On the Degree Distribution of Random Planar Graphs

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Abstract

Let $\mathcal{P}_n$ be the class of all planar graphs with n labeled vertices, and let P_n be a graph drawn uniformly at random from $\mathcal{P}_n$. In this paper we study the degree sequence of P_n. We show that with probability $1 - o(1)$ the number of vertices of degree k in P_n is very close to a quantity $\mu_k n$ that we determine explicitly, for all $k \le c \log n$ and an appropriate $c > 0$. A similar statement is true for random biconnected planar graphs as well.

The main tool in our analysis is a framework that allows us under certain conditions to derive universal results about the degree distribution of random graphs from general classes with structural constraints. In particular, we address so-called critical graph classes, which due to their intricate structure have posed significant technical difficulties in the past.

1 Introduction

Let $\mathcal{P}_n$ be the class of simple labeled planar graphs with n vertices, and denote by P_n a graph drawn uniformly at random from this set. In [20] McDiarmid, Steger, and Welsh showed, among other results, that the probability that P_n is connected is bounded away from 0 and from 1. Observe that this implies that the behavior of a random planar graph is strikingly different from that of a classical Erdős-Rényi graph random $G_{n,p}$ – which is known to satisfy a 0-1 law for all "natural" properties.

From an application point of view, understanding properties of random objects from graph classes with structural constraints, such as planar graphs, is a very desirable target. In real world scenarios, inputs usually come from some restricted set of objects. Consequently, in order to bound the expected or the highly probable running time of an algorithm, gaining a basic understanding of the structural properties of a typical input seems like an unavoidable task. Unfortunately, presently our understanding of structural properties of graph classes with constraints is still very limited. Even more, compared to the wealth of methods and tools that were developed in the classical random graph theory over the last decades, the set of available tools for the analysis of graph classes with structural constraints is comparably small.

Planar graphs have always played an exceptional role within graph theory. One reason, of course, is the four color problem that guided and inspired graph theory for much more than a century. But the importance of planar graphs actually goes well beyond that. The close relation to many real world instances also make it a key class for studying many optimization problems ranging, say, from logistics to visualization of informations. It thus should be no surprise that random planar graphs also became an important and challenging topic for developing and evaluating methods for studying properties of random graphs from classes with structural side constraints.

The first achievements, however, did not address planar graphs per se, but subclasses of them, like triangulations, dissections, outerplanar graphs and series-parallel graphs [12, 13, 3, 4, 7]. A reason for that difference was provided subsequently by the work of Panagiotou and Steger [21]. They showed, among other results, that certain graph classes (here: excluding triangulations) have the property that a random connected object within them is with high probability composed of blocks (i.e., maximal 2-connected components) that all have at most logarithmic size. Properties of a random connected graph from such a graph class thus follow essentially from the law of large numbers. Technically, of course, these results are by far not as easily obtainable as their counterparts within classical Erdős-Rényi random graphs, but nevertheless the underlying reasons are similar in flavor. For planar graphs, however, the work [21] implies that they do not belong to this 'simple' scheme, but instead they belong to a so-called 'critical' scheme that exhibits some sort of significantly more 'complex' behavior. (We will give some relevant details later.) This explains why more sophisticated methods are required.

In this paper we address the problem of obtaining bounds on the degree sequence of a random member of such a 'critical' graph class. In fact, we describe a general framework for obtaining the degree-sequence

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for random connected objects from that of a random 2-connected object, and, similarly, for a random 2-connected object from that of a random 3-connected object. Applied to the class of planar graphs, and using the fact that high probability bounds on the degree sequence of a random 3-connected planar graph are known, see [17], this finally allows us to obtain the bounds on the degree sequence of a random planar graph. Such bounds had always seemed to be obviously true, but long eluded a formal proof. Let $d_\ell(G)$ denote the number of vertices of degree ℓ in a graph G.

THEOREM 1.1. *There is a constant $C > 0$ and a function $d : \mathbb{N} \to \mathbb{R}_+$ such that the following is true for any $\varepsilon > 0$ and sufficiently large n. Let $k = k(n) \leq C \log n$. Then, for a graph P_n, drawn uniformly at random from the class of labeled planar graphs with n vertices,*

$$\Pr[\forall \ell \leq k \;:\; d_\ell(P_n) \in (1 \pm \varepsilon) d(\ell) n] = 1 - o(1).$$

We want to remark that we determine the function d explicitly. Moreover, using a detailed analysis of the generating functions involved in counting the number of graphs, Drmota, Giménez and Noy [6] recently succeeded in determining constants $d_k > 0$ such that the expected number of vertices of degree k, where k is fixed, in P_n is asymptotically equal to $d_k n$. Not surprisingly, we have $d(k) = d_k$, and so our result demonstrates that the highly probable number of vertices of degree k is concentrated around $d_k n$, also for moderately growing k.

As already mentioned above, Theorem 1.1 is essentially a corollary of our main Theorems 3.3 and 4.1. By using them, it is possible to obtain further results, such as for example the probable degree sequence of a random $K_{3,3}$-minor free graph, or a random planar graph with a fixed average degree. We will elaborate further on this issue in a full version of the paper.

1.1 Related Results Random planar graphs were first introduced and studied by Denise, Vasconcellos, and Welsh in [5]. As mentioned above, in [20] it was shown that the probability that a random planar graph is connected is bounded away from 0 and 1. The precise value of this probability is of course given by $\lim_{n\to\infty} |\mathcal{C}_n|/|\mathcal{P}_n|$, where $\mathcal{C}_n$ denotes the class of all labeled connected planar graphs with n vertices. The authors of [20] used some very crude counting arguments to bound this ratio. It took quite a while and required deep methods from combinatorial counting and analytic combinatorics until Giménez and Noy [14], extending earlier work of Bender, Gao, and Wormald [1], were able to determine the sizes of these classes asymptotically.

From [20] and [14] it follows that the number of components of a random planar graph is asymptotically distributed as $1 + Po(\lambda)$, where $\lambda \approx 0.04$. In addition, McDiarmid [18] showed that the expected number of vertices that are not contained within the largest component also converges to a constant $R \approx 0.04$ as n tends to infinity. In order to understand structural properties of a random planar graph it thus clearly suffices to study those of a random *connected* planar graph.

In the following let thus C_n denote a graph drawn uniformly at random from the class $\mathcal{C}_n$ of all labeled connected planar graphs with n vertices. McDiarmid, Steger, and Welsh [20] showed that with high probability C_n contains a linear number of vertices of degree k, for all constants $k \geq 1$, and that the maximum degree is $\Omega(\log n / \log\log n)$. McDiarmid and Reed [19] showed that the maximum degree is actually $\Theta(\log n)$ with high probability.

In a different line of research, Panagiotou and Steger [21] considered the block structure of C_n. They showed that random connected planar graphs belong to a so-called "critical" composition scheme. All such graph classes have the property that a random graph from them has a unique *giant* block that contains a constant fraction of the vertices. Moreover, for such graph classes they determined the typical number of blocks of a certain size in a random graph, and showed that the second largest block has size $O(n^\alpha)$, for some $0 < \alpha < 1$. Such phenomena were further investigated by Gimenez, Noy, and Rue in [15], where also the limiting distribution of the size of the largest block was determined. This subsequent work is an important ingredient of this paper. Finally, Fountoulakis and Panagiotou [11] discovered a similar behavior regarding biconnected graphs.

1.2 Proof Outline As already mentioned, random planar graphs typically consist of a giant block that contains a linear fraction of their vertices. In the initial step of our proof we will exploit this to construct an algorithm, which has the property that its output distribution is *almost* the uniform distribution on all planar graphs with n vertices. More precisely, the algorithm chooses according to the right distribution the size s of the largest block in the graph that it will generate. If s does not lie in the highly probable interval, then the algorithm will just fail and return an undefined object. Otherwise, it will choose a random 2-connected planar graph with s vertices. Then, it will attach to each vertex of it a connected planar graph, according to some well-adjusted distribution that guarantees that the resulting object is selected uniformly among all planar graphs with n vertices and a largest block of size s. So, our algorithm generates all graphs having a largest block of the "right" size with equal probability, and nothing else.

In the second step of our proof we consider the typical structure of the generated graph. Note that vertices of degree k are generated from two sources. First, there are vertices of degree k in the attached planar graphs. Secondly, vertices that had some degree $\ell \leq k$ in the block might have after the attachment step also degree k. So, the big question is how to control the number of vertices generated by both sources. The crucial ingredient of the proof is to replace the "well-adjusted" distribution for generating the attached graphs with some other distribution that i) has a fair chance of mimicking the former distribution and ii) contains enough independence so that it is tractable with tools and concentration inequalities from classical probability theory. It turns out that choosing a special type of the Gibbs distribution, namely the Boltzman model, is sufficient. The actual situation is even more convenient: we are allowed to select the graphs that will be attached *independently* of each other, while maintaining properties i) and ii). The claimed results then follow essentially by controlling the behavior of many independent samples from the same distribution; this turns out to be tractable, although the considered variables have unbounded higher moments.

2 Preliminaries

In this chapter we will collect some basic tools and notation that will become very handy later. We start with some concentration inequalities that form the backbone of our proofs.

Concentration Inequalities The first tools that we shall apply several times are the well-known Chernoff-bounds. We state them here in a simplified form, that is sufficient for our purposes.

THEOREM 2.1. (CHERNOFF-HOEFFDING BOUNDS) *Let $X = \sum_{i=1}^{n} X_i$, where the X_i's are independent and distributed over $[0, 1]$. Let $\mu = \mathbb{E}X$. Then, for all $\varepsilon \geq 0$*

$$\Pr[X < (1-\varepsilon)\mu] \leq e^{-\frac{\varepsilon^2}{2}\mu}$$

and

$$\Pr[X > (1+\varepsilon)\mu] \leq e^{-\frac{\varepsilon^2}{2(1+\varepsilon)}\mu}.$$

A general tool that we shall apply several times is the inequality by Talagrand, see the book [16] for a detailed introduction. Intuitively, it provides strong bounds for the probability that a function defined on a set of independent random variables deviates significantly from its expectation, when the value of the function is not affected much by small changes in each one of its arguments. Talagrand's inequality is therefore a strengthening of Azuma's inequality.

THEOREM 2.2. (TALAGRAND'S INEQUALITY) *Let $Z_1, \ldots, Z_N$ be independent random variables taking values in the sets $\Lambda_1, \ldots, \Lambda_N$ respectively. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_N$. Let $f : \Lambda \to \mathbb{R}$ be a function and set $X = f(Z_1, \ldots, Z_N)$. Assume that there are quantities c_k, $k = 1, \ldots, N$ satisfying the following:*

a. If $z, z' \in \Lambda$ differ only in the kth coordinate, then $|f(z) - f(z')| \leq c_k$.

b. There is an increasing function ψ satisfying the following. Let $z \in \Lambda$ and $r \in \mathbb{R}$ such that $f(z) \geq r$. Then there exists a set $J \subseteq \{1, \ldots, N\}$ with $\sum_{i \in J} c_i^2 \leq \psi(r)$, such that for all $y \in \Lambda$ with $y_i = z_i$ when $i \in J$, we have $f(y) \geq r$.

Then, if $\mathbb{M}X$ denotes the median of X, for every $t \geq 0$ we have

$$\mathbb{P}[|X - \mathbb{M}X| \geq t] \leq 4 \exp\left(-\frac{t^2}{4\psi(\mathbb{M}X + t)}\right). \tag{2.1}$$

The next statement gives a sufficient condition that ensures that the median is very close to the expected value.

PROPOSITION 2.3. (EXAMPLE 2.33 IN [16]) *Let X be a random variable that satisfies the preconditions of Theorem 2.2 with $\psi(r) \leq \alpha\lceil r \rceil$, for some $\alpha > 0$. Then*

$$|\mathbb{M}X - \mathbb{E}X| = O\big(\sqrt{\alpha\mathbb{E}X}\big)$$

and thus

$$\begin{aligned} &\mathbb{P}[|X - \mathbb{E}X| \geq t + O(\sqrt{\alpha\mathbb{E}X})] \\ &\quad \leq 4 \exp\left(-\frac{t^2}{4\psi\big(\mathbb{E}X + O(\sqrt{\alpha\mathbb{E}X}) + t\big)}\right). \end{aligned} \tag{2.2}$$

The presentation of the above inequalities is as in [16], where also many applications are presented.

Notation Let $\mathcal{G}$ be a class of labeled graphs. We denote by $\mathcal{G}_n$ the subset of graphs in $\mathcal{G}$ which have precisely n vertices, and we write $g_n := |\mathcal{G}_n|$. Moreover, we write $G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}$ for its corresponding *exponential generating function (egf).*

We will frequently use the *pointing* and *derivative* operators. Given a labeled class of graphs $\mathcal{G}$, we define $\mathcal{G}^\bullet$ as the class of *pointed* (or *rooted*) graphs, where a vertex is distinguished from all other vertices. The *derived* class $\mathcal{G}'_{n-1}$ is obtained by removing the label n from every object in $\mathcal{G}_n$, such that the obtained objects have $n-1$ labeled vertices, i.e., vertex n can be considered as a distinguished vertex that does not contribute to the size. Consequently, there is a bijection between the classes $\mathcal{G}'_{n-1}$ and $\mathcal{G}_n$. We set $\mathcal{G}' := \bigcup_{n\geq 0} \mathcal{G}'_n$. On generating function level, the pointing operation corresponds to taking the derivative with respect to x, and multiplying by it by x, i.e. $G^\bullet(x) = xG'(x)$. Similarly, the egf of $\mathcal{G}'$ is simply $G'(x)$.

The operators described above appear often in modern theories of combinatorial analysis [9, 2], as well as in systematic approaches to random generation of combinatorial objects [8, 10]. For numerous applications we refer the interested reader to [9].

3 From 2-connected to Connected Random Graphs

The main result of this section, Theorem 3.3, gives sufficient conditions guaranteeing that the number of vertices of a given degree in a random graph from a class of connected graphs is sharply concentrated around a specific value. More specifically, we will deal with classes $\mathcal{C}$ that are *closed under taking biconnected components*, i.e., if G' is obtained from $G \in \mathcal{C}$ by substituting any block in G by any other 2-connected graph in $\mathcal{C}$, then also $G' \in \mathcal{C}$.

Let us collect in the following definition all relevant properties.

Definition 3.1. *Let $\mathcal{C}$ be a class of labeled connected graphs, and $\mathcal{B} = \mathcal{B}(\mathcal{C}) \subset \mathcal{C}$ the subclass of biconnected graphs in $\mathcal{C}$. We say that $\mathcal{C}$ belongs to the* critical connected–biconnected scheme *if it fulfills the following conditions.*

i) $\mathcal{C}$ is closed under taking biconnected components.

ii) The egf $B(x)$ enumerating $\mathcal{B}$ has a unique dominant singularity at $\rho_B > 0$ and satisfies $\rho_B B''(\rho_B) < 1$.

iii) There are constants $c, b, \rho_C > 0$ such that for all $n \geq 1$

$$|\mathcal{C}_n| = (1 + o(1))cn^{-7/2}\rho_C^n n!$$

and

$$|\mathcal{B}_n| = (1 + o(1))bn^{-7/2}\rho_B^n n!.$$

As we shall see in the next section, property *ii)* guarantees that the class is "critical", thus containing a unique linear-sized block. Finally, the last property asserts that the class is "planar-like", as the asymptotic number of connected and biconnected planar graphs satisfy this. However, this condition is not very restrictive, as several other graph classes, like $K_{3,3}$-minor free graphs, have qualitatively similar enumeration sequences.

In [21] the following was shown.

Lemma 3.2. *Suppose that $\mathcal{C}$ belongs to the critical connected–biconnected scheme. Then the egf $C(x)$ has a unique dominant singularity at ρ_C satisfying $\rho_C C'(\rho_C) = \rho_B$.*

Before we state the main result of this section, let us introduce one additional property. Let $\mathcal{G}$ be any class of graphs. We will denote by ConcentratedDegreeSequence($\mathcal{G}$) the following property.

ConcentratedDegreeSequence($\mathcal{G}$) : *There is a function $D_\mathcal{G}(z)$, which is analytic in a complex neighborhood around $z = 0$, and a non-decreasing function $k_0(n)$ such that the following is true. Let G_n denote a graph that is drawn uniformly at random from $\mathcal{G}_n$, and let $\varepsilon > 0$. Then, if we denote by $[z^k]D_\mathcal{G}(z)$ the coefficient of z^k in the Taylor series expansion of $D_\mathcal{G}(z)$ around zero,*

$$\Pr\left[\forall k \leq k_0(n) : d_k(G_n) \in (1\pm\varepsilon)[z^k]D_\mathcal{G}(z)n\right] = 1-o(1).$$

The main result of this section is summarized in the following theorem.

Theorem 3.3. *Let $\varepsilon > 0$ and suppose that $\mathcal{C}$ belongs to a critical connected–biconnected scheme. Assume the property ConcentratedDegreeSequence($\mathcal{B}$), where $\mathcal{B} = \mathcal{B}(\mathcal{C})$. Then there is a function $D_\mathcal{C}(z)$, given in (3.8), and a non-decreasing function $k'_0 = k'_0(n)$, given in (3.9), such that for a random graph C_n from $\mathcal{C}_n$*

$$\Pr\left[\forall k \leq k'_0(n) : d_k(C_n) \in (1\pm\varepsilon)[z^k]D_\mathcal{C}(z)n\right] = 1-o(1).$$

In other words, this theorem implies the property ConcentratedDegreeSequence($\mathcal{C}$) (defined with suitable functions $D_\mathcal{C}(z)$ and $k'_0(n)$), assuming the property only for $\mathcal{B}$. The proof is spread over the next sections. In particular, in Section 3.1 we introduce an approximate sampling algorithm for connected graphs, and prove some important combinatorial properties of it. This will form the backbone of our analysis. Then, in Section 3.2 we show how this algorithm can be used to determine the probable degree sequence for random graphs from $\mathcal{C}_n$.

3.1 Random Sampling The main aim of this section is the development of an almost uniform sampler for nice classes of connected graphs. Before we expose it and prove some of its main properties, let us introduce its two major ingredients.

Let $G \in \mathcal{C}_n$, and denote by $lb(G)$ the size of the largest biconnected block in G, which we will also refer to, if it is unique, as the *(biconnected) core* of G. Moreover, let us write

$$core(n, m) = \{G \in \mathcal{C}_n \ : \ lb(G) = m\}.$$

Using this notation, the probability that the largest block in a random graph C_n from $\mathcal{C}_n$ has m vertices is

$$\Pr[lb(C_n) = m] = \frac{|core(n,m)|}{|\mathcal{C}_n|}. \tag{3.3}$$

Theorem 5.4 from [15] immediately implies the following statement about the probability distribution of $lb(C_n)$.

THEOREM 3.4. *Let $\varepsilon > 0$. There is a $C = C(\varepsilon) > 0$ such that*

$$\Pr\left[|lb(C_n) - (1-\rho_B B''(\rho_B))n| \geq Cn^{2/3}\right] \leq \varepsilon.$$

Moreover, uniformly for $|x| \leq C$

$$\Pr\left[lb(C_n) = (1-\rho_B B''(\rho_B))n + xn^{2/3}\right] = \Theta(n^{-2/3}).$$

In other words, the size of the largest block in a uniform random graph from $\mathcal{C}_n$ is concentrated in an interval of size $O(n^{2/3})$ around $(1-\rho_B B''(\rho_B))n$, and all such sizes have roughly the same probability of being observed.

The second ingredient of our sampler is a probability distribution defined over the whole class $\mathcal{C}$. More precisely, let $\mathcal{C}^\bullet = \cup_{n\geq 1}[n] \times \mathcal{C}_n$, i.e., every graph in $\mathcal{C}$ contains a distinguished vertex, which we will call the *root*. Then the *Boltzmann distribution* over $\mathcal{C}^\bullet$ is given through

$$\forall \gamma \in \mathcal{C}^\bullet \ : \ \Pr[\gamma] = \frac{\rho_C^{|\gamma|}}{|\gamma|! \cdot \rho_C C'(\rho_C)}. \tag{3.4}$$

We want to remark at this point that in general the Boltzmann model can be used to construct a whole *family* of distributions on combinatorial objects, each of which has several distinguished properties. However, the above definition is sufficient for our purposes and we will refer to it as the Boltzmann distribution (over $\mathcal{C}^\bullet$). The interested reader can find in [8] a very detailed treatment of the topic.

Our assumptions on the the sequence $|\mathcal{C}_n|$ imply the following properties of the Boltzmann model. The proof can be found in the appendix.

LEMMA 3.5. *There is a $c > 0$ such that for all $n \geq 1$*

$$\Pr[|\gamma| = n] = (1 + o(1))cn^{-5/2}.$$

Moreover, $\mathbb{E}[|\gamma|] = (1-\rho_B B''(\rho_B))^{-1}$.

With the above tools at hand we are ready to describe our approximate sampling algorithm.

$S_{\mathcal{C}}(n,\varepsilon)$:
  $C \to$ the constant given by Theorem 3.4
  $m \to$ a random value according to the distribution (3.3) $\quad (*)$
  if $|m - (1-\rho_B B''(\rho_B))n| > Cn^{2/3}$
    return $\perp$
  else
    $B \to$ uniformly from $\mathcal{B}_m$ $\quad (**)$
    repeat
      $\forall v \in B$: choose independently $\gamma_v \in \mathcal{C}^\bullet$ according to (3.4)
    until $(\sum_{v\in B} |\gamma_v| = n\ )$
    $\forall v \in B$: identify the root of γ_v with v
    partition the vertex set $[n]$ into blocks $(S_v)_{v\in B}$ of size $|S_v| = |\gamma_v|$
    return the resulting graph, with all γ_v relabeled using labels from S_v in the canonical way

The following lemma summarizes the properties of the algorithm that we will exploit.

LEMMA 3.6. *Let $\varepsilon > 0$ and suppose that $\alpha_B = 1 - \rho_B B''(\rho_B) > 1/2$. Then the following statements are true for sufficiently large n.*

- $\Pr[S_{\mathcal{C}}(n,\varepsilon) = \perp] \leq \varepsilon$.
- *Let $C = C(\varepsilon)$ be the constant from Theorem 3.4, and let G be such that $|lb(G) - \alpha_B n| \leq Cn^{2/3}$. Then*

$$\Pr[S_{\mathcal{C}}(n,\varepsilon) = G] = |\mathcal{C}_n|^{-1}.$$

Proof. The first statement follows immediately from Theorem 3.4, the choice of the constant C in the exposition of the algorithm, and the assumption $\alpha_B > 1/2$, which guarantees the existence of a unique largest block in the output of $S_{\mathcal{C}}(n,\varepsilon)$ for large n. To see the second claim, note first that every graph in $core(n,m)$, where $m > n/2$, can be decomposed in graphs from $\mathcal{B}$ and $\mathcal{C}$ as follows in three steps.

1. Partition the n labels in m non-empty groups $S_1, \ldots, S_m$ of sizes $s_1, \ldots, s_m$, and distinguish a label in each group.
2. Choose a graph from $\mathcal{B}_m$, and replace its labels canonically with the labels in the distinguished set.
3. For $i \in [m]$ choose a graph $G_i \in \mathcal{C}_{s_i}$, and replace its labels canonically with the labels in S_i.

Note that the above three steps establish a bijection between the graphs in $core(n,m)$, and triples $(\Pi, B, \vec{G})$, where Π is a partition as in 1., $B \in \mathcal{B}_m$ is a biconnected graph as in 2., and $\vec{G}$ is a sequence of graphs as in the last step. In other words, we have that

$$
\begin{aligned}
&|core(n,m)| \\
(3.5)\quad &= |\mathcal{B}_m| \cdot \sum_{s_1+\cdots+s_m=n} \binom{n}{s_1,\ldots,s_m} \prod_{i=1}^{m} |[s_i] \times \mathcal{C}_{s_i}| \\
&= |\mathcal{B}_m| \cdot n! \cdot [z^n](zC'(z))^m.
\end{aligned}
$$

With the above fact at hand we can estimate the probability that a particular graph $G \in \mathcal{C}_n$ is generated by $S_{\mathcal{C}}(n,\varepsilon)$, where we may assume that $lb(G) > n/2$. First of all, in the line marked with $(*)$, the algorithm has to choose $m = lb(G)$. Subsequently, following the preceding discussion, in the line marked with $(**)$ it has to pick the biconnected core of G among $|\mathcal{B}_m|$ graphs. Finally, it has to pick the appropriate γ_v's, so that G can be composed out of B and those. The probability for choosing the γ_v's correctly is precisely

$$
\begin{aligned}
&\Pr\left[\forall v \in B \ : \ \gamma_v = G_v \ \Big| \ \sum_{v\in B} |\gamma_v| = n\right] \\
&\overset{(3.4)}{=} \Pr\left[\sum_{v\in B} |\gamma_v| = n\right]^{-1} \cdot \frac{\rho_C^n}{(\rho_C C'(\rho_C))^m} \cdot \prod_{v\in B} \frac{1}{|G_v|!} \\
&= ([z^n](zC'(z))^m)^{-1} \cdot \prod_{v\in B} \frac{1}{|G_v|!}.
\end{aligned}
$$

By combining this with (3.5) we infer that

$$
\begin{aligned}
&\Pr[S_{\mathcal{C}}(n,\varepsilon) = G] \\
&= \frac{1}{\binom{n}{|G_1|,\ldots,|G_m|}} \cdot \frac{n!}{|\mathcal{C}_n|} \cdot \prod_{v\in B} \frac{1}{|G_v|!} = |\mathcal{C}_n|^{-1},
\end{aligned}
$$

as claimed.

We also obtain the following corollary about random graphs from the Boltzmann model.

COROLLARY 3.7. *Let $\varepsilon > 0$ and $C = C(\varepsilon) > 0$ be as in Theorem 3.4. Then, uniformly for all m such that $|m - (1 - \rho_B B''(\rho_B))n| \le Cn^{2/3}$ and graphs $\gamma_1, \ldots, \gamma_m$ drawn independently according to the Boltzmann distribution* (3.4)

$$\Pr\left[\sum_{i=1}^{m} |\gamma_i| = n\right] = \Theta(n^{-2/3}).$$

Proof. The desired probability is equal to the probability that $S_{\mathcal{C}}(n,\varepsilon)$ outputs a graph whose biconnected core contains m vertices. Lemma 3.6 guarantees for the given range of m that all such graphs have the same probability $|\mathcal{C}_n|^{-1}$ of being generated. Hence, the probability is precisely

$$\frac{|core(n,m)|}{|\mathcal{C}_n|} = \Pr[lb(C_n) = m] \overset{(\text{Thm. } 3.4)}{=} \Theta(n^{-2/3}).$$

3.2 Transferring the Degree Sequence: 2-connected $\to$ Connected Graphs Let $G \in \mathcal{C}_n$ be the graph constructed by running the algorithm from the previous section, and let $B(G)$ denote its core, i.e., the graph B generated in the line marked with $(**)$ while building G. For $k \ge \ell$ set

$$
\begin{aligned}
&d_{\ell,k}(G) \\
&= |\{u \in B(G) : deg(B; u) = \ell \text{ and } deg(G;\, u) = k\}|.
\end{aligned}
$$

In words, $d_{\ell,k}(G)$ is the number of degree-k vertices in G that belong to its core and have degree exactly ℓ within the core. Moreover, for $v \in B(G)$ let

$$d'_k(\gamma_v) = |\{u \in \gamma_v, u \text{ is not marked} : deg(\gamma_v;\, u) = k\}|$$

denote the number of non-root vertices of degree k in γ_v. With the above notation we immediately obtain

$$(3.6)\qquad d_k(G) = \sum_{\ell=2}^{k} d_{\ell,k}(G) + \sum_{v\in B(G)} d'_k(\gamma_v).$$

This relation is at the heart of the proof of Theorem 3.3. Before we present the precise argument, we will control the quantities $d_{\ell,k}(G)$ and $d'_k(\gamma_v)$ separately. In particular, the next two lemmas determine the expected values of $d_{\ell,k}(G)$ and $d'_k(\gamma_v)$ and provide appropriate concentration bounds. In order to state them we need some additional notation. Let γ be a graph drawn according to the Boltzmann distribution over $\mathcal{C}^\bullet$, see (3.4), and denote by $rd(\gamma)$ the degree of the distinguished vertex. We write

$$R_{\mathcal{C}}(z) = \sum_{k\ge 0} Pr[rd(\gamma) = k] z^k$$

for the corresponding probability generating function. Moreover, we denote by

$$E_{\mathcal{C}}(z) = \sum_{k\ge 0} \mathbb{E}[d'_k(\gamma)] z^k$$

the function whose kth coefficient denotes the expected number of non-root vertices of degree k in γ. The above notation will be used without any further reference.

LEMMA 3.8. *Let $\varepsilon, \delta > 0$ and let G be a graph constructed by running the algorithm $S_{\mathcal{C}}(n,\delta)$ from Section 3.1. Moreover, set $X_k = \sum_{\ell\ge 2} d_{\ell,k}(G)$. If $\mathcal{B}$ satisfies assumption* ConcentratedDegreeSequence($\mathcal{B}$)*, then uniformly for all $\delta > 0$*

$$
\begin{aligned}
&\Pr\left[\forall k \le k_0 : \ |X_k - \mu_k n| \le \varepsilon \mu_k n + (\log n)^2 \ \Big| \ G \ne \bot\right] \\
&\ge 1 - o(1),
\end{aligned}
$$

where $k_0 := k_0(n/2)$ and $\mu_k = (1 - \rho_B B''(\rho_B))[z^k] D_{\mathcal{B}}(z) R_{\mathcal{C}}(z)$.

Proof. As $G \neq \perp$, it is composed out of the core B and graphs $(\gamma_v)_{v \in B}$, whose distinguished vertices are identified with the corresponding vertices in B. We call B *balanced* if

$$\forall \ell \le k_0 : d_\ell(B) \in (1 \pm \varepsilon/2)[z^\ell]D_{\mathcal{B}}(z) \cdot (1 - \rho_B B''(\rho_B))n.$$

The definition of the sampler implies that the size of B is, with room to spare, within $(1 \pm \varepsilon/3)(1 - \rho_B B''(\rho_B))n$. So, we deduce from assumption ConcentratedDegreeSequence($\mathcal{B}$) that for any $\varepsilon > 0$ and sufficiently large n the core B is balanced with probability $1 - o(1)$. (Here we used that, by assumption, k_0 is monotone and hence $k_0 = k_0(n/2) \le k_0(|B|)$.)

For the remainder of the proof we fix the core B and assume that it is balanced. We distinguish two probability models. Firstly, denoted by $\Pr_{S_{\mathcal{C}}(n,\delta)}[...]$, the model from algorithm $S_{\mathcal{C}}(n, \delta)$ where we repeatedly choose γ_v's until their joint sizes sum up to n. And, secondly, denoted by $\Pr_\gamma[...]$, the space where we just choose γ_v's independently once. Observe that

(3.7)

$$\begin{aligned}
&\Pr_{S_{\mathcal{C}}(n,\delta)}[|X_k - \mu_k n| \ge \varepsilon \mu_k n + \log^2 n \mid B] \\
&= \Pr_\gamma\left[|X_k - \mu_k n| \ge \varepsilon \mu_k n + \log^2 n \;\middle|\; B \wedge \sum_{v \in B} |\gamma_v| = n\right] \\
&\overset{(\text{Cor } 3.7)}{=} O(n^{2/3}) \cdot \Pr_\gamma[|X_k - \mu_k n| \ge \varepsilon \mu_k n + \log^2 n \mid B].
\end{aligned}$$

We will show that the last expression is $o(n^{-5/3})$, which will conclude with a union bound for $k \le n$ the proof of the lemma. To this end we will apply the Chernoff-Hoeffding bounds. Define for $2 \le \ell \le k$ and $v \in B$ the random variables

$$X_{\ell,k,v} = \begin{cases} 1, & \text{if } deg(B; v) = \ell \text{ and } rd(\gamma_v) = k - \ell \\ 0, & \text{otherwise} \end{cases}$$

and observe that $X_k := \sum_{2 \le \ell \le k} \sum_{v \in B} X_{\ell,k,v}$ counts the number of vertices in B that have degree exactly k in G. Linearity and the balancedness of B imply that

$$\begin{aligned}
\mathbb{E} X_k &= \sum_{\ell=2}^{k} d_\ell(B) \Pr[rd(\gamma) = k - \ell] \\
&= \left(1 \pm \frac{\varepsilon}{2}\right) \sum_{\ell=2}^{k} [z^\ell] D_{\mathcal{B}}(z)(1 - \rho_B B''(\rho_B))n \, [z^{k-\ell}] R_{\mathcal{C}}(z) \\
&= \left(1 \pm \frac{\varepsilon}{2}\right) \mu_k n.
\end{aligned}$$

By applying Theorem 2.1 we infer that for $\alpha = \varepsilon/3 + \frac{(\log n)^2}{2\mu_k n}$

$$\Pr_\gamma[|X_k - \mu_k n| \ge \varepsilon \mu_k n + \log^2 n \mid B] \le e^{-\frac{1}{2}\frac{\alpha^2}{1+\alpha}\mu_k n}.$$

Note that if $\mu_k n \gg (\log n)^2$, then the above bound is $e^{-\Omega(\mu_k n)}$, otherwise it is $e^{-\Omega((\log n)^2)}$. In both cases we see that the above probability is $o(n^{-5/3})$, which together with (3.7) completes the proof of the lemma.

Our next lemma deals with the non-root vertices of degree k in the γ_v's. It's proof can be found in the appendix.

LEMMA 3.9. *Let $\varepsilon, \delta > 0$ and let G be a graph constructed by running the algorithm $S_{\mathcal{C}}(n, \delta)$ from Section 3.1. Moreover, set $Y_k = \sum_{v \in B(G)} d'_k(\gamma_v)$. Then, uniformly for all $\delta > 0$ we have*

$$\Pr\left[\forall k \le n : |Y_k - \nu_k n| \le \varepsilon \nu_k n + n^{8/9} \mid G \neq \perp\right] \ge 1 - o(1),$$

where $\nu_k = (1 - \rho_B B''(\rho_B)) \cdot [z^k] E_{\mathcal{C}}(z)$.

With the above lemmas we can finally prove Theorem 3.3.

Proof. [Theorem 3.3] Let

$$D_{\mathcal{C}}(z) = (1 - \rho_B B''(\rho_B))\big(D_{\mathcal{B}}(z) R_{\mathcal{C}}(z) + E_{\mathcal{C}}(z)\big). \tag{3.8}$$

Moreover, let $h(n)$ be equal to

$$\max\left\{k : \forall \ell \le k : [z^\ell] D_{\mathcal{C}}(z) \ge n^{-1/10} \text{ or } [z^\ell] D_{\mathcal{C}}(z) = 0\right\}$$

and define

$$k'_0(n) = \min\left\{k_0(n/2), h(n)\right\}. \tag{3.9}$$

It can easily be checked that $k'_0(n)$ is non-decreasing. Let $\delta > 0$ and denote by $\tilde{\mathcal{C}}_n(\delta)$ the set of graphs in $\mathcal{C}_n$ whose biconnected core has $(1 - \rho_B B''(\rho_B))n \pm C(\delta) n^{2/3}$ vertices, where $C(\delta)$ is the constant from Theorem 3.4. Moreover, denote by $\mathcal{D}_n$ the set of graphs in $\mathcal{C}_n$ such that for all $G \in \mathcal{D}_n$ it holds $\forall k \le k'_0(n) : \quad d_k(G) \in (1 \pm \varepsilon)[z^k] D_{\mathcal{C}}(z) n$. Then Theorem 3.4 guarantees that

$$\Pr[C_n \in \mathcal{D}_n] \ge \Pr[C_n \in \mathcal{D}_n \mid C_n \in \tilde{\mathcal{C}}_n(\delta)] - \delta.$$

To estimate the latter probability, note that due to Lemma 3.6 the distributions "$C_n \mid C_n \in \tilde{\mathcal{C}}_n(\delta)$" and "$S_{\mathcal{C}}(n, \delta) \mid S_{\mathcal{C}}(n, \delta) \neq \perp$" coincide. So, by applying Lemma 3.8 and Lemma 3.9 (where both times we use $\varepsilon/2$ instead of ε), and using (3.6), we infer that with probability $1 - o(1)$

$$\forall k \le k'_0(n) :$$
$$|d_k(C_n) - (\mu_k + \nu_k)n| \le \frac{\varepsilon}{2}(\mu_k + \nu_k)n + n^{8/9} + \log^2 n.$$

However, $\mu_k + \nu_k = [z^k] D_{\mathcal{C}}(z)$. Moreover, the definition of $k'_0(n)$ implies that for large n the right hand side of the above equation is at most $\varepsilon [z^k] D_{\mathcal{C}}(z) n$. The proof is completed, as we may choose δ arbitrarily.

4 From 3-connected to 2-connected Random Graphs

In the previous section we demonstrated how the knowledge about the degree distribution of random 2-connected graphs can help us to understand the degree distribution of random connected graphs from certain families. However, this is in general not sufficient, as a priori, at least considering random planar graphs, we have no information about the number of vertices of degree k in random biconnected graphs. Establishing such bounds is the goal of this section (and, mainly, the Appendix).

In principle, 2-connected planar graphs exhibit a similar 'complex' behavior, as connected planar graphs do. The only difference is that the parameter that we have to look at is the largest 3-connected core, see the works [11, 15]. From there, the main line of reasoning is similar to the previous section, but the actual setting and many of the details are more involved. However, we arrive at the following informal result.

THEOREM 4.1. *Let B_n be a random 2-connected graph having certain closure properties, see Definition B.3. Then, if the degree sequence of a random 3-connected graph satisfies a concentration property, then also the degree sequence of B_n satisfies a concentration property.*

A more precise version of this theorem can be found in the appendix, see Theorem B.5.

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A Missing Proofs

Proof. [Lemma 3.5] The first statement follows immediately from the definition of the Boltzmann model (3.4) and Definition 3.1. To see the second statement, note that the egf $C(x)$ enumerating $\mathcal{C}$ satisfies $C'(x) = e^{B'(xC'(x))}$. By differentiating both sides with respect to x and rearranging we infer that

$$C''(x) = \frac{B''(xC'(x))\,C'(x)^2}{1 - xC'(x)B''(xC'(x))}.$$

By applying Lemma 3.2 we obtain that the expected size of γ is

$$\begin{aligned}\mathbb{E}[|\gamma|] &= \frac{\rho_C \frac{d}{dx}(xC'(x))\big|_{x=\rho_C}}{\rho_C C'(\rho_C)} \\ &= \frac{\rho_C(C'(\rho_C) + \rho_C C''(\rho_C))}{\rho_B} = \frac{1}{1-\rho_B B''(\rho_B)}.\end{aligned}$$

Proof. [Lemma 3.9] As $G \neq \perp$, it is composed out of the core B and graphs $(\gamma_v)_{v\in B}$, whose distinguished vertices are identified with the corresponding vertices in B. Recall that $|B| = (1 - \rho_B B''(\rho_B))n + O(n^{2/3})$. In the sequel we will fix B and distinguish between two probability models, similarly as in the proof of the previous lemma. That is, we denote by $\Pr_{S_\mathcal{C}(n,\delta)}[...]$ the model from algorithm $S_\mathcal{C}(n, \delta)$, where we repeatedly choose γ_v's until their joint sizes sum up to n. And, secondly, denoted by $\Pr_\gamma[...]$, the space where we just choose γ_v's independently once. Using this notation we thus need to show that

$$\Pr_{S_\mathcal{C}(n,\delta)}[|Y_k - \nu_k n| \geq \varepsilon\nu_k n + n^{8/9} \mid B] = o(1/n)$$

uniformly for all $k \leq n$. Proceeding as in the proof of the previous lemma we observe that Corollary 3.7 implies that we it suffices to show the following slightly stronger statement in the probability space $\Pr_\gamma[...]$:

$$\text{(A.1)} \quad \Pr_\gamma[|Y_k - \nu_k n| \geq \varepsilon\nu_k n + n^{8/9} \mid B] = o(n^{-5/3})$$

uniformly for all $k \leq n$. This is what we will now show. Fix a $k \leq n$ arbitrarily, let $s = n^{1/4}$ and write $Y_k = Z + Z'$, where

$$Z = \sum_{v\in B(G)} d'_k(\gamma_v)\mathbf{1}_{[|\gamma_v|\leq s]}$$

and

$$Z' = \sum_{v\in B(G)} d'_k(\gamma_v)\mathbf{1}_{[|\gamma_v|> s]}.$$

That is, we partition Y_k into contributions from 'small' and from 'large' γ_v's. Moreover, set $E'_\mathcal{C}(z) = \sum_{k\geq 0}\mathbb{E}\big[d'_k(\gamma) \mid |\gamma| \leq s\big]z^k$, where γ denotes a graph drawn according to the Boltzmann distribution (3.4), and let $\nu'_k = (1 - \rho_B B''(\rho_B))[z^k]E'_\mathcal{C}(z)$. We will show that

$$\text{(A.2)} \quad \Pr_\gamma\big[|Z - \nu'_k n| \geq \tfrac{1}{2}\varepsilon\nu'_k n + \tfrac{1}{3}n^{8/9} \mid B\big] = o(n^{-5/3}),$$

and

$$\text{(A.3)} \qquad \nu'_k = (1 + o(1))\nu_k \pm O(n^{-1/8}),$$

and

$$\text{(A.4)} \qquad \Pr_\gamma[Z' \geq \tfrac{1}{3}n^{8/9} \mid B] = o(n^{-5/3}).$$

By putting the above bounds together we immediately obtain (A.1), which will thus conclude the proof of the lemma.

Let us make an auxiliary calculation before we proceed with the proof of (A.2)–(A.4). By applying Lemma 3.5 we infer that

$$\text{(A.5)} \quad \begin{aligned}\Pr[|\gamma| > s] &= \sum_{i>s}\Pr[|\gamma| = i] \\ &= (1 + o(1))c\sum_{i>s} i^{-5/2} = \Theta(n^{-3/8}).\end{aligned}$$

(Here the probability is over the choice of a single γ.) Note that this implies that the expected total number of vertices in 'large' γ_v's is in the order $O(n^{5/8})$ – which should make (A.4) plausible.

We start with showing (A.2). We will apply Talagrand's inequality. Note that changing one of the γ_v's will affect Z by at most s, so precondition $a.$ in Theorem 2.2 is satisfied if we choose $c_v = s$ uniformly for all $v \in B$. Moreover, to certify that $Z \geq r$, it suffices to point out at most $\lceil r\rceil$ graphs γ_v that contain a non-root vertex of degree k. In other words, precondition $b.$ in Theorem 2.2 is satisfied with $\psi(r) = s^2\lceil r\rceil$. Thus, together with Proposition 2.3 we obtain for any $t > 0$

$$\text{(A.6)} \quad \begin{aligned}&\Pr_\gamma[|Z - \mathbb{E}Z| \geq t + O(n^{1/4}\sqrt{\mathbb{E}Z}) \mid B] \\ &\leq 4\exp\left\{-\frac{t^2}{4n^{1/2}(\mathbb{E}Z + O(n^{1/4}\sqrt{\mathbb{E}Z}) + t)}\right\}.\end{aligned}$$

Using the bounds on $|B|$, the definition of ν'_k and (A.5) we conclude that

$$\begin{aligned}\mathbb{E}Z &= |B|\Pr\big[|\gamma| \leq s\big]\cdot\mathbb{E}\big[d'_k(\gamma) \bigm| |\gamma| \leq s\big] \\ &= (1 + O(n^{-3/8}))\cdot\nu'_k n.\end{aligned}$$

By plugging this bounds into (A.6), where we use $t = \varepsilon\nu'_k n + n^{8/9}$, we easily infer that

$$\begin{aligned}&\Pr_\gamma\big[|Z - (1 + O(n^{-3/8}))\nu'_k n| \geq \varepsilon\nu'_k n + \tfrac{1}{3}n^{8/9} \bigm| B\big] \\ &= o(n^{-5/3}),\end{aligned}$$

which completes the proof of (A.2).

We next show (A.3). Let G_i be a graph drawn uniformly at random from $\mathcal{C}_i^\bullet$. Then

$$\begin{aligned} &\mathbb{E}\big[d_k'(\gamma) \mid |\gamma| \le s\big] \\ =&\sum_{i\le s} \Pr[|\gamma| = i \mid |\gamma| \le s]\mathbb{E}[d_k'(G_i)] \\ =&\frac{1}{\Pr[|\gamma|\le s]}\sum_{i\le s}\Pr[|\gamma| = i]\mathbb{E}[d_k'(G_i)] \\ \overset{(A.5)}{=}&(1+o(1))\cdot\left(\mathbb{E}[d_k'(\gamma)] - \sum_{i>s}\Pr[|\gamma| = i]\mathbb{E}[d_k'(G_i)]\right). \end{aligned} \tag{A.7}$$

This readily establishes the upper bound in (A.3). To see the lower bound, observe that $\mathbb{E}[d_k'(G_i)] < i$, and so by Lemma 3.5 the ith term in the above sum is $O(i^{-3/2})$. We infer that the whole sum is $O(s^{-1/2})$, which proves the lower bound in (A.3).

Finally, we prove (A.4). Note that for all $v \in B$ we have that $d_k'(\gamma_v) < |\gamma_v|$. Hence,

$$Z' < \sum_{v\in B} |\gamma_v|\mathbf{1}_{[|\gamma_v|>s]}.$$

Let $Z'' = \sum_{v\in B} |\gamma_v|\mathbf{1}_{[|\gamma_v|\le s]}$, and so $Z' < n - Z''$. Therefore, it is sufficient to show that

$$\Pr_\gamma[Z'' < n - \tfrac{1}{3}n^{8/9} \mid B] = o(n^{-5/3}). \tag{A.8}$$

To estimate this probability we again apply Talagrand's inequality. Note that changing one of the γ_v's will affect Z'' by at most s; this guarantees that precondition *a.* in Theorem 2.2 is satisfied with if we set $c_v = s$ for all $v \in B$. Moreover, if $Z'' \ge r$, then this can be certified by exposing at most $\lceil r\rceil$ of the γ_v's, as $|\gamma_v| \ge 1$. So, precondition *b.* in Theorem 2.2 is satisfied with $\psi(r) = s^2\lceil r\rceil$. By applying Theorem 2.2 and Proposition 2.3 we thus infer that for any $t > 0$

$$\begin{aligned} &\Pr_\gamma[Z'' < \mathbb{E}Z'' - t - O(n^{1/4}\sqrt{\mathbb{E}Z''}) \mid B] \\ &\le 4\exp\left\{-\frac{t^2}{4n^{1/2}(\mathbb{E}Z'' + O(n^{1/4}\sqrt{\mathbb{E}Z''}) + t)}\right\}. \end{aligned} \tag{A.9}$$

Note that

$$\begin{aligned} \mathbb{E}Z'' &= |B| \Pr\left[|\gamma| = s\right]\cdot \mathbb{E}\big[|\gamma| \,\big|\, |\gamma| \le s\big] \\ &= |B|\left(\mathbb{E}[|\gamma|] - \sum_{i>s} i\Pr[|\gamma| = i]\right). \end{aligned}$$

By applying Lemma 3.5 we infer that the last sum is $O(s^{-1/2})$, and so $|B| = (1-\rho_B B''(\rho_B))n + O(n^{2/3})$ implies that

$$\mathbb{E}Z'' \ge |B|\left(\frac{1}{1-\rho_B B''(\rho_B)} - O(n^{-1/8})\right) = n - O(n^{7/8}).$$

By using (A.9) with $t = \frac{1}{4}n^{8/9}$ we obtain that the desired probability is $o(n^{-5/3})$, which completes the proof of (A.4) and thus also the proof of the lemma.

B From 3-connected to 2-connected Random Graphs

Let $\mathcal{B}$ be a class of labeled biconnected graphs. In the remainder we assume that the graph consisting of a single edge is contained in $\mathcal{B}$. The main result of this section, Theorem B.5, gives sufficient conditions guaranteeing that the number of vertices of a given degree in $\mathcal{B}_n$ is sharply concentrated around a specific value. The proof is similar to the proof in Section 3. However, many of the preparatory details are different, so we will elaborate more on them.

Networks Before we study the degree sequence of a uniform random graph from $\mathcal{B}_n$, let us introduce an auxiliary graph class that will play subsequently a significant role. Following Trakhtenbrot [22] and Tutte [23] we define a *network* as a connected graph with two designated vertices, called the *left pole* and the *right pole*, such that after adding the edge between the poles and contracting any multiple edges that possibly were created, the resulting graph is in $\mathcal{B}$. The remaining vertices of a network are called *labeled vertices.* Following the standard approach, we will assume that in the exponential generating function of a class of networks the parameter x *always marks the number of labeled vertices.*

The above description provides us with an explicit relationship between the classes of biconnected graphs, and the corresponding networks. We will not provide further details here, and we refer the reader to [22, 23], or to [11] and the references therein for a more detailed treatment. However, we want to remark that an asymptotic study of the properties of random networks suffices for gaining an understanding for the properties of random 2-connected graphs. The following proposition is from [11].

Proposition B.1. *Let $\mathcal{B}$ be a class of biconnected graphs and let $\mathcal{N}$ be the class of the corresponding networks. Moreover, let B_n be a uniform random graph from $\mathcal{B}_n$, and N_n a network that is drawn uniformly at random from $\mathcal{N}_n$. Suppose that* $\Pr[N_{n-2} \in \mathcal{P}] \ge 1 - f(n-2)$, *where $\mathcal{P}$ is any property of graphs that is closed under automorphisms. Then* $\Pr[B_n \in \mathcal{P}] \ge 1 - 2\kappa f(n-2)$, *where κn is the maximum number of edges in a graph in $\mathcal{B}_n$.*

In the sequel we thus only focus on classes of networks. More precisely, following the works [22, 23] we consider classes of networks $\mathcal{N}$ that satisfy certain closure properties. We will say that $\mathcal{N} = \mathcal{N}(\mathcal{T})$ is *closed under* *taking 3-connected components*, if there is a class $\mathcal{T}$ of labeled 3-connected graphs with the following properties.

Any *network* in $\mathcal{N}$ is either an edge, whose endvertices are the poles, or is in class $\mathcal{S}$ (*series network*), or is in class $\mathcal{P}$ (*parallel network*), or is in class $\mathcal{H}$ (*core network*).

Series networks: Any network in $\mathcal{S}$ consists of two networks N_1 and N_2, such that the right pole of N_1 is identified with the left pole of N_2. Here, N_1 is restricted to be either an edge, or a network in $\mathcal{P}$ or in $\mathcal{H}$, and $N_2 \in \mathcal{N}$.

Parallel networks: Any network in $\mathcal{P}$ consists either of an edge and a non-empty set of networks, either in $\mathcal{S}$ or in $\mathcal{H}$, where their right poles (left poles) are identified into a single right pole (left pole), or a set of networks with at least 2 two elements, either in $\mathcal{S}$ or in $\mathcal{H}$ where the identification of the poles is as before.

Core networks: Let $\bar{\mathcal{T}}$ be the class of networks that are created by taking any graph in $\mathcal{T}$, deleting an edge, and then turning its endvertices into poles. Any network in $\mathcal{H}$ consists of a network from $\bar{\mathcal{T}}$, where each edge is replaced by a network whose poles are identified in a unique way with the endvertices of the edges.

In any of the above cases, we do the necessary relabeling in case there are conflicts, when two or more networks are joined through one of the above operations. We will say that a network N *has a (3-connected) core of size* e, if the largest graph from $\mathcal{T}$ that was used in the decomposition of N has e edges. Note that in the previous section "size" referred to the number of vertices of the core. In order to avoid notational overload, we will slightly abuse notation and use "size" here for the number of edges instead. Note that a network can have an empty core, in which case it consists only of series and parallel connections

Note that the notion of the 3-connected core immediately translates to classes of biconnected graphs, through the explicit correspondence between them and classes of networks. So, if a biconnected graph has a non-empty core, then it is obtained by taking a 3-connected graph, and replacing its edges by networks.

Let

$$\Phi(x, y, z) = \frac{2}{x^2} T_y(x, z) - \log\left(\frac{1+z}{1+y}\right) + \frac{xz^2}{1+xz}, \tag{B.10}$$

where for any function F and any variable α we write F_α for the partial derivative of F with respect to α. (From now on, x will always denote the first variable, y the second, and z the third.) In [22, 24] the following was shown.

Lemma B.2. *Let $N(x, y)$ denote the egf enumerating a class of networks $\mathcal{N}$, where x marks the non-pole vertices and y the edges. Then $\Phi(x, y, N(x, y)) = 0$.*

With all the above facts at hand we can define formally the classes of networks that we will consider.

Definition B.3. *Let $\mathcal{N}$ be a class of networks. We say that $\mathcal{N}$ belongs to the* critical networks–3-connected scheme *if it fulfills the following conditions.*

i) $\mathcal{N}$ is closed under taking 3-connected components.

ii) The egf $N(x) = N(x, 1)$ enumerating $\mathcal{N}$ has a unique dominant singularity at $x = \rho_N > 0$ and satisfies $\Phi_z(\rho_N, 1, N(\rho_N)) < 0$.

iii) The function $N(\rho_N, y)$ has a unique dominant singularity at $y = 1$, and a singular expansion of the form

$$N(\rho_N, y) = N_0 + N_2(1-y) + N_3(1-y)^{3/2} + O((1-y)^2),$$

where $N_2 < 0$ and $N_3 \neq 0$, valid in a domain $\Delta = \Delta(\delta, \varepsilon) = \{z \,:\, |z| < \rho_{\bar{T}}(y) + \delta, \arg(z - \rho_{\bar{T}}(y)) > \varepsilon\}$.

Moreover, let $\mathcal{B}$ be the class of biconnected graphs, such that $\mathcal{N}$ is the class of corresponding networks. Then we will also say that $\mathcal{B}$ belongs to the critical networks–3-connected scheme.

The second condition in the above definition guarantees criticality, i.e., a random network from $\mathcal{N}$ will have a large 3-connected core with high probability, see [11]. The third condition is rather technical, and ensures again in a specific way that the class is "planar"-like. The next lemma was shown in [11].

Lemma B.4. *Let $\mathcal{N}$ belong to the critical networks–3-connected scheme. Moreover, let $\mathcal{B}$ be the corresponding class of biconnected graphs, and $\mathcal{T} \subset \mathcal{B}$ be the class of 3-connected graphs in $\mathcal{B}$. If we denote by ρ_B the singularity of the egf enumerating $\mathcal{B}$, and by ρ_T the singularity of the egf enumerating $\mathcal{T}$, then $\rho_N = \rho_B = \rho_T$.*

With the above notation at hand we are ready to present the main result of this section. A main difference compared to the previous section is that now we will not consider random graphs that are drawn according to the uniform distribution with a given number of vertices, but instead graphs that are drawn according to a *weighted* distribution with a given number of *edges*. More precisely, let $\mathcal{G}$ be any class of graphs, and let $\alpha > 0$. We will denote by $\tilde{G}_m = \tilde{G}_m(\alpha)$ a random graph from $\mathcal{G}$ with m edges, where graphs with n vertices are weighted with $\alpha^n/n!$.

(We will always choose α such that this distribution is well-defined.) We will say that the weight of the distribution $\tilde{G}_m$ is α.

Note that the above approach might seem cumbersome, as the uniform distribution on B_n and the weighted distribution $\tilde{B}_m$ have a priori nothing in common. However, it turns out that these distributions are very similar, and studying asymptotic properties of B_n is essentially the same as studying properties of $\tilde{B}_m$, if we choose m close to the expected number of edges in B_n, and set $\alpha = \rho_B$. We refer the reader to [15] for further details.

We will denote by ConcentratedDegreeSequence$(\mathcal{G}, \alpha)$ the following property.

ConcentratedDegreeSequence$(\mathcal{G}, \alpha)$: *There is a function $D_{\mathcal{G}}(z)$, which is analytic in a complex neighborhood of $z = 0$, and a non-decreasing function $k_0(n)$ such that the following is true. Let $\tilde{G}_m$ denote a graph with m edges, that is drawn according to the weighted distribution with weight α from $\mathcal{G}$. Let $\varepsilon > 0$. Then*

$$\Pr\left[\forall k \le k_0(n) : d_k(\tilde{G}_n) \in (1\pm\varepsilon)[z^k]D_{\mathcal{G}}(z)m\right] = 1-o(1).$$

The main result of this section is summarized in the following theorem.

THEOREM B.5. *Let $\varepsilon > 0$ and suppose that $\mathcal{B}$ belongs to a critical biconnected–3-connected scheme. Assume the property ConcentratedDegreeSequence$(\mathcal{T}, \rho_N)$, where $\mathcal{T}$ is the class of 3-connected graphs in $\mathcal{B}$. Then there is a function $D_{\mathcal{B}}(z)$, given in* (B.2)*, and a non-decreasing function $k_0' = k_0'(n)$, given in* (B.2)*, such that for a random graph $\tilde{B}_m(\rho_N)$ from $\mathcal{B}$*

$$\Pr\left[\forall k \le k_0'(n) : d_k(\tilde{B}_m) \in (1\pm\varepsilon)[z^k]D_{\mathcal{C}}(z)m\right] = 1-o(1).$$

The proof of this theorem can be found in the subsequent sections. In particular, the next section presents an approximate sampling algorithm for the weighted distribution on $\mathcal{B}$. Then, as in Section 3.2, we show how this sampler can be used to transfer the bounds for the degree sequence on 3-connected to biconnected graphs.

B.1 Sampling Algorithms The following theorem about the structure of random graphs from $\mathcal{B}$ was shown in [15].

THEOREM B.6. *Let $\varepsilon > 0$, and let $lc(B)$ denote the size of the largest core in a biconnected graph B. Moreover, let $\mathcal{B}$ be a class that belongs to the critical networks–3-connected scheme. Then, if B_m denotes a random graph from $\mathcal{B}$ with m edges and weight ρ_N, then there is a $C = C(\varepsilon) > 0$ such that*

$$\Pr\left[|lc(\tilde{B}_m) - \mu m| \ge Cm^{2/3}\right] \le \varepsilon, \text{ where } \mu = -N_0/N_2.$$

Moreover, uniformly for $|x| \le C$

$$\Pr\left[lc(\tilde{B}_m) = \mu m + xm^{2/3}\right] = \Theta(m^{-2/3}).$$

One further ingredient of our sampler is a probability distribution defined over a class of networks $\mathcal{N}$. More precisely, the *Boltzmann distribution* over $\mathcal{N}$ is given through

$$\forall \gamma \in \mathcal{N} \;:\; \Pr[\gamma] = \frac{\rho_N^{|\gamma|}}{|\gamma|! \cdot N(\rho_N)}. \tag{B.11}$$

With the above tools at hand we are ready to describe our approximate sampling algorithm. In the following we will denote by $\tilde{B}_m$ a random graph from $\mathcal{B}$ with m edges, drawn according to the weighted distribution with weight ρ_N. Moreover, we will denote by $\tilde{T}_m$ a random graph from $\mathcal{T}$, again with m edges and drawn according to the weighted distribution with weight ρ_N. Finally, for a graph or network G, $\|G\|$ denotes the number of edges in G.

$S_{\mathcal{B}}(m, \varepsilon)$: $C \to$ the constant given by Theorem B.6
    $E \to$ a random value according to the distribution of $lc(\tilde{B}_m)$ (*)
    if $|E - \mu m| > Cm^{2/3}$
        return $\perp$
    else
        $T \to$ a random graph from the distribution $\tilde{T}_E$ (**)
        repeat
            $\forall$ edges $e \in T$: choose independently $\gamma_e \in \mathcal{N}$ according to (B.11)
        until ($\sum_{e\in T} \|\gamma_e\| = m$)
        $\forall e \in T$: identify the poles of γ_v with the endpoints of e and remove e
        return the resulting graph, equipped with a random permutation of the labels

The following lemma summarizes the properties of the algorithm that we will exploit, and is the analogue of Lemma 3.6. In particular, it guarantees that the algorithm will fail only with a small probability, and that otherwise it will mimic the weighted distribution on $\mathcal{B}$.

LEMMA B.7. *Let $\varepsilon > 0$ and suppose that $\mu > 1/2$. Then the following statements are true for sufficiently large n.*

- $\Pr[S_{\mathcal{B}}(n, \varepsilon) = \perp] \le \varepsilon$.
- *Let $C = C(\varepsilon)$ be the constant from Theorem B.6, and let B be such that $\|B\| = m$ and $|lc(B) - \mu m| \le Cm^{2/3}$. Then*

$$\Pr[S_{\mathcal{B}}(m, \varepsilon) = B] = \Pr[\tilde{B}_m = B] = \frac{\rho_N^{|B|}}{|B|!} \frac{1}{[y^m]B(\rho_N, y)}.$$

Proof. The proof is very similar to the proof of Lemma 3.6, so we will highlight only the important differences. The bound on the failure probability follows immediately from Theorem B.6. To see the second claim, note first that B is composed out of a core T_B and a sequence networks $(\Gamma_e)_{e\in T}$, which substitute the edges of T_B. The probability that T_B is drawn is

$$\Pr[lb(\tilde{B}_m) = \|T_B\|] \cdot \Pr[T = T_B]$$
$$= \frac{[u^{\|T_B\|}y^m]T(\rho_N, N(\rho,y)u)}{[y^m]B(\rho_N, y)} \cdot \frac{\rho_N^{|T_B|}}{|T_B|!}\frac{1}{[y^{\|T_B\|}]T(\rho_N, y)}.$$

Moreover, the probability that $\gamma_e = \Gamma_e$ for all edges e in T_B is

$$\Pr\left[\forall e \in T_B \;:\; \gamma_e = \Gamma_e \;\middle|\; \sum_{e\in T_B} \|\gamma_e\| = m\right]$$
$$= \Pr\left[\sum_{e\in T_B}\|\gamma_e\| = m\right]^{-1} \cdot \prod_{e\in T}\Pr[\gamma_e = \Gamma_e].$$

But

$$\prod_{e\in T}\Pr[\gamma_e = \Gamma_e] = \frac{\rho_N^{\sum_{e\in T}|\gamma_e|}}{N(\rho_N)^{\|T_B\|}\cdot\prod_{e\in T_B}|\Gamma_e|!}$$

and

$$\Pr\left[\sum_{e\in T_B}\|\gamma_e\| = m\right] = \frac{[y^m]N(\rho_N, y)^{\|T_B\|}}{N(\rho_N)^{\|T_B\|}}.$$

The proof completes by putting all the above together, and observing that the number of ways to distribute the labels is $\binom{|B|}{|T_B|,|\Gamma_1|,\dots,|\Gamma_{\|T_B\|}|}$ and that

$$[y^{\|T_B\|}]T(\rho_N, y)\cdot[y^m]N(\rho_N, y)^{\|T_B\|}$$
$$= [u^{\|T_B\|}y^m]T(\rho_N, N(\rho,y)u).$$

B.2 Transferring the Degree Sequence: 3-connected → 2-connected Graphs Let $B \in \mathcal{B}$ be the graph constructed by running the algorithm from the previous section, and let $T(B)$ denote its core, i.e., the graph T generated in the line marked with $(**)$ while building B. For $k \ge \ell$ set

$$d_{\ell,k}(B) = |\{u \in T(B) : deg(T;u) = \ell \text{ and } deg(B;\,u) = k\}|.$$

In words, $d_{\ell,k}(B)$ is the number of degree-k vertices in B that belong to its core and have degree exactly ℓ within the core. Moreover, if e is an edge of T, then let

$$d'_k(\gamma_e) = |\{u \in \gamma_e, u \text{ is not a pole} : deg(\gamma_e;\,u) = k\}|$$

denote the number of non-pole vertices of degree k in γ_e. With the above notation we immediately obtain

$$d_k(B) = \sum_{\ell=2}^{k} d_{\ell,k}(B) + \sum_{e\in T(B)} d'_k(\gamma_e). \tag{B.12}$$

This relation is at the heart of the proof of Theorem B.5. Now, repeating the work done in Section B.2, we will control the probable values of $d_{\ell,k}$ and d'_k separately. We begin with $d_{\ell,k}$. We say that a class of networks is *symmetric*, if the degree distribution of its left pole is the same as the degree distribution of the right pole.

LEMMA B.8. *Let $\varepsilon,\delta > 0$ and let B be a graph constructed by running the algorithm $S_{\mathcal{B}}(m,\delta)$ from Section B.1. Moreover, set $X_k = \sum_{\ell\ge 2} d_{\ell,k}(B)$. If $\mathcal{T}$ satisfies assumption ConcentratedDegreeSequence($\mathcal{T}, \rho_N$), and $\mathcal{N}$ is symmetric, then uniformly for all $\delta > 0$*

$$\Pr\left[\forall k \le k_0 : |X_k - \mu_k m| \le \varepsilon\mu_k m + (\log m)^2 \middle| B \ne \bot\right] \ge 1 - o(1),$$

where $k_0 := k_0(n/2)$ and $\mu_k = \mu \cdot [z^k]D_{\mathcal{T}}(R_{\mathcal{N}}(z))$.

The proof is essentially the same as the proof of lemma 3.8. The only difference is now that any substituted network may affect the degrees of *two* vertices in the core (instead of one); but this can be handled routinely by Talagrand's inequality instead of the Chernoff bounds.

The next lemma is analogous to Lemma 3.9. Its proof is exactly the same, and thus omitted.

LEMMA B.9. *Let $\varepsilon,\delta > 0$ and let B be a graph constructed by running the algorithm $S_{\mathcal{B}}(m,\delta)$ from Section B.1. Moreover, set $Y_k = \sum_{e\in T} d'_k(\gamma_e)$. Then, uniformly for all $\delta > 0$ we have*

$$\Pr\left[\forall k \le n : |Y_k - \nu_k n| \le \varepsilon\nu_k n + n^{8/9} \middle| B \ne \bot\right] \ge 1 - o(1),$$

where $\nu_k = \mu \cdot [z^k]E_{\mathcal{N}}(z)$.

With the above lemmas at hand, the proof of Theorem B.5 is completed with the same line of reasoning as in the proof of Theorem 3.3, and by setting

$$D_{\mathcal{B}}(z) = \mu\big(D_{\mathcal{T}}(R_{\mathcal{N}}(z)) + E_{\mathcal{N}}(z)\big).$$

and

$$k'_0(n) = \min\Big\{k_0(n/2),$$
$$\max\big\{k \;:\; \forall \ell \le k : [z^\ell]D_{\mathcal{B}}(z) \ge n^{-1/10} \text{ or } [z^\ell]D_{\mathcal{B}}(z) = 0\big\}\Big\}.$$