Component structure of the vacant set induced by a random walk on a random graph.

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September 29, 2010

Abstract

We consider random walks on two classes of random graphs and explore the likely structure of the the set of unvisited vertices (or vacant set). Let $\Gamma(t)$ be the subgraph induced by the vacant set. We show that for random graphs $G_{n,p}$ above the connectivity threshold, and for random regular graphs G_r, for constant $r \geq 3$, there is a phase transition in the sense of the well-known Erdős-Renyi phase transition. Thus for $t \leq (1-\epsilon)t^*$ we have a unique giant plus components of size $O(\log n)$ and for $t \geq (1+\epsilon)t^*$ we have only components of size $O(\log n)$. In the case of G_r we describe the likely degree sequence and structure of the small $(O(\log n))$ size components.

1 Introduction

The problem we consider can be described as follows. We have a finite graph $G = (V, E)$, and a simple random walk $\mathcal{W}_u$ on G, starting at $u \in V$. What is the likely component structure induced by the unvisited vertices of G at step t of the walk?

Initially all vertices V of G are unvisited or *vacant*. We regard unvisited vertices as colored *red*. When $\mathcal{W}_u$ visits a vertex, the vertex is re-colored *blue*. Let $W_u(t)$ denote the position of $\mathcal{W}_u$ at step t. Let $\mathcal{B}(t) = \{W_u(0), W_u(1), \ldots, W_u(t)\}$ be the set of blue vertices at the end of step t, and $\mathcal{R}_u(t) = V \setminus \mathcal{B}_u(t)$. Let $\Gamma(t)$ be the subgraph of G induced by $\mathcal{R}(t)$. Initially $\Gamma(0)$ is connected, unless u is a cut-vertex. As the walk continues, $\Gamma(t)$ will shrink to the empty graph once every vertex has been visited. We wish to determine, as far as possible, the likely evolution of the component structure as t increases.

Our motivation for studying this problem is twofold. Firstly it gives us a precise picture of the likely progress of a random walk up to the point when all vertices have been visited. Secondly we were curious to know how long the graph could resist the walk by maintaining a giant unvisited component.

In this paper we will consider two models of random graph: The random graph $G_{n,p}$ and the random r-regular graph $G_r, r \geq 3$, both on vertex set $V = [n]$.

Recall the typical evolution of the random graph $G_{n,p}$ as p increases from 0 to 1. Initially the graph consists of isolated vertices. As we increase p, or equivalently add random edges, we find that the maximum component size increases from logarithmic size for $p = c/n, c < 1$ and then after the phase transition for $c > 1$ there is jump in maximum component size to linear in n and all other components have logarithmic size. Our aim in this paper is to show that **whp** $\Gamma(t)$ undergoes a reversal of this. In both cases $\mathcal{G}(0)$ is connected and it will start to break up. There will be a critical value t^* such that if $t < t^*$ by a sufficient amount then $\mathcal{G}(t)$ consists of a unique giant component plus a collection of components of size $O(\log n)$. Once we have passed through the critical value, i.e. $t > t^*$ by a sufficient amount then all components are of size $O(\log n)$. The maximum component size will then shrink to zero. We now provide some more details. We say that a graph is *sub-critical* if its maximum component size is $O(\log n)$ and super-critical if is has a *unique* component of size $\Omega(n)$ and all other components are of size $O(\log n)$.

The probability error estimates we make can come from two sources; the space of random graphs, and the random walk on a fixed graph. The space of random graphs does not pose much of a problem. We choose a fixed subset of size $(1 + o(1))$, *nice* graphs, which have the properties we need in order to analyze the

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behavior of the random walk. Choosing a nice graph G, the vacant set of the random walk is still a random set, but, **whp** (in the random walk) the structure is as given above. We note that we use the phrase *choosing randomly* to mean sampling uniformly at random from the given set of available choices.

We first consider a random walk on $G_{n,p}$. We assume that

$$p = \frac{c \log n}{n} \text{ where } (c-1)\log n \to \infty \text{ with } n$$

and

$$c = n^{o(1)}.$$

Let

$$t_\theta = n(\log\log n + (1+\theta)\log c).$$

Theorem 1.1. *Let $\epsilon > 0$ be a small constant. Then* **whp** *we have (i) $\Gamma(t)$ is super-critical for $t \leq t_{-\epsilon}$ and (ii) $\Gamma(t)$ is sub-critical for $t \geq t_\epsilon$.*

The bound $c = n^{o(1)}$ is not optimal. On the other hand there is not going to be a phase transition if p is constant. For example, the vacant set of the complete graph K_n remains connected until the cover time.

Next we consider a random walk on G_r for $r \geq 3$, constant.

Let

$$t^* = \frac{r(r-1)\log(r-1)}{(r-2)^2} n. \tag{1.1}$$

Theorem 1.2. *Let $\epsilon > 0$ be a small constant. Then* **whp** *we have (i) $\Gamma(t)$ is super-critical for $t \leq (1-\epsilon)t^*$, (ii) $\Gamma(t)$ is sub-critical for $t \geq (1+\epsilon)t^*$ and (iii) at some time $t \sim t^*$ the maximum component size in $\Gamma(t)$ is $n^{2/3+o(1)}$.*

For a random walk on G_r, the degree sequence and component size of $\Gamma(t)$ can also be found. Let

$$\rho = \frac{r-1}{r-2}$$

and let

$$N_t = ne^{-\frac{t}{\rho n}}.$$

Let

$$\tau_k = n^{1-1/k} \text{ and } t_k = \frac{\rho r n \log n}{k(r-2)+2}. \tag{1.2}$$

Let $\omega/\log\log n \to \infty$ and $\omega = o(\log n)$.[1]

Theorem 1.3. *Let $D_s(t)$ be the number of vertices of degree s in $\Gamma(t)$ and let*

$$p_t = e^{-\frac{(r-2)t}{\rho r n}}. \tag{1.3}$$

Whp

(a) $|\mathcal{R}(t)| = (1+o(1))N_t$.

(b) *For $0 \leq s \leq r$,*

$$D_s(t) = \begin{cases} (1+o(1))N_t\binom{r}{s}p_t^s(1-p_t)^{r-s} & (i) \\ 0 & (ii) \end{cases}$$

where in (i) we have $\tau_{r-s} \ll t \leq t_{s+1} - \omega n$ and in (ii) we have $t \ll \tau_{r-s}$ or $t \geq t_{s+1} + \omega n$.

The range $t_s - \omega n \leq t \leq t_s + \omega n$ contains the times when the number of vertices of degree s is constant in expectation and unlikely to be concentrated around its mean. We do not think that the asymptotic distribution is Poisson at this point. We can give some more information about the number of small components in $\Gamma(t)$. Again there is a gap containing the times when the expected number of such components is constant.

Theorem 1.4. *Let ϵ be a small positive constant and let $1 \leq k \leq \epsilon \log n$ and $t \leq (1-\epsilon)t_k$. Let $N(k,t)$ denote the number of components of $\Gamma(t)$ with k vertices. Then* **whp**

(a) *If $\tau_{k(r-2)+2} \ll t \leq (1-\epsilon)t_k$ then*
$N(k,t) = \frac{(1+o(1))r}{k((r-2)k+2)} N_t \binom{(r-1)k}{k-1} p_t^{k-1}(1-p_t)^{k(r-2)+2}$

(b) *If $t \ll \tau_{k(r-2)+2}$ or $t \geq (1+\epsilon)t_k$, then $N(k,t) = 0$.*

Note that we do not claim that we can prove that the statements of Theorem 1.4 hold for all t **simultaneously**, although we suspect they do. Our proofs only show this to be true for most values of t.

1.1 Previous work.

The only previous works on this subject that we are aware of are Benjamini and Sznitman [1], Windisch [15] and Černy, Teixeira and Windisch [5]. Papers [1], [15] deal with the component

[1] In what follows, $1+o(1)$ means $1+O(\log^{-K} n)$ for all constant $K > 0$.

structure of the vacant set for a random walk on a d-dimensional torus. Paper [5] deals with random walks on G_r, and shows that **whp** $\Gamma(t)$ is sub-critical for $t \geq (1+\epsilon)t^*$ and there is a unique linear size component for $t \leq (1-\epsilon)t^*$. The authors conjecture that $\Gamma(t)$ is super-critical for $t \leq (1-\epsilon)t^*$ and we prove this conjecture.

2 Uniformity.

The main idea is to realise that the graph $\Gamma(t)$ has a simple distribution. First consider $G_{n,p}$.

LEMMA 2.1.
Consider a random walk on $G_{n,p}$. Conditional on $|\mathcal{R}(t)| = N$, $\Gamma(t)$ is distributed as $G_{N,p}$.

Proof This follows easily from the principle of deferred decisions. We do not have to expose the existence or absence of edges between vertices in $\mathcal{R}(t)$ until one of them is exposed. $\square$

Thus to prove Theorem 1.1 we only need high probability estimates of $|\mathcal{R}(t)|$. These are given in Section 4.

We next consider G_r. We give two structural definitions.

LEMMA 2.2.
Consider a random walk on G_r. Conditional on $N = |\mathcal{R}(t)|$ and $\mathcal{R}(t)$ having degree sequence $\mathbf{d} = d_{\Gamma(t)}(v), v \in \mathcal{R}(t)$, then $\Gamma(t)$ is distributed as $G_{N,\mathbf{d}}$, the random graph with vertex set $[N]$ and degree sequence $\mathbf{d}$.

Proof Suppose that we condition on $\mathcal{R}(t)$ and the *history* $\mathcal{H} = (W_u(0), W_u(1), \ldots, W_u(t))$. If G_1, G_2 are graphs with vertex set $\mathcal{R}(t)$ and if they have the same degree sequence then substituting G_2 for G_1 will not conflict with $\mathcal{H}$ i.e. every extension of G_1 is an extension of G_2 and vice-versa. $\square$

Thus to prove Theorem 1.2 we only need high probability estimates of the degree sequence of $\Gamma(t)$. The proof of Theorem 1.4 can in principle be derived from this, although we do not have a simple way of doing it. Instead we rely on a further characterization of $\Gamma(t)$.

We use the configuration or pairing model of Bollobás [3] and Bender and Canfield [4]. We start with n disjoint sets of points $W_1, W_2, \ldots, W_n$ each of size r. We let $W = \bigcup_{i=1}^n W_i$. A *configuration* F is a partition of W into $rn/2$ pairs i.e. a *pairing*. Ω is the set of configurations. If $F \in \Omega$ we define an r-regular multi-graph $G_F = ([n], E_F)$ where $E_F = \{(i,j) : \exists \{x,y\} \in F : x \in W_i, y \in W_j\}$, i.e. we contract W_i to a vertex i for $i \in [n]$.

It is known that (i) each (simple) graph arises the same number of times as G_F and (ii)if r is constant, the probability that G_F is simple is bounded below by a constant. Thus if F is chosen uniformly at random from Ω then any event that occurs **whp** for G_F will occur **whp** for G_r.

Suppose now that we generate a random F as we do a random walk on $[n]$. We begin with a starting value i_1 and at the start of t-th step we are at some i_t and we have a partition R_t, B_t of W into Red and Blue points respectively. Initially, $R_1 = W$ and $B_1 = \emptyset$. In addition we have a collection F_t of disjoint pairs from W. Initially $F_1 = \emptyset$.

The vertex i_t is in $\mathcal{B}(t)$. To choose a random edge incident with i_t for the next transition, we choose x randomly from W_{i_t}. In either case below, it is possible that $i_{t+1} = i_t$.

Suppose first that $x \in R_t$, then this is a previously unused edge. We choose y randomly from $R_t \setminus \{x\}$ and let $F_{t+1} = F_t \cup \{\{x,y\}\}$. Suppose that $y \in W_j$. This is equivalent to moving from $i_t \in \mathcal{B}(t)$ to $i_{t+1} = j$. We remove x, y from R_t to give R_{t+1} and move them to B_t, to give B_{t+1}. If $i_{t+1} \in \mathcal{R}(t)$, this transition is to an unvisited vertex, and if $i_{t+1} \in \mathcal{B}(t)$, this transition is between visited vertices along an unused edge.

If on the other hand, $x \in B_t$, then it has previously been paired with a $y \in W_j \cap B_t$ and we move from i_t to $i_{t+1} = j$. In this case $R_{t+1} = R_t$, $B_{t+1} = B_t$ and $F_{t+1} = F_t$.

After t steps we will have constructed a random collection F_t of $\leq t$ disjoint pairs from W. We can extend F_t to a random configuration F by adding a random pairing of R_t to it.

LEMMA 2.3.

- **(a)** *F_t plus a random pairing of R_t is a random member of Ω.*
- **(b)** *$i \in \mathcal{R}(t)$ iff $W_i \subseteq R_t$.*

$\square$

3 Vacancy probabilities.

As in our previous papers on random walks on random graphs, we make heavy use of Lemma 3.1 below. Let P be the transition matrix of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(W_u(t) = v)$ be the t-step transition probability. We assume the random walk $\mathcal{W}_u$ on G is ergodic, and thus the random walk has stationary distribution π, where $\pi_v = d(v)/(2m)$.

Let λ_2 be the second eigenvalue of P, and let Φ_G be the conductance of G i.e.

$$\Phi_G = \min_{S \subseteq V, \pi_S \leq 1/2} \frac{\sum_{x \in S} \pi_x P(x, \bar{S})}{\pi_S}.$$

Then,

$$1 - \Phi_G \leq \lambda_2 \leq 1 - \frac{\Phi_G^2}{2} \tag{3.4}$$

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda_2^t. \tag{3.5}$$

A proof of this can be found for example in Jerrum and Sinclair [12]. In addition, Friedman [10] has shown that **whp** $\lambda_2 \leq (2\sqrt{r-1} + \epsilon)/r \leq 29/30$, say. (For (3.5) we need $\lambda_2 = \lambda_{\max}$ which can be achieved by making the chain lazy i.e. by not moving with probability 1/2 at each step. This has no significant effect on the analysis).

Let T be such that, for $t \geq T$

$$\max_{u, x \in V} |P_u^{(t)}(x) - \pi_x| = O\left(\frac{\min_{x \in V} \pi_x}{n^3}\right). \tag{3.6}$$

In which case we can **whp** take

$$T \leq 120 \log n. \tag{3.7}$$

If inequality (3.6) holds, we say the distribution of the walk is in *near stationarity*. Fix two vertices u, v. Let $h_t = \mathbf{Pr}(W_u(t) = v)$ be the probability that the walk $\mathcal{W}_u$ visits v at step t. Let

$$H(z) = \sum_{t=T}^{\infty} h_t z^t \tag{3.8}$$

generate h_t for $t \geq T$.

We next consider the returns to vertex v made by a walk $\mathcal{W}_v$, starting at v. Let $r_t = \mathbf{Pr}(W_v(t) = v)$ be the probability that the walk returns to v at step $t = 0, 1, \ldots$. In particular note that $r_0 = 1$, as the walk starts on v. Let

$$R(z) = \sum_{t=0}^{\infty} r_t z^t$$

generate r_t, and let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \tag{3.9}$$

Thus, evaluating $R_T(z)$ at $z = 1$, we have $R_T(1) \geq r_0 = 1$.

For $t \geq T$ let $f_t = f_t(u \to v)$ be the probability that the first visit made to v by the walk $\mathcal{W}_u$ to v *in the period* $[T, T+1, \ldots]$ occurs at step t. Let

$$F(z) = \sum_{t=T}^{\infty} f_t z^t$$

generate f_t. Then we have

$$H(z) = F(z) R(z). \tag{3.10}$$

The following lemma gives the probability that a walk, starting from near stationarity makes a first visit to vertex v at a given step. For proofs of the lemma and its corollary, see [6], [7].

Lemma 3.1. *Let $R_v = R_T(1)$, where $R_T(z)$ is from (3.9). For some sufficiently large constant K, let*

$$\lambda = \frac{1}{KT}, \tag{3.11}$$

where T satisfies (3.6). *Suppose that*

(i) *For some constant $\theta > 0$, we have*

$$\min_{|z| \leq 1 + \lambda} |R_T(z)| \geq \theta.$$

(ii) *$T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$.*

There exists

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}, \tag{3.12}$$

such that for all $t \geq T$,

(3.13)
$$f_t(u \to v) = (1 + O(T\pi_v)) \frac{p_v}{(1+p_v)^{t+1}} + O(T\pi_v e^{-\lambda t/2}).$$

$$= (1 + O(T\pi_v)) \frac{p_v}{(1+p_v)^t} \qquad \textit{for } t \geq \log^3 n. \tag{3.14}$$

COROLLARY 3.1. *For $t \geq T$ let $\mathcal{A}_v(t)$ be the event that $\mathcal{W}_u$ does not visit v at steps $T, T+1, \ldots, t$. Then, under the assumptions of Lemma 3.1,*

$$\mathbf{Pr}_{\mathcal{W}}(\mathcal{A}_v(t)) = \frac{(1+O(T\pi_v))}{(1+p_v)^t} + O(T^2\pi_v e^{-\lambda t/2}) \tag{3.15}$$

$$= \frac{(1+O(T\pi_v))}{(1+p_v)^t} \qquad \textit{for } t \geq \log^3 n. \tag{3.16}$$

We use the notation $\mathbf{Pr}_{\mathcal{W}}$ when we want to emphasize that we are dealing with the probability space of walks on G.

4 The evolution of $\Gamma(t)$ in $G_{n,p}$

Because $\Gamma(t)$ has the distribution $G_{N,p}$, we only need to get good estimates of N and we can get these from Lemma 3.1. Assume that $c = n^{1/\omega}$ where $\omega \to \infty$. It is shown in [7] that **whp** $R_v = 1 + O(1/\log n)$ for all $v \in V$. Let $\omega_1 = \log^{1/3} n$. For a fixed vertex v we have that its degree $d_G(v)$ satisfies

$$\mathbf{Pr}(|d_G(v) - c\log n| \geq \omega_1(c\log n)^{1/2}) \leq 2e^{-\omega_1^2/3}.$$

This follows from Chernoff bounds on the binomial.

So by the Markov inequality, we see that **whp** all but $O(ne^{-\omega_1^2/4})$ vertices have degrees in the range $c\log n \pm \omega_1(c\log n)^{1/2}$. Denote these vertices by $\mathcal{N}_d$. Fix such a vertex, then (3.16) implies that

$$\mathbf{Pr}(v \in \mathcal{R}(t)) = (1 + O(\log n/n))e^{-(1+O(1/\omega_1))t/n}$$
$$= e^{-(1+O(1/\omega_1))t/n}$$

if we assume that $t \sim n\log\log n$. So if $t = t_\theta$ where $\theta = O(1)$ then

$$\mathbf{E}(|\mathcal{N}_d|) = (1+o(1))\frac{n}{c^{1+\theta}\log n}.$$

We can argue as in the proof (5.33) that if $v, w \in \mathcal{N}_d$ are at distance at least $\omega/2$ in G then

$$\mathbf{Pr}(\mathcal{E}_v \cap \mathcal{E}_w) = (1+o(1))\mathbf{Pr}(\mathcal{E}_v)\mathbf{Pr}(\mathcal{E}_w). \tag{4.17}$$

where $\mathcal{E}_v = \{v \in \mathcal{R}(t)\}$.

Then if $X = |\mathcal{R}(t) \cap \mathcal{N}_d|$ then

$$\mathbf{E}(X(X-1)) \leq \mathbf{E}(X)^2 + O((c\log n)^{\omega/2})\mathbf{E}(X)$$

which implies that $\mathbf{Var}(X) = o(\mathbf{E}(X)^2)$ and then the Chebyshev inequality implies that $X \sim \mathbf{E}(X) \sim \frac{n}{c^{1+\theta}\log n}$ **whp**. The vertices outside $\mathcal{N}_d$ only contribute $o(N)$ **whp** and Theorem 1.1 follows immediately from this.

5 The evolution of $\Gamma(t)$ in G_r

5.1 Estimates of R_v.

Let

$$\ell_1 = \epsilon_1 \log_r n \tag{5.18}$$

for some sufficiently small ϵ_1. A cycle C is *small* if $|C| \leq \ell_1$. A vertex is *nice* if it is at distance at least ℓ_1 from any small cycle. Let $\mathcal{N}$ denote the nice vertices and $\bar{\mathcal{N}}$ denote the vertices that are not nice.

It is straightforward to prove by first moment calculations that for G_r:

(5.19) **Whp** at most $n^{2\epsilon_1}$ vertices are not nice.

(5.20) **Whp** no two small cycles are within distance ℓ_1 of each other.

Let $N_k(v)$ denote the set of vertices at distance at most k from v, $k \geq 1$. A vertex v is *tree-like* to depth k if $N_k(v)$ induces a tree, rooted at v. Thus a nice vertex is tree-like to depth $\ell_1/2$.

LEMMA 5.1.

(a) *If v is nice then*

$$R_v = (1+o(1))\rho \textit{ where } \rho = \frac{r-1}{r-2}.$$

(b) *If v is not nice then*

$$R_v \leq (1+o(1))\frac{r}{r-2}.$$

Proof (a) Let H denote the subgraph of G induced by $N_{\ell_1/2}(v)$. This is a tree and we can embed it into an infinite r-regular tree $\mathcal{T}$ rooted at v. Let $\mathcal{X}$ be the walk on $\mathcal{T}$, starting from v, and let X_t be the distance of $\mathcal{X}$ from the root vertex at step t.

Let $W_0 = 0$, and let W_t be the distance from v of $\mathcal{W}$ in G at step t. We note that we can couple $\mathcal{W}_v, \mathcal{X}$ so that $W_t = X_t$ up until the first time that $W_t > \ell_1/2$..

The values of X_t are as follows: $X_0 = 0$, $X_1 = 1$, and if $X_t = 0$ then $X_{t+1} = 1$. If $X_t > 0$ then

$$X_t = \begin{cases} X_{t-1} - 1 & \text{with probability } q = \frac{1}{r} \\ X_{t-1} + 1 & \text{with probability } p = \frac{r-1}{r}. \end{cases} \tag{5.21}$$

We note the following result (see e.g. [9]), for a random walk on the line $= \{0, ..., a\}$ with absorbing states $\{0, a\}$, and transition probabilities q, p for moves left and right respectively. Starting at vertex z, the probability of absorption at the origin 0 is

$$\rho(z, a) = \frac{(q/p)^z - (q/p)^a}{1 - (q/p)^a} \leq \left(\frac{q}{p}\right)^z, \tag{5.22}$$

provided $q \leq p$.

Let $U_\infty = \{\exists t \geq 1 : X_t = 0\}$, i.e. the event that the particle ever returns to the root vertex in $\mathcal{T}$. It follows from (5.22) with $z = 1$ and $a = \infty$ that

$$\mathbf{Pr}(U_\infty) = \frac{1}{r-1}. \tag{5.23}$$

It follows that the expected number of visits by $\mathcal{X}$ to v is $\frac{1}{1-\frac{1}{r-1}} = \rho$. We write

$$R_v = \sum_{t=0}^{T} r_t \text{ and } \rho = \sum_{t=0}^{\infty} \rho_t$$

where $\rho_t = \mathbf{Pr}(X_t = v)$.

Now $r_t = \rho_t$ for $t \leq \ell_1/2$ and part (a) follows from

$$\sum_{t=\ell_1/2+1}^{T} r_t = o(1) \text{ and } \sum_{t=\ell_1/2+1}^{\infty} \rho_t = o(1). \tag{5.24}$$

The first equation of (5.24) follows from

$$\left| r_t - \frac{1}{n} \right| \leq \lambda_{\max}^t$$

where $\lambda_{\max}$ is the second largest eigenvalue of the walk. This follows from (3.5).

The second equation of (5.24) is proved in Lemma 7 of [6] where it is shown that

$$\sum_{t=\ell_1/2+1}^{\infty} \rho_t \leq \sum_{2j=\ell_1/2+1}^{\infty} \binom{2j}{j} \frac{(r-1)^j}{r^{2j}} \leq \sum_{2j=\ell_1/2+1}^{\infty} \left(\frac{4(r-1)}{r^2}\right)^j.$$

(b) We next note a property of random walks on undirected graphs which follows from results on electrical networks (see e.g. Doyle and Snell [8]). Let v be a given vertex in a graph G and S a set of vertices disjoint from v. Let $p(G)$, *the escape probability*, be the probability that, starting at v, the walk reaches S before returning to v. For an unbiased random walk,

$$p = \frac{1}{d(v) R_{EFF}},$$

where R_{EFF} is the effective resistance between v and S in G. We assume each edge of G has resistance 1. In the notation of this paradigm, deleting an edge corresponds to increasing the resistance of that edge to infinity. Thus by Raleigh's Monotonicity Law, if edges are deleted from G to form a sub-graph G' then $R'_{EFF} \geq R_{EFF}$. So, if we do not delete any edges incident with v then $p' \leq p$.

It follows from (5.20) that H becomes a tree after removing one edge. We can remove an edge not incident with v. By the above discussion on electrical resistance we see that this will not decrease $\mathbf{Pr}(U_\infty^*)$, where this is U_∞ defined with respect to $\mathcal{T}^*$ which is $\mathcal{T}$ less one edge, not incident with v. We can argue crudely that

$$\mathbf{Pr}(U_\infty^*) \leq \frac{1}{r} + \frac{r-1}{r} \cdot \frac{1}{r-1} = \frac{2}{r}.$$

This is because there is an $\frac{r-1}{r}$ chance of a first move to a part of the tree that has branching factor $r-1$ at every vertex.

Let $U_1^* = \{\mathcal{X}$ returns to the root vertex after starting at $\ell_1/2\}$. Then, with f_T equal to the probability of a return by $\mathcal{W}_v$ to v during $[1, T]$, we have

$$f_T \leq \mathbf{Pr}(U_\infty^*) + T\mathbf{Pr}(U_1^*). \tag{5.25}$$

The RHS of (5.25) is at least the probability that $\mathcal{W}_v$ returns before reaching distance $\ell_1/2$ or returns after reaching distance $\ell_1/2$ at some time $t \leq T$.

Now, using (5.22), we see that

$$\mathbf{Pr}(U_1) \leq \frac{1}{(r-1)^{\ell_1/4}}. \tag{5.26}$$

Here we have $\ell_1/4$ in place of $\ell_1/2$ to account for the one place where we move left with probability $\frac{1}{r-2}$. We argue that at least one of the paths from v to w or w to the boundary must be at least $\ell_1/4$ and not use the vertex incident to the deleted edge.

Thus $f_T \leq (2 + o(1))/r$ and since $R_v \leq \frac{1}{1-f_T}$ we have $R_v \leq \frac{r+o(1)}{r-2}$. □

5.2 Proof of Theorem 1.3.

We will assume initially assume that $t \geq \log^3 n$ and deal with the very beginning of the walk in Section 5.5.

Corollary 3.1 gives the probability of not visiting a single vertex in time $[T, t]$. We need to extend this to certain small sets of vertices. In particular we need to consider sets consisting of v and a subset of its neighbours $N(v)$. Let S be such a subset.

Suppose now that S is a subset of V with $|S| = o(n)$. By contracting S to single vertex $\gamma = \gamma(S)$, we form a graph $H = H(S)$ in which the set S is replaced by γ and the edges that were contained in S are contracted to loops. The probability of no visit to S in G can be found (up to a multiplicative error of $1+O(1/n^3)$) from the probability of a first visit to γ in H. This is the content of Lemma 5.2 below.

We first check the mixing time of a walk on H. The conductance of H is at least that of G, because the set of values that we minimize over for H is a subset of the set of values that we minimize over for G. It follows that the mixing time for $\mathcal{W}$ in H is also $O(\log n)$.

LEMMA 5.2. *[7] Let $\mathcal{W}_u$ be a random walk in G starting at $u \notin S$, and let $\mathcal{X}_u$ be a random walk in H starting at $u \neq \gamma$. Let T be a mixing time satisfying* (3.6) *in both G and H. Then*

$$\mathbf{Pr}(\mathcal{A}_\gamma(t); H) = \mathbf{Pr}(\wedge_{v\in S}\mathcal{A}_v(t); G)\left(1 + O\left(\frac{1}{n^3}\right)\right),$$

where the probabilities are those derived from the walk in the given graph.

Proof

Note that $m = rn/2 = |E(G)| = |E(H)|$. Let $W_x(j)$ (resp. $X_x(j)$) be the position of walk $\mathcal{W}_x$ (resp. $\mathcal{X}_x(j)$) at step j. Let $\Gamma = G, H$ and let $P_u^s(x;\Gamma)$ be the transition probability in Γ, for the walk to go from u to x in s steps.

$$\mathbf{Pr}(\mathcal{A}_\gamma(t); H) =$$
$$\sum_{x\neq\gamma} P_u^T(x;H)\,\mathbf{Pr}(X_x(s-T)\neq\gamma,\ T\le s\le t; H) =$$
(5.27)
$$\sum_{x\neq\gamma}\left(\frac{d(x)}{2m}(1+O(n^{-3}))\right)\times$$
$$\mathbf{Pr}(X_x(s-T)\neq\gamma,\ T\le s\le t; H) =$$
(5.28)
$$\sum_{x\notin S}\left(P_u^T(x;G)(1+O(n^{-3}))\right)\times$$
$$\mathbf{Pr}(W_x(s-T)\notin S,\ T\le s\le t; G) =$$
$$\mathbf{Pr}(\wedge_{v\in S}\mathcal{A}_v(t); G)(1+O(1/n^3)).$$

Equation (5.27) follows from (3.6). Equation (5.28) follows as there is a natural measure preserving map ϕ between walks in G that start at $x \notin S$ and avoid S and walks in H that start at $x \neq \gamma$ and avoid γ. □

Fix $v \in V$ and let $N(v) = \{w_1, w_2, \ldots, w_r\}$ and choose $0 \le s \le r$. We estimate

$$\Pi_s(v,t) = \mathbf{Pr}_{\mathcal{W}}(v \text{ is a vertex of degree } s \text{ in } \Gamma(t)) =$$
$$\binom{r}{s}\mathbf{Pr}_{\mathcal{W}}(\{v, w_1, ..., w_s\} \subseteq \mathcal{R}(t), \{w_{s+1}, ..., w_r\} \subseteq \mathcal{B}(t))$$

Thus

$$\mathbf{E}D_s(t) = \sum_{v\in V}\Pi_s(v,t).$$

The next lemma gives us enough information to compute the expected degree sequence of nice vertices in $\mathcal{R}(t)$. We use the subscript G, in $\mathbf{Whp}_G$, when we want to emphasize that the probability space is random r-regular graphs.

LEMMA 5.3. $\mathbf{Whp}_G$ *for all nice vertices v,*

(a) $\mathbf{Pr}_{\mathcal{W}}(v \in \mathcal{R}(t)) = (1+o(1))e^{-\frac{t}{\rho n}}$.
(b)

$$\mathbf{Pr}_{\mathcal{W}}(\{v, w_1, \ldots, w_s\} \subseteq \mathcal{R}(t),\ \{w_{s+1}, \ldots, w_r\} \subseteq \mathcal{B}(t))$$
$$= (1+o(1))e^{-\frac{t}{\rho n}}p_t^s(1-p_t)^{r-s},$$

and p_t is given by (1.3).

Proof Part (a) follows directly from Lemmas 3.1 and 5.1.

For part (b) we can write

$$\mathbf{Pr}_{\mathcal{W}}(\{v, w_1, \ldots, w_s\} \subseteq \mathcal{R}(t), \{w_{s+1}, \ldots, w_r\} \subseteq \mathcal{B}(t))$$
$$= \sum_{X\subseteq[s+1,r]}(-1)^{|X|}\mathbf{Pr}_{\mathcal{W}}((\{v, w_1, \ldots, w_s\}\cup X) \subseteq \mathcal{R}(t))$$
$$= \sum_{X\subseteq[s+1,r]}(-1)^{|X|}\,\frac{1+O(T\pi_{\gamma_X})}{(1+p_{\gamma_X})^t},$$

where $|X| = 0, ..., r-s$. If $s = 0$, we suppose that $\{v, w_1, \ldots, w_s\} = \{v\}$. The term γ_X denotes the contraction of $\{v, w_1, \ldots, w_s\}\cup X$, and

$$\pi_{\gamma_X} = O\left(\frac{1}{n}\right) \text{ and}$$
$$p_{\gamma_X} = (1+o(1))\,\frac{((r-2)(s+|X|)+r)(r-2)}{r(r-1)n}.$$

Explanation: The expression for p_{γ_X} is the product of $\frac{r(s+|X|+1)}{rn}$, the total degree of γ_X divided by the total degree, and the inverse of the expected number of returns to γ_X in the mixing time. Since v is nice, the expected number of returns (up to a factor $1+o(1)$) is $\frac{1}{1-f}$ where f is the probability of return to the root in an infinite tree with branching factor $r-1$ at each non-root vertex. At the root there are $s+|X|$ loops and $(r-2)(s+|X|)+r$ branching edges. This gives

$$f = \frac{2(s+|X|)}{(s+|X|+1)r} + \left(1 - \frac{2(s+|X|)}{(s+|X|+1)r}\right)\frac{1}{r-1}$$

and

$$1-f = \frac{((r-2)(s+|X|)+r)(r-2)}{(s+|X|+1)r(r-1)}.$$

Thus, for $s = 0,1,...,r$,

(5.29)
$$\begin{aligned}&\mathbf{Pr}_{\mathcal{W}}(\{v,w_1,\ldots,w_s\} \subseteq \mathcal{R}(t), \{w_{s+1},\ldots,w_r\} \subseteq \mathcal{B}(t))\\ &= \exp\left\{-(1+o(1))\frac{(r-2)^2 s + r(r-2)}{r(r-1)n}t\right\} \times \\ &\sum_{X\subseteq[s+1,r]} (-1)^{|X|} \exp\left\{-(1+o(1))\frac{(r-2)^2|X|}{r(r-1)n}t\right\}\end{aligned}$$

When $t = O(n)$ we can write (5.29) as $o(1)$ plus

$$\exp\left\{-(1+o(1))\frac{(r-2)^2 s + r(r-2)}{r(r-1)n}t\right\} \times \left(1 - \exp\left\{-\frac{(r-2)^2}{r(r-1)n}t\right\}\right)^{r-s}.$$

and (b) follows, since the terms above are $\Omega(1)$. When $t/n \to \infty$ we go back to (5.29) and observe that the sum is $1 - o(1)$ and thus

$$\begin{aligned}&\mathbf{Pr}_{\mathcal{W}}(\{v,w_1,\ldots,w_s\} \subseteq \mathcal{R}(t), \{w_{s+1},\ldots,w_r\} \subseteq \mathcal{B}(t))\\ &= (1+o(1))\exp\left\{-\frac{(r-2)^2 s + r(r-2)}{r(r-1)n}t\right\}\end{aligned}$$

as required. □

It follows from Lemmas 3.1 and 5.1 that

$$\textbf{whp } \bar{\mathcal{N}} \subseteq \mathcal{B}(t) \text{ } for \text{ } t \geq 10\epsilon_1 n \log n. \tag{5.30}$$

Lemma 5.3 verifies that N_t and $D_s(t)$ have expectations asymptotic to the claimed values in Theorem 1.3. The expressions in this lemma only apply to nice vertices. But (5.30) allows us to use them for $t \geq t_2$. On the other hand, for $t \leq t_2$ we have

$$\mathbf{E}(D_s(t)) \geq n^{1/2} \gg n^{2\epsilon_1}$$

and the vertices of $\bar{\mathcal{N}}$ are asymptotically negligible. We will therefore ignore them in computations concerning $D_s(t)$ from now on.

In particular Lemma 5.3 and the Markov inequality already show that

(5.31)
$D_s(t) = 0$ **whp** for $s > 0$ and $t \geq t_{s+1} + \omega n$ or $t \ll \tau_{r-s}$.

Thus **whp** $\Gamma(t)$ consists of isolated vertices only, from t_2 until the cover time $\sim t_1$.

We now need to prove concentration for N_t and $D_s(t)$.

Let $\mathcal{R}_{\mathcal{N}}(t) = \mathcal{R}(t) \cap \mathcal{N}$ and $Z_{\mathcal{N}}(t) = |\mathcal{R}_{\mathcal{N}}(t)|$. We will use the Chebyshev inequality. Suppose that $t \leq t_1$. We will show that

(5.32)
$$\mathbf{Var}(Z_{\mathcal{N}}(t)) = O(r^{\omega}\mathbf{E}(Z_{\mathcal{N}}(t))) + e^{-a\omega}\mathbf{E}(Z_{\mathcal{N}}(t)))^2.$$

for some constant $a > 0$.

Fix $t \leq t_1$ and let $\mathcal{E}_v$ be the event that vertex $v \in \mathcal{R}(t)$. Let ω be as claimed just before Theorem 1.3. We claim that if v, w are at distance at least ω then

$$\mathbf{Pr}(\mathcal{E}_v \cap \mathcal{E}_w) = (1 + e^{-\Omega(\omega)})\mathbf{Pr}(\mathcal{E}_v)\mathbf{Pr}(\mathcal{E}_w). \tag{5.33}$$

We use Lemma 5.2. Let $S = \{v, w\}$. We argue that for the random walk on the associated H we have

$$R_\gamma = \rho + e^{-\Omega(\omega)} \textit{ and } \pi_\gamma = \frac{2}{n} \textit{ and hence } p_\gamma = (1 + O(Tr^{-\omega}))\frac{2}{\rho n}. \tag{5.34}$$

But the expression for p_γ is clear and the expression for R_γ can be demonstrated using the proof of Lemma 5.1. In this calculation one has to estimate the expected number of returns from v to v, w to w and visits from v to w and vice-versa during the mixing time. The latter contributes $O(Te^{-\Omega(\omega)}) = O(e^{-\Omega(\omega)}) \leq e^{-a\omega}$ because v and w are at distance at least ω.

Equation (5.33) follows from Lemmas 3.1, 5.2 and (5.34). Thus,

$$\begin{aligned}\mathbf{E}(Z^2_{\mathcal{N}}(t)) &= \mathbf{E}(Z_{\mathcal{N}}(t)) + \sum_{v,w:dist(v,w)\geq\omega} \mathbf{Pr}(\mathcal{E}_v \cap \mathcal{E}_w) + \\ &+ \sum_{v,w:dist(v,w)<\omega} \mathbf{Pr}(\mathcal{E}_v \cap \mathcal{E}_w)\\ &\leq \mathbf{E}(Z_{\mathcal{N}}(t)) + (1+e^{-a\omega})\mathbf{E}(Z_{\mathcal{N}}(t))^2 + r^{\omega}\mathbf{E}(Z_{\mathcal{N}}(t))\end{aligned}$$

and (5.32) follows.

Applying the Chebyshev inequality we see that

$$\mathbf{Pr}(|Z_{\mathcal{N}}(t)-\mathbf{E}(Z_{\mathcal{N}}(t))| \geq \mathbf{E}(Z_{\mathcal{N}}(t))e^{-a\omega/3}) \leq \frac{2r^{\omega}e^{a\omega}}{\mathbf{E}(Z_{\mathcal{N}}(t))}+e^{-a\omega/3}=o(1)$$

for $t \leq t_2$.

This completes the proof of part (a) of Theorem 1.3.

The proof of part (b) is similar to that of (a). Observe first that Lemma 5.3 implies

$$\mathbf{E}(D_s(t))=(1+o(1))N_t\binom{r}{s}p_t^s(1-p_t)^{r-s}. \tag{5.35}$$

It follows from (5.35) that

$$\mathbf{E}(D_s(t))=\Omega(r^{\omega/\theta_{r,s}}) \text{ for } t \leq t_{s+1}-\omega n$$

We can use the Chebyshev inequality to prove concentration. We let $\mathcal{F}_v(s)$ be the event that v is a vertex of degree s in $\Gamma(t)$. We prove that

$$\mathbf{Pr}(\mathcal{F}_v(s)\cap\mathcal{F}_w(s))=(1+e^{-\Omega(\omega)})\mathbf{Pr}(\mathcal{F}_v(s))\mathbf{Pr}(\mathcal{F}_w(s))$$

whenever v,w are at least ω apart. We can argue for this by a small change in the argument for (a). This proves concentration for $t \leq t_{s+1}-\omega n$ and proves part (b). □

5.3 Proof of Theorem 1.2.

We combine Lemma 2.2 and Theorem 1.3 with the results of Molloy and Reed [13, 14]. We summarise what we need from [13, 14]:

THEOREM 5.1. *Let $\lambda_0,\lambda_1,\dots,\lambda_r \in [0,1]$ be such that $\lambda_0+\lambda_1+\cdots+\lambda_r=1$. Suppose that $\mathbf{d}=d_1,d_2,\dots,d_N$ satisfies $|\{j : d_j=s\}| = (1+o(1))\lambda_s N$ for $s=0,1,\dots,r$. Let $G_{n,\mathbf{d}}$ be chosen randomly from graphs with vertex set $[N]$ and degree sequence $\mathbf{d}$. Let*

$$L=\sum_{s=1}^{r}s(s-2)\lambda_s.$$

(a) *If $L<0$ then* **whp** *$G_{n,\mathbf{d}}$ is sub-critical.*

(b) *If $L>0$ then* **whp** *$G_{n,\mathbf{d}}$ is super-critical. Furthermore the unique giant component has size θN where θ is defined as follows: Let $\Lambda=\sum_{s=1}^{r}s\lambda_s$. Define α to be the smallest positive solution to*

$$\Lambda-2\alpha-\sum_{s=1}^{r}s\lambda_s\left(1-\frac{2\alpha}{\Lambda}\right)^{s/2}=0.$$

Then

$$\theta=1-\sum_{s=1}^{r}\lambda_s\left(1-\frac{2\alpha}{\Lambda}\right)^{1/2}.$$

□

We now evaluate Λ in the context of $\Gamma(t)$. Then Theorem 1.3 implies that we can take

$$L=\sum_{s=1}^{r}\binom{r}{s}p_t^s(1-p_t)^{r-s}s(s-2)=rp_t((r-1)p_t-1).$$

Thus the critical value for t is the one that gives $p_t=\frac{1}{r-1}$. One can easily check that this is indeed the case for t^* as defined in (1.1). parts (i) and (ii) of Theorem 1.1 follow immediately.

To prove (iii) we use the result of Hatami and Molloy [11] that when $|L|=O(n^{-1/3})$ the size of the giant is $n^{2/3+o(1)}$. At each step, L changes by $O(1/n)$ and so at some time $t\sim t^*$ the conditions of [11] will be satisfied. At this point **whp** there are $\Theta(n)$ vertices in $\Gamma(t)$ and (iii) follows. □

5.4 Proof of Theorem 1.4.

In this section we study the number of components of a given size. In principle one should be able to work this out from Lemma 2.2 and Theorem 1.3. This has proven more difficult than we anticipated. Instead, we try to estimate the number directly. We can use these lemmas though to argue that almost all small components are trees. Indeed if we fix t and condition on the values $D_s=D_s(t)$ satisfying Theorem 1.3 then we have the following:

LEMMA 5.4.

(a) *If $t \leq \epsilon_1 n\log n$ then* **whp** *there are at most $n^{3\epsilon_1}$ components of size $k \leq \epsilon_1\log n$ that are not trees.*

(b) *If $t \geq \epsilon_1 n\log n$ then* **whp** *there are no components of size $k \leq \log^2 n$ that are not trees.*

Proof

Let $N=|\mathcal{R}(t)|$. Applying Lemma 2.2 we see that the expected number of sets of k vertices that contain at least k edges is bounded by

$$\binom{N}{k}\binom{\binom{k}{2}}{k}\left(\frac{r}{n}\right)^k \leq \left(\frac{rNe^2}{2n}\right)^k.$$

To prove (a) we take $N=n$ and apply the Markov inequality. To prove (b) we take $N=N_t$. □

With this in mind we concentrate on the number of tree components of size k for some $k \leq \epsilon_1 \log n$. Since there are **whp** at most $n^{2\epsilon_1}$ vertices that are not nice, we will concentrate on counting the number of components that are made up of nice vertices only. We will also assume that $t \leq t_1$, see (5.31).

The following is proved as Lemma 4 of [2].

LEMMA 5.5. *Let b_k be the number of subtrees of size k rooted at a vertex v in an infinite r-regular tree $\mathcal{T}$. Then*

$$b_k = \frac{r}{(r-2)k+2}\binom{(r-1)k}{k-1}.$$

□

Now consider the situation described in Lemma 2.3. Fix $v \in \mathcal{R}(t)$ and consider the k-neighbourhood of v in the multi-graph on $[n]$ induced by a random pairing on $R(t)$. It is **whp** a tree. Now delete any edge that corresponds to an edge (x, y) with $x \in \mathcal{R}(t), y \in \mathcal{B}(t)$. Let T be the component that contains v. If T has k vertices then T corresponds to a tree component of $\Gamma(t)$ with k vertices.

So, fix a tree T^* as counted in Lemma 5.5 and let us determine the probability that $T = T^*$. The total degree of $\mathcal{R}(t)$ is $(1 + o(1))rN_t$ and it follows from Theorem 1.3 that that the number of $\mathcal{R}(t) : \mathcal{B}(t)$ edges is $(1+o(1))N_t r(1-p_t)$. So if we choose $x \in W_{\mathcal{R}(t)}$ then it is paired with something in $W_{\mathcal{R}(t)}$ with probability $(1 + o(1))p_t$, where the $o(1)$ term is $o(1/\log n)$.

It follows that

$$\text{(5.36)}\quad \mathbf{Pr}(T = T^*) = (1 + o(1))p_t^{k-1}(1 - p_t)^{(r-2)k+2}.$$

To see this, start at the root v and examine the points paired with W_v. Each point will have probability $(1 + o(1))p_t$ of being paired with an element of $W_{\mathcal{R}(t)}$ and the $o(1)$ term will not change by more than $O(\log n/N_t) = o(n^{-1/5})$ as we proceed with this argument. The count in Lemma 5.5 assumes an ordering of the neighbours of each vertex and by implication an ordering of W_v and a statement about which members of W_v are paired with $W_{\mathcal{R}(t)}$ and which not. Suppose we pair W_v with points from $W_{x_i}, i = 1, 2, \ldots, d(v)$, where $d(v)$ is the degree of v in T^*. Then we continue by pairing up W_{x_1} and then W_{x_2} and so on. The factor p_t^{k-1} is from the $k-1$ times we have to pair with $W_{\mathcal{R}(t)}$ and the factor $(1-p_t)^{(r-2)k+2}$ is from the number of times we do not. Note that the sizes of unpaired $W_{\mathcal{R}(t)}$ and $W_{R,\mathcal{B}(t)}(t)$ change by $O(\log n)$ as we proceed.

It follows from (5.36) that

$$\mathbf{E}(N(k, t)) = (1 + o(1))N_t \frac{b_k}{k} p_t^{k-1}(1 - p_t)^{(r-2)k+2}.$$

It remains to prove concentration around the mean. We use the Chebyshev inequality. We fix two vertex disjoint trees T_1, T_2 in G. Arguing as above we see that

$$\mathbf{Pr}(T_1, T_2 \text{ are components of } \Gamma(t)) \leq (1 + o(1)) \prod_{i=1}^{2} \mathbf{Pr}(T_i \text{ is a component of } \Gamma(t)).$$

We get what we want, provided $\mathbf{E}(N(k, t)) \to \infty$ and it does so for $t \leq (1 - \epsilon)t_k$.

5.5 In the beginning. Using Lemma 2.2 we see that for $1 \leq t \leq \log^3 n$ we have that $\Gamma(t)$ is a random graph with a degree sequence of the following form: There are $n - s$ vertices of degree r, where $s \leq rt$, and s vertices of degree $< r$.

If the minimum degree in $\Gamma(t)$ is at least one then we next prove that $\Gamma(t)$ is connected **whp**. Indeed, let V_r be the set of vertices of degree r in $\Gamma(t)$. We argue that **whp**

(5.37) V_r induces a connected subgraph of $\Gamma(t)$.

(5.38) Each $x \in \mathcal{R}(t) \setminus V_r$ is adjacent to V_r.

For k even let

$$\phi(k) = \frac{k!}{(k/2)!2^{k/2}}$$

be the number of ways of partitioning $[k]$ into $k/2$ pairs.

Let $m = O(\log^3 n)$ be the sum of the degrees, in $\Gamma(t)$, of the vertices in $\mathcal{R}(t) \setminus V_r$. Working in the configuration model,

$$\begin{aligned}
&\mathbf{Pr}((5.37) fails)\\
&\leq \sum_{k=3}^{n/2} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l} \frac{\phi(kr + l)\phi((n - s - k)r + m - l)}{\phi((n - s)r + m)}\\
&= \sum_{k=3}^{n/2} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l} \frac{\binom{(r(n-s)+m)/2}{(kr+l)/2}}{\binom{r(n-s)+m}{kr+l}}\\
&\leq \sum_{k=3}^{n/2} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l} \frac{1}{\binom{(r(n-s)+m)/2}{(kr+l)/2}}\\
&\leq \sum_{k=3}^{n/2} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l} \left(\frac{kr + l}{r(n - s) + m}\right)^{(kr+l)/2}\\
&= o(1).
\end{aligned}$$

Explanation: If the subgraph induced by V_r is disconnected, let S be one component of this subgraph. Choose a set of k vertices S of degree r and l points from the m points in W associated with vertices T of degree less than r. The l points contain the edges between S and T. Now pair up the $kr + l$ points of $S \cup T$ randomly and pair the remaining points randomly.

The probability that (5.38) fails is $O(\log^3 n/n)$. A vertex of $\mathcal{R}(t) \setminus V_r$ of degree d has an $O(n^{-d})$ chance of not being connected to V_r.

Next, we have to deal with the possibility that there are isolated vertices in $\Gamma(t)$ for $t \leq \log^3 n$. So consider the event

$$\mathcal{A}(t) = \{\exists v \in N(W_t) : \ v \in \mathcal{R}(t) \text{ and } N(v) \subseteq \mathcal{B}(t)\} .$$

We claim that

$$\mathbf{Pr}(\mathcal{A}(t)) = O\left(\frac{\log^3 n}{(r-1)^{\ell_1/2}}\right), \tag{5.39}$$

where ℓ_1 is given by (5.18). It follows that

$$\mathbf{Pr}((5.38)\ fails) \leq \sum_{t=1}^{\log^3 n} \mathbf{Pr}(\mathcal{A}(t)) = o(1).$$

To prove (5.39) fix t and a neighbour v of W_t. Property (5.20) implies that there is at least one neighbour w of v that is not contained in a small cycle. If $w \neq X_t$ then to reach w the walk $\mathcal{W}$ must emulate a walk on the infinite tree $\mathcal{T}$ that starts at distance $\ell_1/2$ from the root and visits it. This has probability $1/(r-1)^{\ell_1/2}$ and this must be inflated by $\log^3 n$ to account for $\log^3 n$ possible starting times. If $w = X_t$ then to visit another neighbour of v then we must first reach distance at least $\ell_1/2$ and then we can repeat the argument and use inequality (5.26).

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