\( u \): displacement
\( v = \partial_t u \): particle velocity
\( \sigma \): stress tensor

Inverse problem: least squares formulation

1. Data: \( d_{r,s,t} \)
2. source: \( f_s(x,t) = \delta(x-s)w(t) \), \( \omega \): source wavelet (signature)

Def: \( J[c] = \frac{1}{2} ||d_{r,s,t} - u_s(r,t)||^2_2 \), where \( u_s \) comes from solving the corresponding wave equation. Problem:

\[
\min_{c(x)} J[c] \text{ s.t. } \begin{cases}
\frac{1}{c_0^2(x)} \frac{\partial^2}{\partial t^2} u_s - \Delta u_s = f_s, \\
u_s(x,0) = \partial_t u_s(x,0) = 0 \\
\text{proper boundary conditions}
\end{cases}
\]

Write \( u = F[c] \); \( F[c] = F[c_0] + \frac{\delta F}{\delta c^2} [c_0] (c^2 - c_0^2) + \ldots \). Note, \( \frac{\delta F}{\delta c^2} \) is an operator, Frechet derivative.

Assume know background wave speed \( c_0(x) \). Want to call \( F[c_0] \) the incident wave, often assume \( c_0 \) is smooth. This approach works well when \( c^2 - c_0^2 \) is oscillatory, (has zero mean, etc). Want to call \( \frac{\delta F}{\delta c^2} [c_0] \) primary reflected waves; further terms would correspond to multiples. If these conditions are not met, then further terms in the Taylor expansion are significant, otherwise not.

\[
\begin{cases}
\frac{1}{c_0^2(x)} \frac{\partial^2}{\partial t^2} u_{\text{inc}} - \Delta u_{\text{inc}} = f_s \\
\frac{1}{c_0^2(x)} \frac{\partial^2}{\partial t^2} u_s - \Delta u_s = f_s \\
\text{→ (subtract) } \begin{cases}
\frac{1}{c_0^2(x)} \frac{\partial^2}{\partial t^2} u_{\text{scat}} - \Delta u_{\text{scat}} = -V(x) \frac{\partial^2}{\partial t^2} u \\
\text{zero initial conditions, proper boundary conditions.}
\end{cases}
\end{cases}
\]

\[
u_{\text{inc}}(x,t) = \int \int G(x,y;t\tau) f_s(y,\tau) dy d\tau = Gf
\]

if \( C_0 = 1 \), \( G(x,y,y) = \frac{\delta(t-|x-y|/c)}{ct-|x-y|} \).

\[
\rightarrow u_{\text{scat}} = -G(V \frac{\partial^2}{\partial t^2} u), u = u_{\text{inc}} - GV \frac{\partial^2}{\partial t^2} u
\]

\[
(I + GV \frac{\partial^2}{\partial t^2}) u = u_{\text{inc}}.
\]

Lippman-Schwinger equations.

Born series:

\[
u = u_{\text{inc}} - GV \frac{\partial^2}{\partial t^2} u_{\text{inc}} + GV \frac{\partial^2}{\partial t^2} GV u_{\text{inc}} + \ldots
\]

Write \( \mathbf{F} = \frac{\delta F}{\delta c^2} [c_0] \), which is the forward / linearized modeling operator, \( \mathbf{F}V = u^{B}_x \) (superscript stands for Born). Weak scattering: require that \( \|G(V \frac{\partial^2}{\partial t^2})\|_2 < 1 \), which implies the series converges; usually this is too strong a condition, and is not practical.

Linearized Inverse Problem: minimize over \( V \)

\[
J_L[V] = \frac{1}{2} \|d_{r,s,t} - u_{\text{inc},s}(r,t) - u^{B}_{x,s}(r,t)\|_2^2 = \frac{1}{2} \|d - \mathbf{F}V\|_2^2 \text{ s.t. } \begin{cases}
\frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} u - \Delta u_{x,s} = -V(x) \frac{\partial^2}{\partial t^2} u_{\text{inc},s}, \\
\text{zero initial conditions} \\
\text{proper boundary conditions}
\end{cases}
\]

(removes multiples, compares to primaries).
Solution of Linearized Inverse Problem: is 
\[ V = F^{-1}u_{\text{scat}} = (F^*F)^{-1}F^*u_{\text{Bscat}} \] 
\[ \delta J = F^*(FV - \tilde{d}), \] 
\[ \delta V = -\alpha \frac{\delta J}{\delta V}. \] Gradient descent works for full non-linear problem: 
\[ \frac{\delta J}{\delta V} = F^*(FV - \tilde{d}), \]

Adjoint-State Method:
\[ \langle d, FV \rangle_{r,t,s} = \langle F^*d, V \rangle_x \Rightarrow \sum_r \int d_r(t)FV(r,t)dt = \int F^*d(x)V(x)dx \quad \text{(fix s)} \]
one construction:
\[ d_{\text{ext}}(x,t) = \sum_r d(r,t)\delta(x - r) \]
then,
\[ \sum_r \int d_r(t)FV(r,t)dt = \int \int d_{\text{ext}}(x,t)FV(x,t)dtdx \]
now, \( u^B_x(x,t) = FV(x,t) \), which solves the wave equation, so let
\[ d_{\text{ext}}(x,t) = \left( \frac{1}{c_0^2} \partial_t^2 - \Delta \right)q(x,t) \]
called the adjoint wavefield. Now integrate by parts. Boundary terms:
\[ \int \frac{1}{c_0^2} \partial_t q u^T_0 dx, \int \frac{1}{c_0^2} q \partial_t u^T_0 dx, \int \frac{1}{c_0^2} \partial u^T_0 S_z dt, \int \frac{1}{c_0^2} \frac{\partial u}{u} dS_x dt \]
The last two terms are zero, due to finite speed of propagation. The lower boundary terms in the frist two terms are zero due to initial conditions. Require \( q(x,T) = \frac{\partial q}{\partial t}(x,T) = 0 \), then the upper boundary terms are zero aswell. Final value problem (backward in time). This is the adjoint state problem:
\[
\begin{cases}
\left\{ d_{\text{ext}}(x,t) = \left( \frac{1}{c_0^2} \partial_t^2 - \Delta \right)q(x,t) \\
q(x,T) = \frac{\partial q}{\partial t}(x,T) = 0
\end{cases}
\]
LHS:
\[ \int \int q(x,t) \left( \frac{1}{c_0^2} \partial_t^2 - \Delta \right)u^B_{\text{scat}}(x,t) dx dt = \int V(x) \left[ \left( -1 \right) \int q(x,t) \partial_t^2 u_{\text{inc}}(x,t) dt \right] dx \]
Fix s.
\[ \hat{u}_{\text{inc}}(x,\omega) = \int \hat{G}(x, y, \omega) \hat{f}(y, \omega) dy \]
\[ u^B_{\text{scat}}(x, \omega) = \int \hat{G}(x, y, \omega)(-V(y))(-\omega^2)\hat{u}_{\text{inc}}(y, \omega) dy = \]
\[ \omega^2 \int \int \hat{G}(x, y, \omega)V(y)\hat{G}(y, z, \omega) \hat{f}(z, \omega) dy dz \]
Parseval:
\[ \langle d, u^B_x \rangle_{r,t} = \langle \hat{d}, \hat{d}^B_x \rangle = \sum_r \int \hat{d}(r, \omega)\hat{u}^B_{\text{scat}}(r, \omega) d\omega \]
\[
\sum_r \int \hat{d}(r, \omega) \omega^2 \int \int \bar{g}(r, y, \omega)V(y) \bar{G}(y, z, \omega) \hat{f}(z, \omega) dy dz d\omega \\
= \int dy V(y) \int d\omega^2 \sum_r \bar{G}(r, y, \omega) \hat{d}(r, \omega) \int dz \bar{G}(y, z, \omega) \hat{f}(z, \omega)
\]

\[
\mathbf{F}^* d(x) = \int d\omega^2 \sum_r \bar{G}(r, x, \omega) \hat{d}(r, \omega) \int dz \bar{G}(x, z, \omega) \hat{f}(z, \omega) \\
= \int d\omega^2 \hat{q}(x, \omega) = -\langle \hat{q}, (-\omega^2) \hat{u}_{inc}\rangle_{\omega}
\]

Connection with RADAR:

\[
\hat{d} = \mathbf{F} V(s, \omega) = \int \exp(2i\omega \| \gamma(s) - y \| / c) A(\omega, s, y) V(y) dy
\]

\[
\mathbf{F}^* \hat{d}(x) = \int \exp(2i\omega \| \gamma(s) - y \| / c) A(\omega, s, y) \hat{d}(s, \omega) dy
\]

\[
\mathbf{F}^{-1} \hat{d}(x) = \int \exp(2i\omega \| \gamma(s) - y \| / c) B(\omega, s, y) \hat{d}(s, \omega) dy
\]

RADAR:

\[
d(s, t) \approx \int \delta(t - \frac{2}{c} \| \gamma(s) - x \|) V(x) dx
\]

\[
I_{KM}(x) = \int \delta(t - \frac{2}{c} \| \gamma(s) - x \|) d(s, t) ds dt
\]

CT:

\[
g(\theta, s) = \int \delta(s - x \cdot \theta) f(x) dx
\]

\[
I_{KM} = \int \delta(s - x \cdot \theta) g(\theta, s) d\theta ds = \int g(\theta, x \cdot \theta) d\theta
\]

(unfiltered back projection)

Seismology:

\[
d(r, t) \approx \int \delta(t - \tau(r, x) - \tau(s, x)) V(x) dx
\]

\[
I_{KM}(x) = \sum_r \int \delta(t - \tau(r, x) - \tau(s, x)) d(r, t) dt = \sum_r d(r, \tau(r, x) + \tau(s, x))
\]

Assume \( c(x) \) is smooth (no reflection, refraction). \( y \): take off point. (Eikonal, Transport Equation)

Examples, with \( c(x) = c_0 \): \( \tau(x, y) = \| x - y \| / c_0 \), \( \tau(x, y) = x_1 / c_0 \), distance function to any curve.

Iso-phase lines of \( e^{i\omega \tau} \) = iso-level lines of \( \tau \) = wavefronts. See viscosity solution of the eikonal equation, progressive wave expansion.
Eikonal:
\[ \|\nabla \tau(x, y)\| = \frac{1}{c(x)} \]

Rays: characteristic curves. Wavefronts: level curves of \( \tau \), rays: perpendicular to wavefronts. So
\[ \dot{X}(t) = c(X(t)) \frac{\nabla \tau(X(t), y)}{\|\nabla \tau(X(t), y)\|} \]

chain rule:
\[ \frac{d}{dt} \tau(X(t), y) = \dot{X}(t) \nabla \tau(X(t), y) = c(X(t)) \frac{\nabla \tau(X(t), y)}{\|\nabla \tau(X(t), y)\|} \cdot \nabla \tau(X(t), y) = c(X(t), y) \|\nabla \tau(X(t), y)\| = 1 \]

\( \tau \): travel time; \( \tau(X(t), y) - \tau(X(0), y) = t, \tau(X(0), y) = 0 \). Example: for \( X(t) \) still depends on \( \tau \).

\[ p(t) = \nabla \tau(X(t), y), \dot{p}(t) = \nabla \nabla \tau \cdot \dot{X} \]

\[ = \nabla \nabla \tau \cdot \nabla c^2 = \frac{1}{2} \nabla \|\nabla \tau \nabla c^2 \| = -\frac{1}{2} \nabla (c^2)\|p(t)\|^2 \]

get

\[ \left\{ \begin{array}{l}
\dot{X}(t) = c^2(X(t))p(t), \quad X(0) = x_0 \\
\dot{p}(t) = -\frac{1}{2}(\nabla c^2)(X(t))\|p(t)\|^2 \quad p(0) = p_0
\end{array} \right. \]

and \( \tau(X(t), y) = t \).

WAVE:
\[ \left( \frac{1}{c^2} \partial_t^2 - \Delta \right) G = \delta(x - y) \]
\[ G(x, y, t) \approx a_0(x, y)\delta(t - \tau(x, y)) \]

Hamilton-Jacobi

FINISH

\( \tau(x, y) = \inf \{ T : \exists X \text{ s.t. } X(0) = 0, X(T) = y, \|X(t)\| = c(X(t))\forall 0 < t < T \} \)

argument is \( X(t) \) that solves Hamilton system for some \( p_0 \).

\[ \tau(x, y) = \inf \left\{ \int_0^s \mathcal{L}(X(s), \dot{X}(s))ds : X(0) = x, X(s) = y \right\} \]

(any parameterization), \( \mathcal{L}(x, \dot{x}) = \frac{1}{x(c)} \|\dot{x}\|, \int_0^1 \frac{1}{c(x)}d\ell \) and any min is \( X(t) \) for some \( p_0 \). Euler-Lagrange:

\[ \frac{\delta \mathcal{L}}{\delta x} - \frac{d}{ds} \frac{\delta \mathcal{L}}{\delta \dot{x}} = 0. \]

Amplitude:

\[ 2\nabla \tau \cdot \nabla a_0 + a_0 \Delta \tau = 0 \]

\[ \frac{d}{dt} a_0(X(t), y) = \dot{X}(t) \nabla_x a_0 = c^2 \nabla \tau \nabla a_0 = -\frac{c^2}{2} a_0 \Delta \tau \]

\[ \nabla \cdot a^2 \nabla \tau = 0 \]
Model Velocity Estimator:

\[ J[c] = \frac{1}{2} \|d - F[c]\|^2 \]

\[ \frac{1}{c^2} = \frac{1}{c_0^2} + V \]

\[ u_s(r, t) = F[c] \iff \left( \frac{1}{c^2} \partial_t^2 - \Delta \right) u_s = f_s \]

\[ u_{scat,s}^B(r, t) = \frac{\delta F}{\delta c^{-2}}[c_0]V = FV \iff \left( \frac{1}{c^2} \partial_t^2 - \Delta \right) u_{scat,s}^B = -V \partial_t^2 u_{scat,s}^B \]

\[ - \frac{\delta J}{\delta c^{-2}}[c_0] = \mathbf{F}^*(d - \mathbf{F}[c]) \]

\[ \mathbf{F}_s^* d_s(x) = - \int_0^T q_s(x, t) \partial_t^2 u_{inc,s}(x, t) dt \]

(1) Landweber iterations:

\[ \left\{ \begin{align*}
\delta V_{k+1} &= -\alpha \frac{\delta J}{\delta c^{-2}}[c_k] \\
\frac{1}{c_{k+1}^2} &= \frac{1}{c_k^2}
\end{align*} \right. \]

\[ \max_{c_0} |\langle d, f(r - c_0 \cdot) \rangle| \]

Full Waveform Inversion: new data as \( \hat{d}_{r,s}(\omega) \) add freq \( \omega \) from low to high (Karzmarz), \( \lambda \) from large to small. Requires knowledge of \( \hat{d}_{r,s}(\omega) \) for small \( \omega \).

(2) Extension Principles (symes): \( d_{r,s,t}, \mathbf{F}_s^* d_s(x) = \sum_s \mathbf{F}_s^* d_s(x) \) (stack). \( \mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_{N_s}), \mathbf{F}_s^* = \sum_s \mathbf{F}_s^* \) (adjoint and operator-sum commute). Idea: let \( V_s(x) = \mathbf{F}_s^* d_s(x) \), look at images before summing. If \( c_0 \) is good, then \( V_s \approx V_s \), else not. Example: Differential semblance optimization (Symes).

\[ J_{DSO}[c_0, (V_s, \ldots, V_{N_s})] = \frac{1}{2} \sum_s |V_{s+1} - V_s| + \frac{1}{2} \sum_s \|d_s - \mathbf{F}_s V_s\|^2_2, \quad \min J_{DSO} \]

\( \Omega \subseteq \mathbb{R}^3 \), Extension: manifold \( \Omega \), operator \( \chi \), such that (see notes).

Example: (Standard extension), \( \Omega = \Omega \times S \), \( (x \in \Omega, s \in S) \). \( \nabla \in \mathcal{D}'(\Omega) \) in \( \nabla(x, s) \).

\[ (\chi V)(x, s) = V(c) \]

\[ \mathbf{F} \mathbf{V}_{r,s,t} = \int dx V(x) \int d\tau G(x, x_r, \tau) \partial_t^2(x_s, x, t - \tau) \]

\[ (\chi V)(x, s) = V(c) \]

\[ \mathbf{F} \mathbf{V}_{r,s,t} = \int dx V(x) \int d\tau G(x, x_r, \tau) \partial_t^2(x_s, x, t - \tau) \]

(insert)
(3) Travel Time Tomography:

\[ J_{TT}[c] = \frac{1}{2}\|\tau_{r,s} - \tau(x_r, x_s)\|, \min_{c} J_{TT}[c] \text{ s.t. } \|\nabla_x \tau(x_r, x_r)\| = \frac{1}{c(x)} \]

Solution in layered media (Hergoltz-Wiechert (1910)). Data: \( \tau(r) \), fix \( s \). Horizontal slowness \( T'(x) = p(x) = \frac{1}{\epsilon_o} \cos(\theta_0) \).