $\sigma \in \mathcal{M}_{G_1}$. Let $z = ie^{-\beta}$, then from Proposition 5.1, we get

$$\begin{aligned}\mathbb{P}_G(m \subset S) &= \frac{\sum_{\sigma\in\mathcal{M}_{G_1}} e^{-\beta|I(\sigma\cup m)|}}{\sum_{\sigma\in\mathcal{M}_G} e^{-\beta|I(\sigma)|}} \\ &= \frac{i^{|V_1|}e^{-\beta|V_0|}\det(A-zI)_{|V_1}}{i^{|V|}\det(A-zI)},\end{aligned}$$

indeed $U_G(\sigma \cup m) = U_{G_1}(\sigma) \cup V_0$. Now from Jacobi Formula, $\det(A - zI)_{|V_1} = \det(A - zI)\det R(z)_{|V\backslash V_1}$ and the conclusion follows. □

For example, we have the formula

$$\mathbb{P}_G(u \subset U(S)) = -ie^{-\beta}R(ie^{-\beta})_{uu}. \tag{5.22}$$

Now when β tends to infinity, the left-hand side tends to $P_G(u \in U)$, the probability that u is uncovered by a random maximal matching and the right-hand term to the spectral measure associated to the vertex u.

References

[1] M. Aizenman, R. Sims, and S. Warzel. Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. *Probab. Theory Related Fields*, 136(3):363–394, 2006.

[2] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electronic Journal of Probability*, 12:1454–1508, 2007.

[3] D. Aldous and J. M. Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 1–72. Springer, Berlin, 2004.

[4] M. Bauer and O. Golinelli. On the kernel of tree incidence matrices. *J. Integer Seq.*, 3(1):Article 00.1.4, 1 HTML document (electronic), 2000.

[5] M. Bauer and O. Golinelli. Exactly solvable model with two conductor-insulator transitions driven by impurities. *Phys. Rev. Lett.*, 86:2621–2624, 2001.

[6] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13 pp. (electronic), 2001.

[7] S. Bhamidi, S. Evans, and A. Sen. Spectra of large random trees. *ArXiv*, 0903.3589, 2009.

[8] C. Bordenave and M. Lelarge. Resolvent of large random graphs. *To appear in Random Structures and Algorithms*, 0801.0155, 2008.

[9] C. Bordenave and M. Lelarge. The rank of diluted random graphs. *submitted*, arXiv:0907.4244, 2009.

[10] K. Costello and V. Vu. The rank of random graphs. *Random Structures and Algorithms*, to appear.

[11] K. P. Costello, T. Tao, and V. Vu. Random symmetric matrices are almost surely nonsingular. *Duke Math. J.*, 135(2):395–413, 2006.

[12] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of graphs.* Johann Ambrosius Barth, Heidelberg, third edition, 1995. Theory and applications.

[13] R. Karp and M. Sipser. Maximum matchings in sparse random graphs. *Proc. of the Twenty-second Annual Symposium on Foundations of Computer Science*, IEEE:364–375, 1981.

[14] P. W. Kasteleyn. Graph theory and crystal physics. In *Graph Theory and Theoretical Physics*, pages 43–110. Academic Press, London, 1967.

[15] R. Kenyon. Local statistics of lattice dimers. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(5):591–618, 1997.

[16] H. Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, 92:336–354, 1959.

[17] O. Khorunzhy, M. Shcherbina, and V. Vengerovsky. Eigenvalue distribution of large weighted random graphs. *J. Math. Phys.*, 45(4):1648–1672, 2004.

[18] A. Klein. Extended states in the Anderson model on the Bethe lattice. *Adv. Math.*, 133(1):163–184, 1998.

[19] B. D. McKay. The expected eigenvalue distribution of a large regular graph. *Linear Algebra Appl.*, 40:203–216, 1981.

[20] L. Nirenberg. *Topics in nonlinear functional analysis*, volume 6 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2001. Chapter 6 by E. Zehnder, Notes by R. A. Artino, Revised reprint of the 1974 original.

[21] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York, 1972.

[22] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[23] L. Zdeborová and M. Mézard. The number of matchings in random graph. *Journal of Statistical Mechanics*, 2006(5):P05003, 2006.

Copyright © by SIAM.
Unauthorized reproduction of this article is prohibited.