Partition constrained covering of a symmetric crossing supermodular function by a graph

Attila Bernáth * Roland Grappe † Zoltán Szigeti ‡

October 12, 2009

Abstract

Given a symmetric crossing supermodular set function p on V and a partition $\mathcal{P}$ of V, we solve the problem of finding a graph with ground set V having edges only between the classes of $\mathcal{P}$ such that for every subset X of V the cut of the graph defined by X contains at least $p(X)$ edges. The objective is to minimize the number of edges of the graph.

This problem is a common generalization of the global edge-connectivity augmentation of a graph with partition constraints, which was solved by Bang-Jensen, Gabow, Jordán and Szigeti [1] and the problem of covering a symmetric crossing supermodular set function solved by Benczúr and Frank [3]. Our problem can be considered as an abstract form of the problem of global edge-connectivity augmentation of a hypergraph by a multipartite graph, which was earlier solved by the authors [5].

1 Introduction

This paper is concerned with edge-connectivity augmentation problems in graphs, hypergraphs and abstract forms of the problems for "connectivity" set functions. For a survey, we refer to [7].

Our starting point is the problem of *global edge-connectivity augmentation of a graph*, where we have to add a minimum number of new edges to a given graph $G = (V, E)$ in order to obtain a k-edge-connected graph, for a given $k \geq 2$. A natural lower bound can be obtained as follows: for a set X of degree $d(X)$ less than k, the deficiency of X is $k - d(X)$, that is, we must add at least $k - d(X)$ edges between X and $V - X$. The deficiency of a subpartition of V is the sum of the deficiencies of its sets. By adding a new edge we may decrease the deficiency of at most two sets of this subpartition so we may decrease the deficiency of the subpartition by at most two, hence we obtain the so-called *subpartition lower bound*: $\alpha_G :=$ $\lceil$half of the maximum deficiency of a subpartition of $V\rceil$. The *minimax theorem* due to Watanabe and Nakamura [8] says that this lower bound α_G can always be achieved.

The next step is the problem of *global edge-connectivity augmentation of a hypergraph*, where we have to add a minimum number of new graph edges to a given hypergraph $\mathcal{G} = (V, \mathcal{E})$ in order to obtain a k-edge-connected hypergraph, for a given k. Of course the subpartition lower bound holds also for hypergraphs. However, a new lower bound arises: after deleting $k - 1$ hyperedges the connected components must be connected by the new graph edges, hence we obtain the *components lower bound*: $\omega_{\mathcal{G}} - 1$, where $\omega_{\mathcal{G}}$:=maximum number of connected components after deleting $k - 1$ hyperedges. The *minimax theorem* due to Bang-Jensen and Jackson [2] says that the lower bound $\max\{\alpha_{\mathcal{G}}, \omega_{\mathcal{G}} - 1\}$ can always be achieved.

Benczúr and Frank [3] considered the abstract form of this problem, namely *covering of a symmetric crossing supermodular function by a graph*: given a symmetric, crossing supermodular set function p on V, what is the minimum number of edges of a graph that covers p, that is for all subsets X of V the cut defined by X contains at least $p(X)$ edges? The *subpartition* and the *component lower bounds* can be extended for this problem: $\alpha_p :=$ $\lceil$half of the maximum of the sum of the values of the sets in a subpartition$\rceil$ and $dim(p) :=$ maximum size of a p-full partition, where a partition is p-full if each union of some of its sets, has value at least one. The *minimax theorem* due to Benczúr and Frank [3] says that the lower bound $\max\{\alpha_p, dim(p) - 1\}$ can always be achieved.

Now we consider the partition constrained versions

*MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. E-mail: bernath@cs.elte.hu. Supported by OTKA grant K60802.

†Laboratoire G-SCOP, CNRS, Grenoble INP, UJF, 46, Avenue Félix Viallet, Grenoble, France, 38000. E-mail: roland.grappe@g-scop.inpg.fr.

‡Laboratoire G-SCOP, CNRS, Grenoble INP, UJF, 46, Avenue Félix Viallet, Grenoble, France, 38000. E-mail: zoltan.szigeti@g-scop.inpg.fr.

of the above problems.

Bang-Jensen et al. [1] introduced the problem of *partition constrained global edge-connectivity augmentation of a graph*: given a graph $G = (V, E)$, an integer $k \geq 2$ and a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of V, what is the minimum number of new edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected graph? We have a new *partition constrained lower bound* because we can not add a new edge in P_i for every i: $\beta_G^i :=$ maximum deficiency of a subpartition of P_i. The *minimax theorem* due to Bang-Jensen et al. [1] says that the lower bound $\max\{\alpha_G, \beta_G^1, ..., \beta_G^r\}$ can be achieved, except if the graph contains a C_4- or a C_6-configuration, in which case one more edge is needed.

Grappe et al. [5] considered the problem of *partition constrained global edge-connectivity augmentation of a hypergraph*: given a hypergraph $\mathcal{G} = (V, \mathcal{E})$, an integer k and a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of V, what is the minimum number of new graph edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected hypergraph? The *minimax theorem* due to Grappe et al. [5] says that the lower bound $\max\{\alpha_\mathcal{G}, \beta_\mathcal{G}^1, ..., \beta_\mathcal{G}^r, \omega_\mathcal{G} - 1\}$ can be achieved, except if the hypergraph contains a $\mathcal{C}_4$- or a $\mathcal{C}_6$-configuration, extension of the above configurations, in which case one more edge is needed.

We emphasize that the above mentioned papers contain polynomial algorithms solving the corresponding problems.

In this paper we solve the abstract version of the above problem, namely the *partition constrained covering of a symmetric crossing supermodular function by a graph*: given a symmetric, crossing supermodular set function p on V and a partition $\mathcal{P}$ of V, what is the minimum number of edges, between different members of $\mathcal{P}$, resulting in a graph that covers p? We show that the lower bound $\max\{\alpha_p, \beta_p^1, ..., \beta_p^r, dim(p)-1\}$ can be achieved except if a C_4^*-, C_5^*- or a C_6^*-configuration exists for $(p, \mathcal{P})$, in which case one more edge is needed. This result strictly generalizes the partition constrained problem for hypergraphs. Indeed, a new configuration arises, and it extends an application of Benczúr and Frank [3] that can not be treated in the framework of hypergraphs, see Section 4.3.

We will follow the classical approach of Frank [4]. First we treat the so called *degree-specified version* of the above problem in Section 3 which is the following: given a symmetric, crossing supermodular set function p on V, a partition $\mathcal{P}$ of V, and a function $m : V \to \mathbb{Z}_+$ (also called *degree-specification*) and the task is to decide whether a graph G covering p exists that has only edges connecting different members of $\mathcal{P}$ and satisfies $d_G(v) = m(v)$ for every $v \in V$. We show the natural necessary conditions of the existence of such a graph and we characterize the exceptional structures (called *obstacles*): these are the only cases that satisfy these conditions but still there does not exist a solution. Then in Section 4 we turn to the above given *minimization version* of our problem and we solve it the following way. Firstly, in Section 4.1 we try to find a degree-specification m satisfying the necessary conditions given earlier and with $m(V)$ as small as possible, but avoiding the obstacles: these necessary conditions correspond to natural lower bounds for $m(V)$. Finally, in Section 4.2 we exhibit the structures (configurations) where we can only avoid creating obstacles if we allow one more edge than what is given by the natural lower bounds. This approach provides a polynomial algorithm, see Section 5. We mention that many of the proofs are omitted here.

2 Definitions

Graphs and set functions: Let us be given a finite ground set V. For a graph $G = (V, E)$ and $X, Y \subseteq V$, $\boldsymbol{d_G(X, Y)}$ denotes the number of edges between $X - Y$ and $Y - X$, $\boldsymbol{\overline{d}_G(X, Y)}$$= d_G(X, V - Y)$, and $\boldsymbol{d_G(X)}$$= d_G(X, V - X)$. It is well-known that the following equalities hold for all $X, Y \subseteq V$.

$$(2.1) \quad d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y),$$

$$(2.2) \quad d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2\overline{d}_G(X, Y).$$

Two subsets X and Y of V are **crossing** if none of $X - Y$, $Y - X$, $X \cap Y$ and $V - (X \cup Y)$ is empty. A set function $p : 2^V \to \mathbb{Z}$ is **symmetric** if $p(X) = p(V - X)$ for all $X \subseteq V$, and is called **crossing supermodular** if it satisfies (2.3) for all crossing sets $X, Y \subseteq V$ with $p(X), p(Y) > 0$. A symmetric crossing supermodular set function p also satisfies (2.4) for such set pairs X, Y.

$$(2.3) \quad p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y).$$
$$(2.4) \quad p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

Let $G = (V, E)$ be a graph and p a symmetric crossing supermodular set function on V. By (2.1), the function $-d_G$ is symmetric crossing supermodular, hence so is $\boldsymbol{p_G}$$= p - d_G$. Moreover, by (2.3) and (2.1) and respectively by (2.4) and (2.2), for crossing p-positive subsets X and Y of V, the following hold.

$$(2.5) \quad p_G(X) + p_G(Y) \leq p_G(X \cap Y) + p_G(X \cup Y) - 2d_G(X, Y),$$

$$(2.6)\quad p_G(X)+p_G(Y) \le p_G(X-Y)+p_G(Y-X) - 2\bar{d}_G(X,Y).$$

The graph G is said to **cover** the function p if (2.7) holds and it is said to **satisfy** the degree specification $m : V \to \mathbb{Z}_+$ if (2.8) holds. Moreover, m is called ***p*-feasible** if (2.9) holds, where $\boldsymbol{m(X)} = \sum\{m(x) : x \in X\}$.

$$(2.7)\quad d_G(X) \ \ge\ p(X) \ \text{ for all } \emptyset \ne X \subset V,$$
$$(2.8)\quad d_G(v) \ =\ m(v) \ \text{ for all } v \in V,$$
$$(2.9)\quad m(X) \ \ge\ p(X) \ \text{ for all } \emptyset \ne X \subset V.$$

A partition $\{X_1, \ldots, X_t\}$ of V is called ***p*-full** if $p(\cup_{i\in I} X_i) > 0$ for all nonempty $I \subsetneq \{1, \ldots, t\}$ and $p(X_j) = 1$ for some $j \in \{1, \ldots, t\}$. The maximum cardinality of a p-full partition is the **dimension** of p and is denoted by $\mathbf{dim}(\boldsymbol{p})$.

For an element $v \in V$, $\boldsymbol{\chi_v}$ denotes the incidence vector of the set $\{v\}$.

Operations: Let $m : V \to \mathbb{Z}_+$ be a p_G-feasible degree specification. An element $v \in V$ is called ***m*-positive** if $m(v) > 0$. Let x, y be two different m-positive elements and $uv \in E$ an edge of G. The **splitting off** at x, y consists of replacing m by $\bar{m}$ and p_G by $p_{\bar{G}}$, where

$$(2.10)\qquad \bar{m} = m - \chi_x - \chi_y \text{ and } \bar{G} = G + xy.$$

The **unsplitting** of uv is the reverse of splitting off: replace m by $\hat{m} = m + \chi_u + \chi_v$, and p_G by $p_{\hat{G}}$, where $\hat{G} = G - uv$. Note that $\hat{m}$ is $p_{\hat{G}}$-feasible. The **one-change** at x, u, v, y is defined as unsplitting uv and splitting off at x, u and at v, y, that is replacing m by $m'' = m - \chi_x - \chi_y$ and p_G by $p_{G''}$, where $G'' = G - uv + ux + vy$. Any of the above operations is called ***p*$_G$-admissible** if the new degree specification is feasible with the new set function.

A set $X \subset V$ is called **tight** if $m(X) = p(X)$ and **dangerous** if $m(X) \le p(X) + 1$.

Partition constraint: Let $\mathcal{P} = \{P_1, \ldots P_r\}$ be a partition of V. An element $v \in V$ that belongs to some P_i is said to be of color i. The notation $\boldsymbol{c(v)} = i$ will also be used for $v \in P_i$. A p_G-feasible degree specification m is called $\boldsymbol{(p_G, \mathcal{P})}$**-feasible** if

$$(2.11)\qquad m(V) \quad \text{is} \quad \text{even},$$
$$(2.12)\qquad m(P_i) \ \le\ \frac{m(V)}{2} \ \text{ for all } P_i \in \mathcal{P}.$$

We call $P_i \in \mathcal{P}$ **dominating** if $m(P_i) = \frac{m(V)}{2}$. A pair of m-positive elements is called **rainbow** if they are of different colors and any dominating color class contains one of them. A splitting off, or a one-change operation is called ***p*$_G$-allowed** if it is p_G-admissible and it uses only rainbow pairs. A **complete *p*$_G$-allowed splitting off** is a sequence of allowed splitting off that decreases $m(V)$ to zero.

3 Degree specified version

In this section we solve the degree specified version of our problem: given a symmetric crossing supermodular function $p : 2^V \to \mathbb{Z}$, a degree specification $m : V \to \mathbb{Z}_+$ and a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of V, find a graph G on V covering p and satisfying m such that the edges of G connect distinct classes of $\mathcal{P}$. Note that finding such a graph is equivalent to finding a complete p-allowed splitting off.

3.1 Necessary conditions First we provide necessary conditions for the existence of a complete p-allowed splitting off.

LEMMA 1. *Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular function, $m : V \to \mathbb{Z}_+$ a degree specification and $\mathcal{P} = \{P_1, \ldots, P_r\}$ a partition of V. If there exists a complete p-allowed splitting off, then m is $(p, \mathcal{P})$-feasible and* (3.13) *holds.*

$$(3.13)\qquad m(V) \ \ge\ 2(\dim(p) - 1).$$

Proof. Let $G = (V, E)$ be a graph obtained by a complete p-allowed splitting off. Note that $m(V) = 2|E|$, hence (2.11) holds. Since every splitting off is rainbow, we get (2.12). By (2.8) and (2.7), we have $m(X) = \sum\{m(x) : x \in X\} = \sum\{d_G(x) : x \in X\} \ge d_G(X) \ge p(X)$, hence we obtain (2.9). Let $\mathcal{V} = \{V_1, \ldots, V_{\dim(p)}\}$ be a p-full partition of V. Let G' be the graph obtained from G by contracting each set of $\mathcal{V}$ into a singleton. Since G covers p and $\mathcal{V}$ is p-full, G' is connected. Therefore G' and hence G has at least $\dim(p) - 1$ edges. By $m(V) = 2|E|$, we get (3.13).

3.2 Obstacles It turns out that the above necessary conditions are not sufficient. Exceptional structures must be forbidden in order to get the sufficiency. We describe these structures below. An obstacle is a partition $\mathcal{A}$ of V satisfying two types of conditions. On the one hand, the p- and m-values of the sets in the partition $\mathcal{A}$ fulfill rigorous conditions. On the other hand, the partition $\mathcal{A}$ is closely related to the partition $\mathcal{P}$. Note that the C_5^*-obstacle arises only for our abstract form of the problem, it does not exist in the framework of graphs or hypergraphs.

Let $p : 2^V \to \mathbb{Z}$ be a symmetric crossing supermodular function, $m : V \to \mathbb{Z}_+$ a degree specification and $\mathcal{P} = \{P_1, \ldots, P_r\}$ a partition of V. We assume that m is

$(p, \mathcal{P})$-feasible: note that we don't have to assume (3.13) since it will follow from the properties of the obstacles.

DEFINITION 2. *A partition* $\mathcal{A} = \{A_1, \ldots, A_4\}$ *of* V *is called a* $\boldsymbol{C_4^*}$***-obstacle for*** $\boldsymbol{(p, \mathcal{P}, m)}$ *if*

1. (a) $p(A_i) = m(A_i)$ *for* $i = 1, \ldots, 4$,
   (b) $m(A_i \cup A_{i+1}) - p(A_i \cup A_{i+1})$ *is odd for* $i = 1, \ldots, 4$,
   (c) $p(A_{i-1} \cup A_i) + p(A_i \cup A_{i+1}) = p(A_{i-1}) + p(A_{i+1})$ *for* $i = 1, \ldots, 4$,
   (d) $p(A_i \cup A_{i+2}) \le 0$ *for* $i = 1, 2$,
2. *there exist* $l \in \{1, 2\}$ *and a dominating* $P \in \mathcal{P}$ *such that the* m*-positive elements of* $A_l \cup A_{l+2}$ *are the* m*-positive elements of* P.

DEFINITION 3. *A partition* $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ *of* V $(t \ge 1)$ *is called a* $\boldsymbol{C_5^*}$***-obstacle for*** $\boldsymbol{(p, \mathcal{P}, m)}$ *if*

1. (a) $p(A_i) = m(A_i) = 1$ *for* $i = 1, \ldots, 4$,
   (b) $p(B_j) = m(B_j) = 2$ *for* $j = 1, \ldots, t$,
   (c) $p(A_i \cup B_j) = p(A_i \cup A_{i+1}) = 1$ *for* $i = 1, \ldots, 4$ *and* $j = 1, \ldots, t$,
   (d) $p(A_i \cup A_{i+2}) \le 0$ *for* $i = 1, 2$,
2. (a) *either there exist* $l \in \{1, 2\}$ *and a dominating* $P \in \mathcal{P}$ *such that for* $j = 1, \ldots, t$,
   $m(P \cap A_l) = m(P \cap A_{l+2}) = m(P \cap B_j) = 1$,
   (b) *or there exist* $j_0 \in \{1, \ldots, t\}$ *and distinct* $P_{k_1}, P_{k_2} \in \mathcal{P}$ *such that for* $i = 1, 2$,
   $m(P_{k_i}) = m(V)/2 - 1$,
   $m(P_{k_i} \cap A_i) = m(P_{k_i} \cap A_{i+2}) = m(P_{k_i} \cap B_j) = 1$ *for* $j \in \{1, \ldots, t\} - j_0$.

DEFINITION 4. *A partition* $\mathcal{A} = \{A_1, \ldots, A_6\}$ *of* V *is called a* $\boldsymbol{C_6^*}$***-obstacle for*** $\boldsymbol{(p, \mathcal{P}, m)}$ *if*

1. (a) $m(A_i) = p(A_i) = 1$ *for* $i = 1, \ldots, 6$,
   (b) $p(A_i \cup A_{i+1}) = 1$ *for* $i = 1, \ldots, 6$,
   (c) $p(A_i \cup A_{i+j}) \le 0$ *for* $i = 1, \ldots, 6$ *and* $j = 2, 3, 4$,
2. *there exist distinct* $P_{k_i} \in \mathcal{P}$ *such that the* m*-positive elements of* $A_i \cup A_{i+3}$ *are the* m*-positive elements of* P_{k_i} *for* $i = 1, 2, 3$.

The following lemma shows that two distinct obstacles may not exist simultaneously.

LEMMA 5. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be two partitions of* V. *If each satisfies one of Definition 2.1, 3.1, and 4.1 then* $\mathcal{A} = \mathcal{A}'$ *is composed of maximal tight sets and it satisfies exactly one of these conditions.*

The following results motivate the definition of the obstacles.

LEMMA 6. *If a partition of* V *satisfies Definition 2.1 (resp. Definition 3.1 or Definition 4.1) but does not satisfy Definition 2.2 (resp. Definition 3.2 or Definition 4.2), then there exists a complete* p*-allowed splitting off.*

LEMMA 7. *Let* $p : 2^V \to \mathbb{Z}$ *be a symmetric crossing supermodular function,* $m : V \to \mathbb{Z}$ *a degree specification and* $\mathcal{P} = \{P_1, \ldots, P_r\}$ *a partition of* V. *If* $\mathcal{A}$ *is an obstacle for* $(p, \mathcal{P}, m)$, *then there is no complete* p*-allowed splitting off.*

Proof. In this proof, a_i will denote an m-positive element of A_i.

1. $\mathcal{A}$ is a C_4^*-obstacle. By Definition 2.2, every rainbow splitting off involves one m-positive element of $A_1 \cup A_3$ and one of $A_2 \cup A_4$. Thus, after any sequence of allowed splitting off, $m(A_i \cup A_{i+1}) - p(A_i \cup A_{i+1})$ decreases by an even number for $i = 1, \ldots, 4$, and then, by (2.9) and Definition 2.1b, it remains at least 1. Then, by Definition 2.1a and Definition 2.1c, so does $m(A_i)$ for $i = 1, \ldots, 4$, hence $m(V) > 0$ and the lemma follows.
2. $\mathcal{A}$ is a C_5^*-obstacle. It can be shown that after a p-allowed splitting off, we obtain a C_4^*- or a C_5^*-obstacle. Therefore, Case 2 reduces to Case 1.
3. $\mathcal{A}$ is a C_6^*-obstacle. Note that by Definition 4.1a and 4.1b, the splitting off at a_i, a_{i+1} is not p-admissible, and by Definition 4.2, the splitting off at a_i, a_{i+3} is not rainbow. Therefore the only p-allowed splitting off are at a_i, a_{i+2}, for $i = 1, \ldots, 6$. Perform a p-allowed splitting off at a_i, a_{i+2} for some $i = 1, \ldots, 6$. Applying (2.3) to $A_i \cup A_{i+1}$ and $A_{i+1} \cup A_{i+2}$ implies that their union becomes tight and we are back to Case 1 with the C_4^*-obstacle $\{A_i \cup A_{i+1} \cup A_{i+2}, A_{i+3}, A_{i+4}, A_{i+5}\}$.

3.3 Splitting off theorem We will now prove our new splitting off result, which may be stated as follows.

THEOREM 8. *Let* $p : 2^V \to \mathbb{Z}$ *be a symmetric crossing supermodular set function,* $m : V \to \mathbb{Z}_+$ *a degree specification and* $\mathcal{P}$ *a partition of* V. *There exists a complete* p*-allowed splitting off if and only if* m *is* $(p, \mathcal{P})$*-feasible,* $m(V) \ge 2(dim(p) - 1)$ *and there exists no* C_4^**-,* C_5^**-, or* C_6^**-obstacle.*

Proof. [Sketch of proof] Necessity is given by Lemmas 1 and 7. We outline an algorithm to show the sufficiency. It consists of two steps, first we perform allowed splitting off as long a possible, secondly, we perform allowed one-changes as long as possible. When we get stuck, unsplitting edges is necessary. Depending on the position of the edges, many distinct cases occur.

Splitting off algorithm:

Step 1. Perform an arbitrary sequence of allowed splitting off as long as there exists one. Let m_G and p_G be the resulting degree specification and function.

Step 2. In this step, execute exactly one the following cases.

1. If there is no p_G-admissible splitting off, then perform an arbitrary longest sequence of allowed one-changes. If necessary, perform a final allowed splitting off, then we are done by Lemma 9.

2. Otherwise by Lemma 10, there exists a C_4^*-obstacle $\mathcal{A}$ for $(p_G, \mathcal{P}, m_G)$ with $m_G(A_i) = 1$ for $i = 1, \dots, 4$.

   a. If there exists a p_G-allowed one-change, perform it. Perform a final allowed splitting off and we are done.

   b. Otherwise, there are three cases.

      i. If an edge of G connects distinct members of $\mathcal{A}$ and another one lies within a member of $\mathcal{A}$, then after unsplitting these edges there exists a complete allowed splitting off by Lemma 11.

      ii. If all the edges of G connect distinct members of $\mathcal{A}$, then there exist two edges such that after their unsplitting there exists a complete allowed splitting off by Lemma 12.

      iii. The remaining case is when no edge of G connects distinct members of $\mathcal{A}$.

         A. If there exists a p_G-admissible one-change then, by Lemma 13, G contains at least two edges, and by Lemma 14, there exist two edges such that after their unsplitting there exists a complete allowed splitting off.

         B. If there exists no p_G-admissible one-change, then, by Lemma 15, there exist a partition of V satisfying Definition 3.1. As no obstacle exists for $(p, \mathcal{P}, m)$, by Lemma 6, it is easy to find a complete p-allowed splitting off.

The correctness of the above algorithm is implied by the following lemmas. The proofs of these lemmas are complicated and omitted.

Lemma 9. *If there is no p_G-admissible splitting off, then after any longest sequence of allowed one-changes we have $m'(V) \leq 2$.*

Lemma 10. *If there is a p_G-admissible splitting off but no p_G-allowed splitting off, then there exists a C_4^*-obstacle $\mathcal{A}$ for $(p_G, \mathcal{P}, m_G)$ with $m_G(A_i) = 1$ for $i = 1, \dots, 4$.*

Suppose $\mathcal{A}$ is a C_4^*-obstacle for $(p_G, \mathcal{P}, m_G)$.

Lemma 11. *If there is an edge of G between distinct members of $\mathcal{A}$ and another one within a member of $\mathcal{A}$, then after unsplitting these edges there exists a complete allowed splitting off.*

Lemma 12. *If all the edges of G connect distinct members of $\mathcal{A}$, then either there exist two edges such that after their unsplitting there exists a complete allowed splitting off or $\mathcal{A}$ is a C_4^*-obstacle for $(p, \mathcal{P}, m)$.*

Now we may assume that no edge of G connects distinct members of $\mathcal{A}$.

Lemma 13. *If there exists a p_G-admissible one-change and G contains exactly one edge, then there exists a C_6^*-obstacle for $(p, \mathcal{P}, m)$.*

Lemma 14. *If there exists a p_G-admissible one-change and G contains at least two edges, then there exist two edges such that after their unsplitting there exists a complete allowed splitting off.*

Lemma 15. *If there exists no p_G-admissible one-change, then there exists a partition of V satisfying Definition 3.1.*

4 Minimization version

In this section, we are given a symmetric crossing supermodular function p on V and a partition $\mathcal{P} = \{P_1, \dots P_r\}$ of V. We show how to find algorithmically a minimum set of edges connecting distinct members of $\mathcal{P}$ such that the resulting graph covers p.

First, we provide the lower bound. Secondly, we explain how to find a minimum $(p, \mathcal{P})$-feasible degree specification. Then, we describe the instances for which the lower bound may not be achieved. Finally, we prove our main result, see Theorem 23.

4.1 Extension Let $\boldsymbol{OPT(p, \mathcal{P})}$ be the minimum number of edges between different members of $\mathcal{P}$ resulting in a graph that covers p. Let $\boldsymbol{\Phi}$ be the maximum of the following values. By Lemma 16, Φ is a lower bound for $OPT(p,\mathcal{P})$.

$$\alpha_p = \max\{\lceil \frac{1}{2}\sum_{X\in\mathcal{X}} p(X)\rceil : \mathcal{X} \text{ subpartition of } V\},$$
$$\beta_p^i = \max\{\sum_{Y\in\mathcal{Y}} p(Y) : \mathcal{Y} \text{ subpartition of } P_i\}, \quad \text{for } i = 1,\dots r,$$
$$dim(p) - 1 = \max\{|\mathcal{V}| : \mathcal{V}\ p\text{-full partition of } V\} - 1.$$

LEMMA 16. $\Phi \le OPT(p,\mathcal{P})$.

Below, we describe how to find a $(p,\mathcal{P})$-feasible degree specification m for which $m(V) = 2\Phi$. For $u \in V$, let X_u be a minimal tight set containing u (if u is not contained in a tight set then $X_u := V$). We will use that if m is p-feasible degree specification, u is an m-positive element and $u' \in X_u$, then $m - \chi_u + \chi_{u'}$ is p-feasible.

EXTENSION ALGORITHM:

1. Let $m : V \to \mathbb{Z}_+$ be a p-feasible degree specification such that $m(V)$ is minimum applying [4, Theorem 7.1].
2. If $m(V)$ is odd, let $m := m + \chi_u$ for some $u \in V$. Then $m(V) = 2\alpha_p$.
3. If $m(V) < 2(\dim(p)-1)$, let $m := m + (2(\dim(p)-1) - m(V)) \cdot \chi_u$ for some u.
4. If some $P \in \mathcal{P}$ satisfies $m(P) > \frac{m(V)}{2}$,
   (a) If there exists an m-positive element $u \in P$ such that $X_u \nsubseteq P$, then let $m' := m - \chi_u + \chi_{u'}$, for some $u' \in X_u - P$. Now $m'(P) = m(P) - 1$. Repeat 4.
   (b) Otherwise for all m-positive element $u \in P$, we have $X_u \subseteq P$. Then let $m := m + (2m(P) - m(V)) \cdot \chi_u$, for some $u \in V - P$.
5. Stop.

The resulting degree specification m is called an **optimal extension for $(p,\mathcal{P})$**. Note that m is $(p,\mathcal{P})$-feasible and satisfies (3.13). Note that, $m(V) \ge 2\Phi$. In fact, equality holds.

LEMMA 17. *Any optimal extension m for $(p,\mathcal{P})$ satisfies $m(V) = 2\Phi$.*

4.2 Configurations

In this section we describe the functions and the partitions for which the lower bound may not be achieved. They may be classified in three different types of structures, called configurations. Two of them are natural generalizations of the configurations arising for graphs and hypergraphs. A new kind of configuration arises, it exists only in the abstract form of the problem. A configuration is a partition $\mathcal{A}$ of V satisfying two kinds of conditions. First, the p-values of the sets in the partition $\mathcal{A}$ and the lower bound Φ satisfy strict conditions. Secondly, the partition $\mathcal{A}$ is intimately related to the partition $\mathcal{P}$.

A pair (X_1, X_2) of disjoint sets of V is called a ***P*-pair** if $P \in \mathcal{P}$ and there exist a subpartition $\mathcal{X}_i$ of X_i such that $\sum_{X\in\mathcal{X}_i} p(X) = p(X_i)$ for $i = 1,2$ and $\mathcal{X}_1 \cup \mathcal{X}_2$ is a subpartition of P. A subpartition $\mathcal{X}$ of V is called ***P*-subpartition** if $P \in \mathcal{P}$ and there exist a set $X' \subseteq X$ for every $X \in \mathcal{X}$ such that $p(X') = 1$ and $\bigcup_{X\in\mathcal{X}} X' \subseteq P$.

DEFINITION 18. *A partition $\mathcal{A} = \{A_1,\dots,A_4\}$ of V is a **C_4^*-configuration for $(p,\mathcal{P})$** if*

1. (a) *$p(A_i) > 0$ for every $i = 1,\dots,4$,*
   (b) *$p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1})$ is odd for every $i = 1,\dots,4$,*
   (c) *$p(A_i \cup A_{i-1}) + p(A_i \cup A_{i+1}) = p(A_{i-1}) + p(A_{i+1})$ for every $i = 1,\dots,4$,*
   (d) *$\Phi = \frac{1}{2}\sum_{A\in\mathcal{A}} p(A) = p(A_i) + p(A_{i+2})$ for every $i = 1,\dots,4$.*
2. *There exist $P \in \mathcal{P}$ and $i \in \{1,2\}$ such that (A_i, A_{i+2}) is a P-pair.*

DEFINITION 19. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1,\dots, B_t\}$ of V $(t \ge 1)$ is a **C_5^*-configuration for $(p,\mathcal{P})$** if*

1. (a) *$p(A_i) = 1$ for every $i = 1,\dots,4$,*
   (b) *$p(B_j) = 2$ for every $j = 1,\dots t$,*
   (c) *$p(A_i \cup B_j) = p(A_i \cup A_{i+1}) = 1$ for every $i = 1,\dots,4$ and $j = 1,\dots t$,*
   (d) *$p(A_i \cup A_{i+2}) \le 0$ for $i = 1,2$,*
   (e) *$\Phi = \frac{1}{2}\sum_{A\in\mathcal{A}} p(A) = 2 + t$,*
2. (a) *Either there exist $P \in \mathcal{P}$ and $i \in \{1,2\}$ such that (A_i, A_{i+2}) is a P-pair and $\{B_1,\dots B_t\}$ is a P-subpartition,*
   (b) *Or there exist $P_{k_1}, P_{k_2} \in \mathcal{P}$ and $j \in \{1,\dots t\}$ such that (A_1, A_3) is a P_{k_1}-pair, (A_2, A_4) is a P_{k_2}-pair, and $\{B_i, i \ne j\}$ is both a P_{k_1}- and a P_{k_2}-subpartition.*

Definition 20. *A partition* $\mathcal{A} = \{A_1, \ldots, A_6\}$ *of* V *is a* $\boldsymbol{C_6^*}$***-configuration for*** $(\boldsymbol{p}, \boldsymbol{\mathcal{P}})$ *if*

1. *(a)* $p(A_i) = 1$ *for every* $i = 1, \ldots, 6$,

   (b) $p(A_i \cup A_{i+1}) = 1$ *for every* $i = 1, \ldots, 6$,

   (c) $\Phi = \frac{1}{2}\sum_{A\in\mathcal{A}} p(A) = 3$.

2. *there exist distinct* $P_{k_i} \in \mathcal{P}$ *such that* (A_i, A_{i+3}) *is a* P_{k_i}*-pair for* $i = 1, 2, 3$.

There is a strong relation between configurations and obstacles, which is shown in the two following lemmas.

Lemma 21. *If a configuration exists for* $(p, \mathcal{P})$*, then for every optimal extension* m*, there exists an obstacle for* $(p, \mathcal{P}, m)$.

Proof. Let $\mathcal{A}$ be a configuration for $(p, \mathcal{P})$ and m an optimal extension for $(p, \mathcal{P})$. Since $\sum_{A\in\mathcal{A}} p(A) \leq \sum_{A\in\mathcal{A}} m(A) = m(V) = 2\Phi = \sum_{A\in\mathcal{A}} p(A)$, we have $m(A) = p(A)$ for all $A \in \mathcal{A}$. This implies the lemma.

Lemma 22. *If no configuration exists for* $(p, \mathcal{P})$*, then there is an optimal extension* m *such that no obstacle exists for* $(p, \mathcal{P}, m)$.

Proof. [Sketch of proof:] Let m be an optimal extension for $(p, \mathcal{P})$. If there exists no obstacle for $(p, \mathcal{P}, m)$, then the lemma is proved. Suppose that there is an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$. We will use that if m is p-feasible and $u \in A \in \mathcal{A}$ is an m-positive element, then $m' = m - \chi_u + \chi_{u'}$ is p-feasible for $u' \in X_u$, $X_u \subseteq A$ and A is still a maximal tight set.

We have $2\Phi \leq m(V) = \sum_{A\in\mathcal{A}} m(A) = \sum_{A\in\mathcal{A}} p(A) \leq 2\alpha \leq 2\Phi$. Therefore there is equality everywhere and $\Phi = \frac{1}{2}\sum_{A\in\mathcal{A}} p(A)$. Definition 2.1 (resp. Definition 3.1 and Definition 4.1) for the obstacle $\mathcal{A}$ directly imply that Definition 18.1 (resp. Definition 19.1, and Definition 20.1) holds for the partition $\mathcal{A}$. Since no configuration exists for $(p, \mathcal{P})$, Definition 18.2 (resp. Definition 19.2 and Definition 20.2) does not hold for $\mathcal{A}$. We show below how it helps to destroy the obstacles.

1. If $\mathcal{A}$ is a C_4^*-obstacle, then Definition 18.2 does not hold and for every dominating $P \in \mathcal{P}$ there exists an m-positive element $u \in P$ such that $X_u \nsubseteq P$, replace m by m' for some $u' \in X_u - P$.

2. If $\mathcal{A}$ is a C_5^*-obstacle, then there are four cases depending on the classes of $\mathcal{P}$ maximizing $m(P)$ over $P \in \mathcal{P}$. We mention that some of these cases are complicated, however modifications similar to the case of the C_4^*-obstacle can be applied to destroy Definition 3.2.

3. If $\mathcal{A}$ is a C_6^*-obstacle, then there exists an m-positive element $u \in P$ such that $X_u \nsubseteq P$, then replace m by m' for some $u' \in X_u - P$.

After these modifications, Definition 2.2 (resp. Definition 3.2 and Definition 4.2) does not hold, hence $\mathcal{A}$ is not an obstacle for $(p, \mathcal{P}, m)$ anymore. As Definition 2.1 (resp. Definition 3.1 and Definition 4.1) still holds for $\mathcal{A}$, Lemma 5 implies that there is no obstacle for $(p, \mathcal{P}, m')$, so we have found the desired optimal extension.

4.3 Main theorem By exploiting the relations between configurations and obstacles and by applying our splitting off result, we may now prove our main theorem. It states that the lower bound Φ may always be achieved unless there exists a configuration.

Theorem 23. *Let* $p : 2^V \to \mathbb{Z}$ *be a symmetric crossing supermodular set function and* $\mathcal{P} = \{P_1, \ldots P_r\}$ *a partition of* V*. Then the minimum number of edges between different members of* $\mathcal{P}$ *resulting in a graph that covers* p *is* Φ *unless a configuration exist, in which case it is* $\Phi + 1$.

Proof. The following lemmas prove the theorem.

Lemma 24. $OPT(p, \mathcal{P}) \geq \Phi$*. If there exists a configuration for* p*, then the inequality is strict.*

Proof. By Lemma 16, $OPT(p, \mathcal{P}) \geq \Phi$.

Suppose there exists a configuration for $(p, \mathcal{P})$ and the inequality is not strict. Let F be a minimum set of edges such that (V, F) covers p and satisfies the partition constraint, and let m be the degree specification obtained from $m := 0$ by unsplitting every edge of F. By the minimality of F, m is an optimal extension for $(p, \mathcal{P})$. Since there is a configuration for $(p, \mathcal{P})$, by Lemma 21, there is an obstacle for $(p, \mathcal{P}, m)$. But now Theorem 8 contradicts the existence of a complete p-allowed splitting off.

Lemma 25. $OPT(p, \mathcal{P}) \leq \Phi + 1$*. If there exists no configuration for* $(p, \mathcal{P})$*, then the inequality is strict.*

Proof. If there exists no configuration for $(p, \mathcal{P})$, then by Lemma 22 there exists an optimal extension m for $(p, \mathcal{P})$ which contains no obstacle. Hence by Theorem 8 there exists a complete p-allowed splitting off and the strict inequality follows. If there exists a configuration for $(p, \mathcal{P})$, let m be an optimal extension for $(p, \mathcal{P})$. By Lemma 21, there exists an obstacle for $(p, \mathcal{P}, m)$. Replace m by $m' := m + \chi_u + \chi_v$ for some u, v without violating $m(P) \leq \frac{m(V)}{2}$ for every $P \in \mathcal{P}$. Now, the sets containing u and v may not be tight, therefore there is no obstacle for $(p, \mathcal{P}, m')$. By Theorem 8, there exists a complete allowed splitting off and the inequality follows.

Application: For an integer k and a subset T of V, a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called k-edge-connected in T if $d_{\mathcal{H}}(X) \geq k$ for all $X \subset V$ such that $X \cap T$ and $T - X$ are non empty. The problem of making a given hypergraph k-edge-connected in T by adding a minimum set of edges was solved by Bang-Jensen and Jackson [2] when $T = V$, and by Benczúr and Frank [3] for arbitrary T. Theorem 23 solves the following partition constrained version of this problem: given a hypergraph $\mathcal{H}_0 = (V, \mathcal{E}_0)$, a subset T of V, a partition $\mathcal{P}$ of T and an integer k, find a graph $G = (T, E)$ with a minimum number of edges to be added to $\mathcal{H}_0$ between distinct members of $\mathcal{P}$ such that the resulting hypergraph is k-edge-connected in T. Indeed, if we define the function $p : T \to \mathbb{Z}$ with $p(X) = \max\{k - d_{\mathcal{H}_0}(X \cup Y) : Y \subseteq V - T\}$ for any nonempty $X \subsetneq T$ and $p(\emptyset) = p(T) = 0$, then it is easy to see that p is symmetric and positively crossing supermodular, so Theorem 23 can be applied. The special case $T = V$ was earlier solved by the authors [5].

5 Algorithm

In this section, we describe an algorithm that, given $p : 2^V \to \mathbb{Z}$ a symmetric crossing supermodular set function and $\mathcal{P} = \{P_1, \dots P_r\}$ a partition of V, finds a minimum number of edges between different members of $\mathcal{P}$ resulting in a graph that covers p. It consists of three major steps, extension, then splitting off, and finally determining if a configuration exists.

Augmentation algorithm:

1. Find an optimal extension m for $(p, \mathcal{P})$ applying the extension algorithm of Section 4.1. Recall that $m(V) = 2\Phi$.
2. Apply the splitting off algorithm described in the proof of Theorem 8 to $(p, \mathcal{P}, m)$.
   (a) If there is a complete allowed splitting off, then we have found the desired graph having $\frac{m(V)}{2} = \Phi$ edges.
   (b) Otherwise, we have found an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$.
3. Apply the proof of Lemma 22 to $\mathcal{A}$.
   (a) If it finds another optimal extension m' for $(p, \mathcal{P})$ such that no obstacle exists for $(p, \mathcal{P}, m')$, then Theorem 8 provides a complete allowed splitting off and thereby the desired graph with Φ edges.
   (b) Otherwise, there exists a configuration for $(p, \mathcal{P})$. The algorithmic proof of Lemma 25 provides the desired graph with $\frac{m(V)}{2} = \Phi + 1$ edges.

The subroutine needed for finding an optimal extension and for executing the two operations used in our algorithm — splitting off and one-change — is the minimization of a crossing submodular function $m - p$, where m is modular and p is symmetric crossing supermodular. One can reduce crossing submodular function minimization to fully submodular one. Recently, many combinatorial algorithms that find a minimizer for a submodular function in polynomial time were developed. For example, [6] provides a minimizer in $O((|V|^5\gamma + |V|^6) log|V|)$, where γ is the time needed to call the function oracle. Our algorithm computes an optimal solution for the partition constrained covering of a symmetric supermodular function using at most $|V|^3$ times a submodular function minimization algorithm. Note that, in our application for hypergraphs, evaluating the function can be done in polynomial time. In fact, in this case the minimization problem can be solved with network flow techniques [2].

6 Conclusion

In this paper we proposed an abstract form for the problem of partition constrained global edge-connectivity augmentation of a hypergraph. We provided a minimax theorem for this problem and we sketched a polynomial algorithm to find an optimal solution, when the function is given by a polynomial oracle. This theorem implies the main theorems of [3] and [5], and consequently the results in [2], [1] and [8]. Our abstract form also provides a new application given in Section 4.3.

References

[1] J. Bang-Jensen, H. Gabow, T. Jordán, Z. Szigeti, Edge-connectivity augmentation with partition constraints, *SIAM J. Discrete Math.* Vol. 12, No. 2 (1999) 160-207.

[2] J. Bang-Jensen, B. Jackson, Augmenting hypergraphs by edges of size two, *Math. Program.* Vol. 84, No. 3 (1999) 467-481.

[3] A. Benczúr, A. Frank, Covering symmetric supermodular functions by graphs, *Math. Program.* Vol. 84, No. 3 (1999) 483-503.

[4] A. Frank, Augmenting graphs to meet edge-connectivity requirements, *SIAM J. Discrete Math.* Vol. 5, No. 1 (1992) 22-53.

[5] R. Grappe, A. Bernáth, Z. Szigeti, Edge-connectivity augmentation of a hypergraph by adding a multipartite graph, *Electronic Notes in Discrete Mathematics* Vol. 34, No. 1 (2009) 173-177.

[6] S. Iwata, J. Orlin, A simple combinatorial algorithm for submodular function minimization, *SODA 2009* (2009) 1230-1237.

[7] Z. Szigeti, On edge-connectivity augmentations of graphs and hypergraphs, W. Cook, L. Lovász, J. Vygen (Editors): Research Trends in Combinatorial Optimization. Springer, Berlin 2009.
[8] T. Watanabe, A. Nakamura, Edge-connectivity augmentation problems, *J. Comput. Syst. Sci.* 35 (1987) 96-144.