On Succinct Convex Greedy Drawing of 3-Connected Plane Graphs

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Abstract

Geometric routing by using virtual locations is an elegant way for solving network routing problems. In its simplest form, greedy routing, a message is simply forwarded to a neighbor that is closer to the destination. It has been an open conjecture whether every 3-connected plane graph has a greedy drawing in $\mathbf{R}^2$ (by Papadimitriou and Ratajczak [23]). Leighton and Moitra [20] recently settled this conjecture positively. One main drawback of this approach is that the coordinates of the virtual locations requires $\Omega(n \log n)$ bits to represent (the same space usage as traditional routing table approaches). This makes greedy routing infeasible in applications. A similar result was obtained by Angelini et al. [2]. However, neither of the two papers give the time efficiency analysis of their algorithms. In addition, as pointed out in [16], the drawings in these two papers are not necessarily planar nor convex.

In this paper, we show that the classical Schnyder drawing in $\mathbf{R}^2$ of plane triangulations is greedy with respect to a simple natural metric function $H(u,v)$ over $\mathbf{R}^2$ that is equivalent to Euclidean metric $D_E(u,v)$ (in the sense that $D_E(u,v) \leq H(u,v) \leq 2\sqrt{2}D_E(u,v)$.) The drawing is succinct, using two integer coordinates between 0 and $2n-5$.

For 3-connected plane graphs, there is another conjecture by Papadimitriou and Ratajczak (as stated in [16]):

> Convex Greedy Embedding Conjecture: Every 3-connected planar graph has a convex greedy embedding in the Euclidean plane.

In a recent paper [6], Cao et al. provided a plane graph G and showed that any convex greedy embedding of G in Euclidean plane must use $\Omega(n)$-bit coordinates. Thus, if we add the succinctness requirement, the Convex Greedy Embedding Conjecture is false.

In this paper, we show that the classical Schnyder drawing in $\mathbf{R}^2$ of 3-connected plane graphs is *weakly greedy* with respect to the same metric function $H(*,*)$. The drawing is planar, convex, and succinct, using two integer coordinates between 0 and f (where f is the number of internal faces of G).

1 Introduction

Routing is one of the most important algorithmic problems in networking. Extensive research has been devoted to discover efficient routing algorithms (see [8, 27]). Previously, routing was done via routing protocols (e.g., see [8, 27]). This approach is space inefficient and requires considerable setup overhead, which makes it infeasible for some networks (e.g. wireless sensor networks). Recently, an alternative approach *geometric routing* has been proposed. Geometric routing uses geometric coordinates of the vertices to compute the routing paths. The simplest geometric routing is *greedy routing*, in which a vertex simply forwards messages to a neighbor that is closer to the destination.

Greedy routing is simple, but also imposes some problems. First, GPS devices, typically used to determine geometric coordinates, are expensive and increase energy consumption. (This should be avoided, especially, for sensor networks). More importantly, a bad network topology and geographical location of network nodes can lead to routing failures because a *void position* has been reached. (Namely, a packet has reached a node all whose neighbors are farther from the destination than the node itself). For example, for a star-shaped network $K_{1,7}$ embedded in $\mathbf{R}^2$, as in Figure 1, greedy routing fails due to the fact that the vertex u is at a void position: all neighbors (only one) of u are farther from the destination w than itself in the embedding.

To solve these problems, an elegant solution was proposed by Rao et al. in [24]: Instead of using the real geometric coordinates (e.g., GPS coordinates), one could use graph drawing, based on the structure of a network G, to compute vertex coordinates in the drawing. The drawing coordinates are used as the *virtual coordinates* of the vertices of G. Then geometric routing algorithms rely on virtual coordinates to compute routes. *Greedy drawing* is introduced as a

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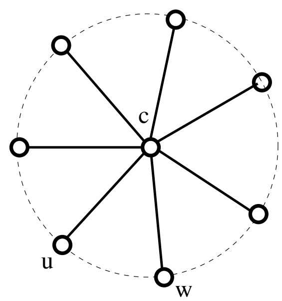

Figure 1: No greedy drawing for $K_{1,7}$: u is at a void position.

solution for greedy routing. Simply speaking, a greedy drawing is a drawing in which greedy routing works. More precisely:

DEFINITION 1. [23] Let S be a set and $H(*,*)$ a metric function over S. Let $G=(V,E)$ be a graph.

1. A *drawing* of G into S is a mapping $d: V \to S$ such that $u \neq v$ implies $d(u) \neq d(v)$.
2. The drawing d is a *greedy drawing* with respect to H if for any two vertices u, w of G ($u \neq w$), u has a neighbor v such that $H(d(u), d(w)) > H(d(v), d(w))$.
3. The drawing d is a *weakly greedy drawing* with respect to H if for any two vertices u, w of G ($u \neq w$), u has a neighbor v such that $H(d(u), d(w)) \geq H(d(v), d(w))$.

After [24], there has been intense research on greedy routing schemes that assign network nodes to virtual coordinates in a natural metric space. The feasibility of the greedy routing depends heavily on the metric space used to define the notion of "closer to the destination". As pointed out by Goodrich and Strash in [17], similar to the graph $K_{1,7}$ in Fig 1, star graphs (consisting of a central vertex adjacent to an arbitrarily large number of leaves) cannot support greedy geometric routing in any fixed-dimensional Euclidean space. They observed that, by a simple packing argument, there has to be two leaves in such a graph, that are closer to each other than to the central vertex. Even for 2-connected or 3-connected planar graphs embedded in the Euclidean plane $\mathbf{R}^2$, a network may have "holes" where greedy routing algorithms could get stuck in a local metric minimum (see [15] for related work on hole detection in sensor networks). Several researchers (e.g., see [11, 19, 22]) have shown that greedy geometric routing is possible, for any connected graph, in fixed-dimensional *hyperbolic spaces*. Obviously, people are especially interested in greedy routing in metric spaces over $\mathbf{R}^2$, since this space more closely matches the geometry of wireless sensor networks.

In their inspiring work, Papadimitriou and Ratajczak [23] showed that any 3-connected planar graph can be embedded in R^3 that supports greedy geometric routing, with respect to a non-standard metric function. They conjectured that greedy drawing is possible in Euclidean plane $\mathbf{R}^2$. This conjecture has drawn a lot of interests [2, 7, 9, 17, 20, 21, 22]. Greedy drawings in $\mathbf{R}^2$ were first discovered only for graphs containing power diagrams [7], then for graphs containing Delaunay triangulations [21]. The existence of greedy drawing for plane triangulations were shown in [9] by using Schnyder's *realizer* concept. However, the proof was not constructive. Leighton and Moitra [20] eventually settled this conjecture positively by giving an algorithm to produce a greedy drawing of any 3-connected planar graph in $\mathbf{R}^2$. A similar result was independently found by Angelini et al. [2]. However, neither of the two papers give the time efficiency analysis of their algorithms. In addition, the drawings in these two papers are not necessarily planar [16].

Furthermore, as pointed out by Eppstein and Goodrich in [11], greedy routing is still not yet practically feasible, due to the fact that in the worst case, the virtual coordinates produced by greedy drawing require $\Omega(n \log n)$ bits to represent them. Thus, these greedy routing algorithms have the same space usage as traditional routing table approaches. The main obstacle for the applicability of greedy routing is not only the existence of a greedy drawing, but also the existence of a *succinct greedy drawing*, in which the virtual coordinates are represented in $O(\log n)$ bits. Angelini et al. [1] proved that succinct greedy drawing does not always exist in $\mathbf{R}^2$. They proved that there are infinitely many trees which are greedy drawable, but all greedy drawings need exponential size grids. In their recent work [17], Goodrich and Strash used the property of Leighton-Moitra embedding to formulate a *new* coordinate systems (not the underlying coordinates used for embedding), where coordinate corresponds to certain information on the Leighton-Moitra embedding, and each vertex is assigned a triplet of three such coordinates for the routing purposes. Since each element of the triple can take on values in the range $[0..n]$, the triple can be stored using $O(\log n)$ bits. This $O(\log n)$ representation is totally different from the real underlying geometric embedding, which still needs $\Omega(n \log n)$ bits, according to [11]. In [18], a *generalized greedy* routing algorithm based on Schnyder drawing was introduced. The routing decision is based on rules other than decreasing distance. So it is not a true greedy drawing

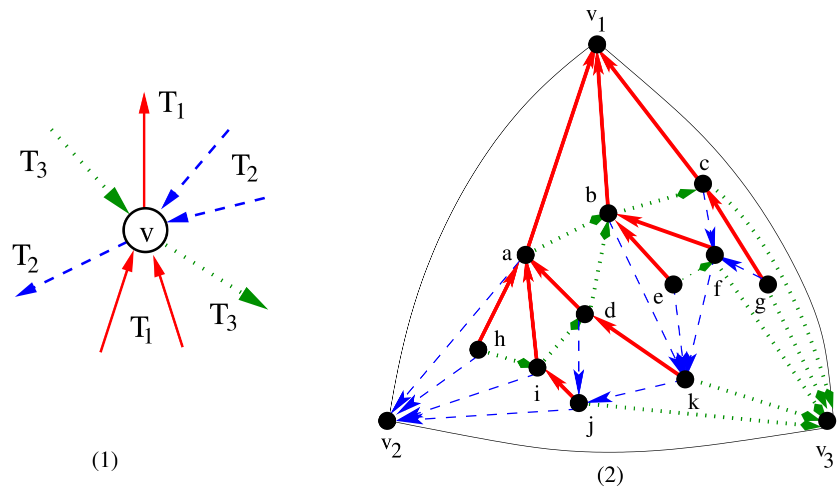

Figure 2: (1) Edges around an internal vertex v. (2) A plane triangulation G and a realizer $\mathcal{R}$.

either.

The following is another open conjecture by Papadimitriou and Ratajczak (as stated in [16]):

> Convex Greedy Embedding Conjecture: Every 3-connected planar graph has a convex greedy embedding in the Euclidean plane.

In a recent paper [6], Cao et al. provided a plane graph G and showed that any convex greedy embedding of G in Euclidean plane must use $\Omega(n)$-bit coordinates. (The graph G described in [6] is not 3-connected. However, it is easy to make G 3-connected by adding three additional edges.) Thus, if we add the succinctness requirement, the Convex Greedy Embedding Conjecture is false. In another recent paper [4], Berchenko and Teicher claimed an algorithm that finds a convex greedy embedding for 3-connected planar graphs. Their algorithm is randomized and works only for large n. Not enough details are given in [4]. Moreover, the length of the embedding coordinates was not specified in [4]. In any case, their embedding cannot be succinct due to the counter example in [6].

In this paper, we show that the classical Schnyder drawing of plane triangulation in $\mathbf{R}^2$ is greedy with respect to a simple natural metric function $H(*,*)$ that is equivalent to Euclidean distance. It is succinct, using two integer coordinates between 0 and $2n-5$.

For 3-connected plane graphs, because of the counter-example in [6], there exists no drawing that is greedy, convex, succinct, with respect to Euclidean distance. If one wants the drawing to be greedy, convex and succinct, we must give up the Euclidean distance. We show that the grid drawing based on Schnyder wood is *weakly greedy* with respect to the same metric function $H(*,*)$. The drawing is planar, convex, and succinct using two integer coordinates between 0 and f (where f is the number of internal faces of G).

The present paper is organized as follows. In Section 2, we show the drawing for plane triangulations based on Schnyder realizer is greedy with respect to $H(*,*)$. In Section 3, we show the drawing of 3-connected plane graphs based on Schnyder woods is weakly greedy with respect to the same metric function $H(*,*)$. Section 4 concludes the paper.

2 Greedy Drawing of Plane Triangulations

Most definitions we use are standard. A *planar graph* is a graph G such that the vertices of G can be drawn in the plane and the edges of G can be drawn as non-intersecting curves. Such a drawing is called an *embedding*. The embedding divides the plane into a number of connected regions. Each region is called a *face*. The unbounded face is the *external face*. The other faces are *internal faces*. The vertices and edges not on the external face are *internal edges* and *vertices*. A *plane graph* is a planar graph with a fixed embedding. A *plane triangulation* is a plane graph where every face is a triangle (including the external face). We abbreviate the words “counterclockwise” and “clockwise” as ccw and cw respectively. The following concept was introduced in [25, 26].

Definition 2. Let G be a plane triangulation of n vertices with three external vertices v_1, v_2, v_3 in ccw order. A *Schnyder realizer* (or simply realizer) $\mathcal{R} = \{T_1, T_2, T_3\}$ of G is a partition of its internal edges into three sets T_1, T_2, T_3 of directed edges such that the following hold:

- For each $i \in \{1, 2, 3\}$, the internal edges incident to v_i are in T_i and directed toward v_i.

- For each internal vertex v of G, v has exactly one edge leaving v in each of T_1, T_2, T_3. The ccw order of the edges incident to v is: leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, and entering in T_2. Each entering block may be empty. (See Figure 2 (1)).

Figure 2 (2) shows a realizer of a plane triangulation G. The solid (dashed and dotted, respectively) lines are the edges in T_1 (T_2 and T_3, respectively).

In [25, 26], Schnyder showed that every plane triangulation has a realizer which can be constructed in linear time. It was also shown that each T_i of a realizer is a tree rooted at v_i, spanning all vertices of G except v_{i-1} and v_{i+1}. (We assume a cyclic structure on the set $\{1, 2, 3\}$ so that $i-1$ and $i+1$ are always defined. Namely if $i = 3$ then $i+1 = 1$; if $i = 1$ then $i-1 = 3$).

For each internal vertex u of G and $i \in \{1, 2, 3\}$, $P_i(u)$ denotes the path in T_i from u to the root v_i of T_i. We also use $P_i(u)$ to denote the set of vertices in the path $P_i(u)$. It was shown in [26] that $P_1(v), P_2(v)$ and $P_3(v)$ have only the vertex v in common, and for two vertices $u \neq v$ and two indices $i \neq j$, $P_i(u)$ and $P_j(v)$ can have at most one common vertex. Let $p_i(w)$ denote the parent of w in T_i. If $u = p_i(w)$, then w is an *i-child* of u. If there is a path in T_i from w to u, then u is an *i-ancestor* of w, and w is an *i-descendant* of u.

For each internal vertex u of G and $i \in \{1, 2, 3\}$, $R_i(u)$ denotes the region of G bounded by the paths $P_{i-1}(u), P_{i+1}(u)$ and the exterior edge (v_{i-1}, v_{i+1}). We also use $R_i(u)$ to denote the set of vertices in the region $R_i(u)$. Let $R_i^\circ(u)$ denote the interior of $R_i(u)$. Namely $R_i^\circ(u) = R_i(u) - (P_{i-1}(u) \cup P_{i+1}(u))$. We further partition $R_i^\circ(u)$ into three subsets:

1. The *i-Descendants*: $D_i(u) = \{v \mid v$ is an i-descendent of $u\}$.

2. The *i-Left-Cousins*: $LC_i(u) = \{v \mid v$ is an i-descendent of a vertex $w \in P_{i+1}(u)$ where $w \neq u\}$.

3. The *i-Right-Cousins*: $RC_i(u) = \{v \mid v$ is an i-descendent of a vertex $w \in P_{i-1}(u)$ where $w \neq u\}$.

For example, consider the realizer shown in Fig 2 (2). We have $P_2(a) = \{a, v_2\}$. $P_3(a) = \{a, b, c, v_3\}$. $R_1(a)$ consists of all vertices of G, except v_1. $R_1^\circ(a) = \{d, e, f, g, h, i, j, k\}$. $D_1(a) = \{d, h, i, j, k\}$. $LC_1(a) = \emptyset$. $RC_1(a) = \{e, f, g\}$.

Definition 3. Let u be a vertex of G and $i \in \{1, 2, 3\}$.

1. $x_i(u)$ denotes the number of faces of G in the region $R_i(u)$.

2. The *Schnyder coordinate* of u is: $S(u) = (x_1(u), x_2(u), x_3(u))$. (Note that $x_1(u) + x_2(u) + x_3(u) = 2n - 5 =$ the number of internal faces of G.)

For example, in the realizer shown in Fig 2 (2), we have $S(a) = (19, 3, 1)$ and $S(e) = (4, 12, 7)$.

Definition 4. *Let u and w be two vertices of G with Schnyder coordinates $S(u) = (x_1(u), x_2(u), x_3(u))$ and $S(w) = (x_1(w), x_2(w), x_3(w))$ respectively.*

1. $D(u, w) = |x_1(u) - x_1(w)| + |x_2(u) - x_2(w)| + |x_3(u) - x_3(w)|$.

2. $H(u, w) = |x_1(u) - x_1(w)| + |x_2(u) - x_2(w)| + |(x_1(u) + x_2(u)) - (x_1(w) + x_2(w))|$.

Note that $D(u, w)$ is the Manhattan distance between $S(u)$ and $S(w)$ in $\mathbf{R}^3$. Since $x_1(v) + x_2(v) + x_3(v) = 2n - 5$ for all vertices v, we have $H(u, w) = D(u, w)$. However, $H(*, *)$ is defined by using $x_1(*)$ and $x_2(*)$ values only. It is straightforward to verify that $D(*, *)$ is a metric function over $\mathbf{R}^3$ and $H(*, *)$ is a metric function over $\mathbf{R}^2$. In addition, we have:

Fact 2.1. Let $D_E(*, *)$ be the Euclidean distance over $\mathbf{R}^2$. The following holds for any u, v:

$$D_E(u, v) \leq H(u, v) \leq 2\sqrt{2} D_E(u, v)$$

This is because:

- $H(u, v) \leq 2 \times (|x_1(u) - x_1(v)| + |x_2(u) - x_2(v)|) \leq 2\sqrt{2} D_E(u, v)$.

- $H(u, v) \geq (|x_1(u) - x_1(v)| + |x_2(u) - x_2(v)|) \geq D_E(u, v)$.

Lemma 2.1. *For any two vertices u and w ($u \neq w$) in G, u has a neighbor v such that $D(u, w) > D(v, w)$.*

Proof: If w is a neighbor of u, let $v = w$ and the lemma is trivially true. So we assume w is not a neighbor of u. Without loss of generality, assume $w \in R_1(u)$. The other cases are symmetric.

Case 1. $w \in D_1(u)$. Let v be the 1-child of u that is a 1-ancestor of w. Let a (b, respectively) be the first common vertex of $P_2(u)$ and $P_2(v)$, ($P_3(u)$ and $P_3(v)$, respectively). Let c (d, respectively) be the first common vertex of $P_2(u)$ and $P_2(w)$, ($P_3(u)$ and $P_3(w)$, respectively). (See Figure 3 (1)). Let C_1, C_2, C_3, C_4 be the regions bounded by the paths $P_i(u), P_i(v)$ and

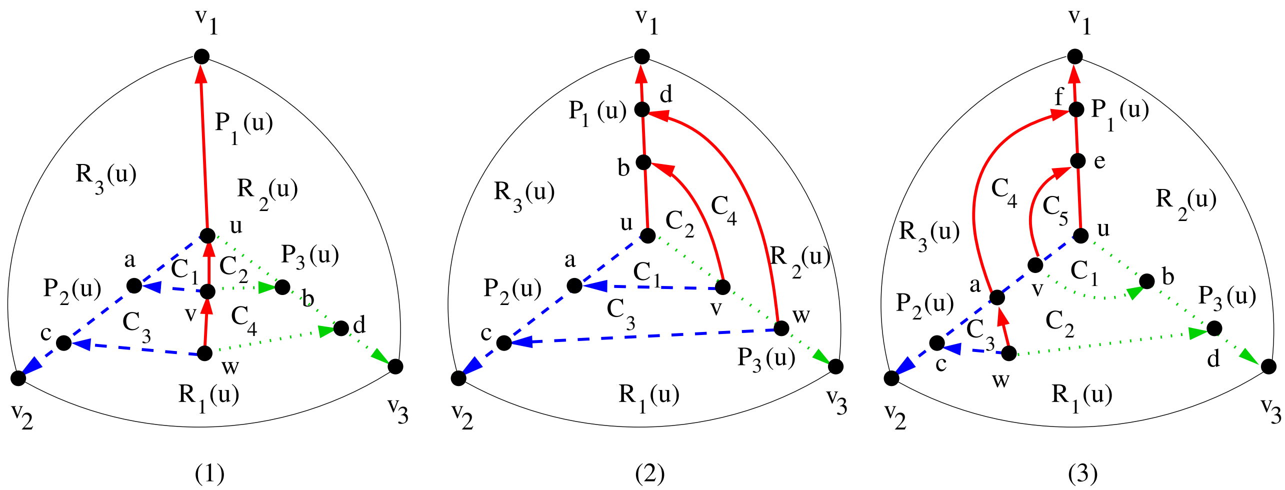

Figure 3: The proof of Lemma 2.1. (1) Case 1; (2) Case 2; (3) Case 4.

$P_i(w)$ as shown in Figure 3 (1). By the properties of the realizer, these regions are non-empty. Let $\#(C_i)$ be the number of faces in the region C_i $(1 \le i \le 4)$. We have:

$$\begin{aligned}
D(u,w) &= |x_1(u)-x_1(w)|+|x_2(u)-x_2(w)| \\
&\quad +|x_3(u)-x_3(w)| \\
&= (\#(C_1)+\#(C_2)+\#(C_3)+\#(C_4)) \\
&\quad +(\#(C_2)+\#(C_4))+(\#(C_1)+\#(C_3)) \\
D(v,w) &= |x_1(v)-x_1(w)|+|x_2(v)-x_2(w)| \\
&\quad +|x_3(v)-x_3(w)| \\
&= (\#(C_3)+\#(C_4))+\#(C_4)+\#(C_3)
\end{aligned}$$

Clearly $D(u,w) > D(v,w)$ is equivalent to $2(\#(C_1)+\#(C_2)) > 0$, which is true.

Case 2. $w \in P_3(u)$. Let $v = p_3(u)$. Let a (b, respectively) be the first common vertex of $P_2(u)$ and $P_2(v)$, ($P_1(u)$ and $P_1(v)$, respectively). Let c (d, respectively) be the first common vertex of $P_2(u)$ and $P_2(w)$, ($P_1(u)$ and $P_1(w)$, respectively). (See Figure 3 (2)). Let C_1, C_2, C_3, C_4 be the regions bounded by the paths $P_i(u), P_i(v)$ and $P_i(w)$ as shown in Figure 3 (2). We have:

$$\begin{aligned}
D(u,w) &= |x_1(u)-x_1(w)|+|x_2(u)-x_2(w)| \\
&\quad +|x_3(u)-x_3(w)| \\
&= (\#(C_1)+\#(C_3))+(\#(C_2)+\#(C_4)) \\
&\quad +(\#(C_1)+\#(C_2)+\#(C_3)+\#(C_4)) \\
D(v,w) &= |x_1(v)-x_1(w)|+|x_2(v)-x_2(w)| \\
&\quad +|x_3(v)-x_3(w)| \\
&= \#(C_3)+\#(C_4)+(\#(C_3)+\#(C_4))
\end{aligned}$$

Then $D(u,w) > D(v,w)$ is equivalent to $2(\#(C_1)+\#(C_2)) > 0$, which is true.

Case 3. $w \in P_2(u)$. Similar to Case 2.

Case 4. $w \in LC_2(u)$. Let $v = p_2(u)$. Let $a \in P_2(u)$ be the 1-ancestor of w on $P_2(u)$. (See Figure 3 (3)). Let b be the first common vertex of $P_3(u)$ and $P_3(v)$. Let c (d, respectively) be the first common vertex of $P_2(u)$ and $P_2(w)$, ($P_3(u)$ and $P_3(w)$, respectively). (See Figure 3 (3)). Let e (f, respectively) be the first common vertex of $P_1(u)$ and $P_1(v)$, ($P_1(u)$ and $P_1(a)$, respectively). Let C_1, C_2, C_3, C_4, C_5 be the regions bounded by the paths $P_i(u), P_i(v)$ and $P_i(w)$ as shown in Figure 3 (3). We have:

$$\begin{aligned}
D(u,w) &= |x_1(u)-x_1(w)|+|x_2(u)-x_2(w)| \\
&\quad +|x_3(u)-x_3(w)| \\
&= (\#(C_1)+\#(C_2)+\#(C_3)) \\
&\quad +(\#(C_1)+\#(C_2)+\#(C_4)+\#(C_5)) \\
&\quad +|(\#(C_4)+\#(C_5))-\#(C_3)| \\
D(v,w) &= |x_1(v)-x_1(w)|+|x_2(v)-x_2(w)| \\
&\quad +|x_3(v)-x_3(w)| \\
&= (\#(C_2)+\#(C_3))+(\#(C_2)+\#(C_4)) \\
&\quad +|\#(C_4)-\#(C_3)|
\end{aligned}$$

Case 4a: $\#(C_4) > \#(C_3)$:

$$\begin{aligned}
D(u,w) &= (\#(C_1)+\#(C_2)+\#(C_3)) \\
&\quad +(\#(C_1)+\#(C_2)+\#(C_4)+\#(C_5)) \\
&\quad +(\#(C_4)+\#(C_5))-\#(C_3) \\
D(v,w) &= (\#(C_2)+\#(C_3))+(\#(C_2)+\#(C_4)) \\
&\quad +(\#(C_4)-\#(C_3))
\end{aligned}$$

Then $D(u,w) > D(v,w)$ is equivalent to $2 \times (\#(C_1) + \#(C_5)) > 0$, which is true.

Case 4b: $\#(C_4) \leq \#(C_3)$ and $\#(C_4)+\#(C_5) > \#(C_3)$:

$$\begin{aligned} D(u,w) &= (\#(C_1) + \#(C_2) + \#(C_3)) \\ &\quad +(\#(C_1) + \#(C_2) + \#(C_4) + \#(C_5)) \\ &\quad +(\#(C_4) + \#(C_5)) - \#(C_3) \\ D(v,w) &= (\#(C_2) + \#(C_3)) + (\#(C_2) \\ &\quad +\#(C_4)) + (\#(C_3) - \#(C_4)) \end{aligned}$$

Then $D(u,w) > D(v,w)$ is equivalent to $2 \times (\#(C_1) + \#(C_4) + \#(C_5)) > 2 \times \#(C_3)$. This is true because $\#(C_4) + \#(C_5) > \#(C_3)$.

Case 4c: $\#(C_4) + \#(C_5) \leq \#(C_3)$:

$$\begin{aligned} D(u,w) &= (\#(C_1) + \#(C_2) + \#(C_3)) \\ &\quad +(\#(C_1) + \#(C_2) + \#(C_4) + \#(C_5)) \\ &\quad +\#(C_3) - (\#(C_4) + \#(C_5)) \\ D(v,w) &= (\#(C_2) + \#(C_3)) \\ &\quad +(\#(C_2) + \#(C_4)) + (\#(C_3) - \#(C_4)) \end{aligned}$$

Then $D(u,w) > D(v,w)$ is equivalent to $2 \times \#(C_1) > 0$, which is true.

Case 5. $w \in RC_2(u)$. Similar to Case 4. □

THEOREM 2.1. *Let $G = (V,E)$ be a plane triangulation with n vertices. Then G has a greedy drawing with respect to the metric function $H(*,*)$ on a $(2n-5) \times (2n-5)$ grid. The drawing can be constructed in linear time.*

Proof: In [26], Schnyder showed that if we take any two numbers in the Schnyder coordinates (for example, take $x_1(u)$ and $x_2(u)$) as the x- and y-coordinates, we obtain a straight-line drawing of G on a $(2n-5) \times (2n-5)$ grid. By Lemma 2.1, for any two vertices u and w ($u \neq w$) in G, u has a neighbor v such that $D(u,w) > D(v,w)$. Since $H(v,w) = D(v,w)$, we have $H(u,w) > H(v,w)$. □

Note: The term $|(x_1(u)+(x_2(u))-(x_1(w)+(x_2(w))|$ in the definition of $H(u,w)$ is necessary. If we drop this term from the definition, there are examples for which the drawing is not greedy.

3 Weakly Greedy Drawing of 3-Connected Plane Graphs

In this section, we show the convex drawing of 3-connected plane graphs, based on Schnyder wood, is *weakly greedy* with respect to $D(*,*)$. Schnyder wood generalizes the realizer concept to 3-connected plane graphs as follows [10, 12].

DEFINITION 5. Let G be a 3-connected plane graph with three external vertices v_1, v_2, v_3 in ccw order. A *Schnyder wood* of G is a triplet of rooted spanning trees $\{T_1, T_2, T_3\}$ of G with the following properties:

1. For $i \in \{1,2,3\}$, the root of T_i is v_i, the edges of G are directed from children to parent in T_i.
2. Each edge e of G is contained in at least one and at most two spanning trees. If e is contained in two spanning trees, then it has different directions in the two trees.
3. For each vertex $v \notin \{v_1, v_2, v_3\}$ of G, v has exactly one edge leaving v in each of T_1, T_2, T_3. The ccw order of the edges incident to v is: leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, and entering in T_2. Each entering block may be empty. An edge with two opposite directions is considered twice. The first and the last incoming edges are possibly coincident with the outgoing edges. (Figure 4 (1) and (2) show two examples of edge pattern around an interval vertex v. In the second example, the edge leaving v in T_3 and an edge entering v in T_2 are the same edge).
4. For $i \in \{1,2,3\}$, all the edges incident to v_i belong to T_i.

Figure 4 (3) shows an example of Schnyder wood. The edges in T_1, T_2, T_3 are drawn as solid, dashed and dotted lines respectively. We call the edges in T_1 as red, T_2 blue, and T_3 green. According to the definition, each edge is assigned one or two colors, and is said to be *1-colored* or *2-colored*, respectively.

It was shown in [10] that every 3-connected plane graph has a Schnyder wood, which can be computed in linear time. For each vertex v of G and $i \in \{1,2,3\}$, $P_i(v)$ denotes the path in T_i from v to the root v_i of T_i. The sub-path of the external face of G with end vertices v_1 and v_2 and not containing v_3 is denoted by $ext(v_1, v_2)$. The sub-paths $ext(v_2, v_3)$ and $ext(v_3, v_1)$ are defined similarly.

For each vertex $u \neq v_1, v_2, v_3$, the three paths $P_1(u), P_2(u)$ and $P_3(u)$ divide G into three regions $R_1(u), R_2(u), R_3(u)$. Define $R_i^\circ(u) = R_i(u) - (P_{i-1}(u) \cup P_{i+1}(u))$. As before, $R_i^\circ(u)$ can be partitioned into three subsets $D_i(u), RC_i(u)$ and $LC_i(u)$ as follows. (Since the edges of G may belong to two different trees, the definition is slightly more complicated.)

1. The *i-Descendants*: $D_i(u) = \{v \in R_i^\circ(u) \mid$ the first i-ancestor of v in $P_{i-1}(u) \cup P_{i+1}(u)$ is $u\}$.
2. The *i-Left-Cousins*: $LC_i(u) = \{v \in R_i^\circ(u) \mid$ the first i-ancestor of v in $P_{i-1}(u) \cup P_{i+1}(u)$ is in $P_{i+1}(u) - \{u\}\}$.

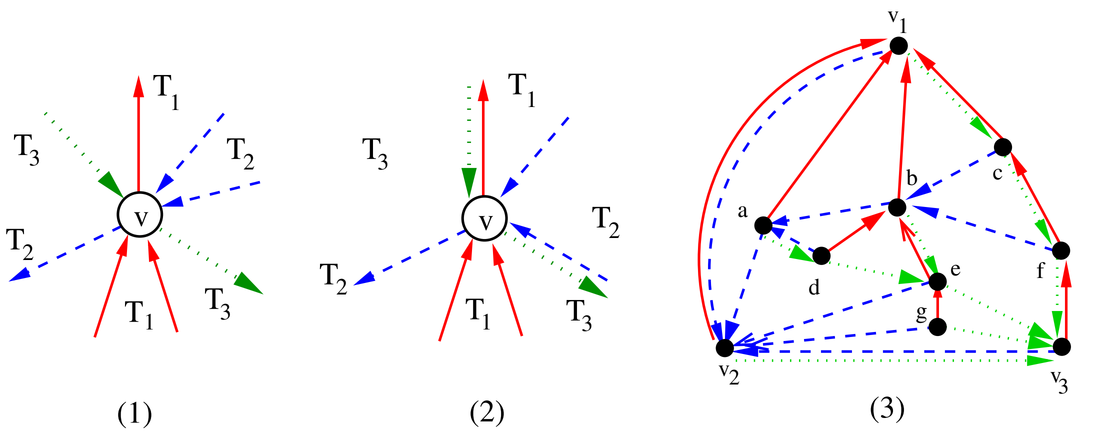

Figure 4: (1) and (2) Two examples of edge pattern around an internal vertex v; (3) A 3-connected graph G with its Schnyder wood.

3. The *i-Right-Cousins*: $RC_i(u) = \{v \in R_i^\circ(u) \mid$ the first i-ancestor of v in $P_{i-1}(u) \cup P_{i+1}(u)$ is in $P_{i-1}(u) - \{u\}\}$.

For example, consider the graph shown in Fig 4 (3). We have $R_1(b) = \{v_2, a, b, d, e, g, v_3\}$. $P_2(b) = \{b, a, v_2\}$, $P_3(b) = \{b, e, v_3\}$. $D_1(b) = \{d\}$, $LC_1(b) = \emptyset$ and $RC_1(b) = \{g\}$. (Note that although g is a 1-descendent of b, since the first 1-ancestor of g is $e \in P_3(b) - \{b\}$, $g \notin D_1(b)$.)

The properties of Schnyder wood have been studied extensively in [3, 5, 10, 12, 13, 14] and are summarized in the following lemma.

Lemma 3.1. *Let $G = (V, E)$ be a 3-connected plane graph with $|V| = n$ and $|E| = m$. Let $\mathcal{R} = (T_1, T_2, T_3)$ be a Schnyder wood of G, where T_i is rooted at the vertex v_i for $i \in \{1, 2, 3\}$.*

1. *The number of 2-colored edges of G is $m_b = 3n - m - 3$.*

2. *For each vertex v of G, $P_1(v)$, $P_2(v)$ and $P_3(v)$ have only the vertex v in common.*

3. *For $i, j \in \{1, 2, 3\}$ $(i \neq j)$ and two vertices u and v, the intersection of $P_i(u)$ and $P_j(v)$ is either empty or a common sub-path.*

4. *For vertices v_1, v_2, v_3 the following hold: $P_1(v_2) = P_2(v_1) = ext(v_1, v_2)$; $P_2(v_3) = P_3(v_2) = ext(v_2, v_3)$; $P_3(v_1) = P_1(v_3) = ext(v_3, v_1)$.*

Let f be the number of internal faces of G. Let $(x_1(u), x_2(u), x_3(u))$ be the Schnyder coordinates, where $x_i(u) =$ the number of internal faces of G in the region $R_i(u)$. If we take any two Schnyder coordinates as x- and y-coordinates, we obtain a straight-line convex drawing of G on an $f \times f$ grid [10, 12]. It's tempting to show this drawing is greedy with respect to the metric function $D(*, *)$. Unfortunately, it is not. In the following, we prove that this drawing is weakly greedy with respect to $D(*, *)$.

Lemma 3.2. *For any two vertices u and w $(u \neq w)$ in G, u has a neighbor v such that $D(u, w) \geq D(v, w)$.*

Proof: If w is a neighbor of u, let $v = w$ and the theorem is trivially true. So we assume w is not a neighbor of u. Without loss of generality, assume $w \in R_1(u)$. The other cases are symmetric.

The proof is similar to the proof of Lemma 2.1. We divide the proof into the same five cases (see Figure 3.) However, in the 3-connected case, some regions C_i may be empty because of the existence of 2-colored edges. In these cases, we would only be able to show $D(u, w) = D(v, w)$.

Case 1: This case is illustrated as in Figure 3 (1). (However, some region C_i might be empty).

The argument in the proof of Lemma 2.1 still works here. We have shown that $D(u, w) > D(v, w)$ is equivalent to $\#(C_1) + \#(C_2) > 0$. We show C_1 and C_2 cannot be empty. If C_1 is empty, then the edge (v, u) is in T_2 and directed from v to u in T_2 (see Figure 5 (1).) However, we already know (v, u) is in T_1 and directed from v to u in T_1. This contradicts the requirement of Schnyder woods. Similarly, C_2 is not empty. Hence $\#(C_1) + \#(C_2) > 0$, and $D(u, w) > D(v, w)$.

Case 2 (see Figure 3 (2)): We have shown that $D(u, w) > D(v, w)$ is equivalent to $\#(C_1) + \#(C_2) > 0$. Although C_1 and C_2 can be empty individually, we show that they cannot be empty at the same time. If C_1 is empty, then the edge $e = (v, u)$ is in T_2 and directed from v to u in T_2. If C_2 is empty, then e is in T_1 and

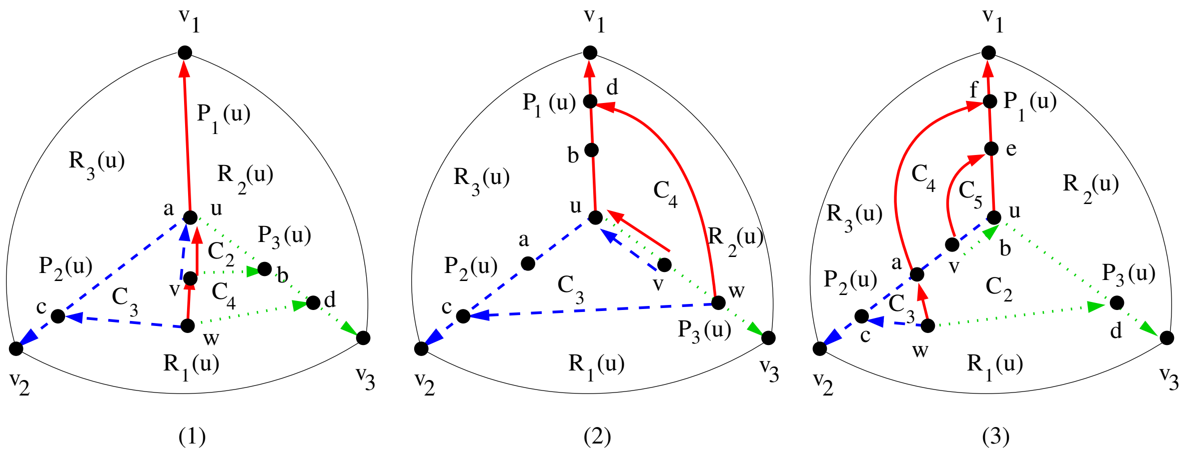

Figure 5: The proof of Lemma 3.2. (1) Case 1; (2) Case 2; (3) Case 4.

directed from v to u in T_1 (see Figure 5 (2).) However, we already know e is in T_3 and directed from u to v in T_3. If both C_1 and C_2 are empty, then the edge e has to be in three trees. By the properties of Schnyder wood, this is impossible. So $D(u, w) > D(v, w)$.

Case 3: Similar to Case 2.

Case 4 (see Figure 3 (3).)

Case 4a: We have shown that $D(u, w) > D(v, w)$ is equivalent to $\#(C_1) + \#(C_5) > 0$. Similar to Case 2, C_1 and C_5 cannot be both empty. So this is true and $D(u, w) > D(v, w)$.

Case 4b: We have shown $D(u, w) > D(v, w)$ is equivalent to $2 \times (\#(C_1) + \#(C_4) + \#(C_5)) > 2 \times \#(C_3)$. This is true because we have assumed $\#(C_4) + \#(C_5) > \#(C_3)$ in this sub-case.

Case 4c: We have shown $D(u, w) > D(v, w)$ is equivalent to $\#(C_1) > 0$. If C_1 is not empty, then $\#(C_1) > 0$ and $D(u, w) > D(v, w)$. If the edge $e = (v, u)$ is a 2-colored edge, belonging to both T_2 and T_3 (see Figure 5 (3)), then C_1 is empty and $\#(C_1) = 0$. In this case, we have $D(u, w) = D(v, w)$

Case 5: Similar to Case 4. □

From Lemma 3.2 and the results in [10, 12], we have:

THEOREM 3.1. *Let $G = (V, E)$ be a 3-connected plane graph with n vertices and f internal faces. Then G has a weakly greedy drawing with respect to metric function $H(*, *)$ on a $f \times f$ grid. The drawing is planar and convex, and can be constructed in linear time.*

By Theorem 3.1, Schnyder drawing is only weakly greedy. The graph in Figure 6 (1) shows that this result cannot be improved. The Schnyder coordinates of the vertices are given in the following table.

	$x_1(*)$	$x_2(*)$	$x_3(*)$
v_1	15	0	0
v_2	0	15	0
v_3	0	0	15
a	2	3	10
b	10	1	4
c	10	4	1
d	3	9	3
e	2	11	2
f	13	1	1
g	1	7	7
h	2	1	12
u	10	2	3
v	10	3	2
w	4	7	4

If we take $x_3(*)$ as the x- and $x_1(*)$ as the y-coordinate, the drawing is shown in Figure 6 (2). For the origin u and destination w, we have a case 4c, and $D(u, w) = D(v, w) = D(a, w) = D(b, w) = 12$. So u has no neighbor that is strictly closer to w.

4 Conclusion

In this paper, we showed that the classical Schnyder drawing on a $(2n - 5) \times (2n - 5)$ grid of plane triangulation is greedy with respect to a simple natural metric function $H(*, *)$. We also showed that the classical Schnyder drawing on an $f \times f$ grid of 3-connected plane graphs is weakly greedy with respect to the same metric function H.

When the drawing is greedy, the routing rule is very simple: u just forwards the message to a neighbor v such that $H(d(u), d(w)) > H(d(v), d(w))$. When the drawing is only weakly greedy, the routing rule is more complicated: in the case that u has no neighbors

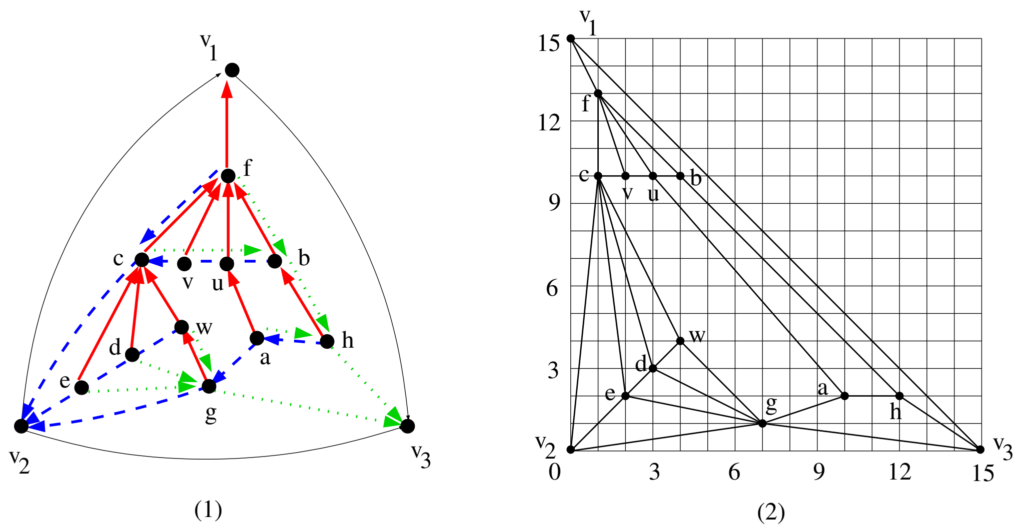

Figure 6: An example.

strictly closer to the destination w, u must forward the message to a neighbor v such that $H(d(u), d(w)) = H(d(v), d(w))$. In this case, we must make sure the message does not enter a loop consisting of vertices with equal distance to the destination. In [18], a set of simple routing rules was presented for Schnyder drawing, based on Schnyder coordinates. It is simple to use. However, since the routing decision is not based on decreasing distance alone, it is not a true greedy routing algorithm. It remains an open problem to find a convex, succinct, true greedy drawing with respect to a natural metric function for 3-connected plane graphs.

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