Waves and Imaging in Random Media

Josselin Garnier (Université Paris 7 and Ecole Normale Supérieure)

- Principle of imaging with waves: waves are used to probe an unknown medium.
  1) record the waves generated by sources on a sensor array.
  2) process the data (backpropagation) in order to estimate relevant features of the medium (source or reflector locations, ...).

- Applications: medical imaging, geophysical exploration, non-destructive testing.

- In the presence of clutter noise (i.e. due to the scattering medium):
  - one usually observes a loss of resolution and signal-to-noise ratio in imaging.
  - one also observes “super-resolution” phenomena and “enhanced refocusing” in time-reversal experiments.

- It is possible to image with noisy signals; even with pure noise!
Waves in Random Media: Inverse Problems and Imaging

Let us start with some unexpected effects due to randomness:

- enhanced refocusing for a time reversal experiment.
- imaging using ambient noise.
Time reversal experiment (1/4)


Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and S records the reconstructed pressure field.
Time reversal experiment (2/4)


Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and S records the reconstructed pressure field.
Experimental observations

The source emits a short 1 $\mu$s pulse.

The TRM records a long scattered signal.

One can observe a time recompression at the original source location.
One can observe a spatial refocusing at the original source location.

**Scan of the pressure field**

**Time-reversed waves**

1. Multiple scattering medium
2. Water
Classical problem in geophysics: Travel time estimation (for background velocity estimation).

- Method 1: Use of earthquake signals.
- Method 2: Use of seismic noise and cross correlation techniques in order to estimate the Green’s function.
Travel time estimation by cross correlation of ambient noise signals

- Ambient noise sources (○) emit stationary random signals.
- The waves propagate in the (inhomogeneous) medium.
- The signals \( u(t, x_1) \) and \( u(t, x_2) \) are recorded at two sensors \( x_1 \) and \( x_2 \).

What information (about the medium) can possibly be in these signals?
Travel time estimation by cross correlation of ambient noise signals

- Ambient noise sources (○) emit stationary random signals.
- The waves propagate in the (inhomogeneous) medium.
- The signals \( u(t, \mathbf{x}_1) \) and \( u(t, \mathbf{x}_2) \) are recorded at two sensors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \).

Compute the empirical cross correlation:

\[
C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{T} \int_0^T u(t, \mathbf{x}_1)u(t + \tau, \mathbf{x}_2)dt
\]

- \( C_T(\tau, \mathbf{x}_1, \mathbf{x}_2) \) is related to the Green’s function from \( \mathbf{x}_1 \) to \( \mathbf{x}_2 \)!
- The singular component of the Green’s function from \( \mathbf{x}_1 \) to \( \mathbf{x}_2 \) gives the travel time from \( \mathbf{x}_1 \) to \( \mathbf{x}_2 \).

G2S3 July, 2011
Record *during one month* the seismic noise at a sensor array (triangles). For each pair of sensors cross-correlate the recorded noise data.
Process the cross-correlation matrix to obtain a map of the background velocity.

[Shapiro et al, Science 307 (2005), 1615.];
• Application of ambient noise imaging: volcano monitoring.

Piton de la Fournaise (volcano in La Réunion island):
- Detection of changes in surface wave travel times during mid September 2010.
- Alert sent out on September 23, 2010.
- Eruption on October 14, 2010.
Waves in Random Media: Inverse Problems and Imaging

- Introduction and motivations (time-reversal experiment and travel time tomography using ambient noise).

- Some useful tools: Fourier identities, Green’s functions, Helmholtz-Kirchhoff theorem, geometric optics.

- Migration and least squares imaging.

- Resolution and stability analysis of Reverse-Time migration and time reversal:
  - homogeneous medium,
  - additive (measurement) noise,
  - random travel time model, random paraxial model.

- Coherent interferometric imaging.

- Passive sensor imaging using cross correlations of ambient noise signals.
Some useful tools
Fourier identities

Let $f(t)$ be a “nice” real-valued function. Its Fourier transform is

$$\hat{f}(\omega) = \int f(t) e^{i\omega t} dt$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int \hat{f}(\omega) e^{-i\omega t} d\omega$$

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$\hat{f}(\omega)$</th>
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<tbody>
<tr>
<td>$\frac{d^n f}{dt^n}$</td>
<td>$(-i\omega)^n \hat{f}(\omega)$</td>
</tr>
<tr>
<td>$f \ast g(t) = \int f(s)g(t - s) ds$</td>
<td>$2\pi \hat{f}(\omega)\hat{g}(\omega)$</td>
</tr>
<tr>
<td>$f(-t)$</td>
<td>$\hat{f}(\omega)$</td>
</tr>
<tr>
<td>$\int f(s)g(t + s) ds$</td>
<td>$2\pi \hat{f}(\omega)\hat{g}(\omega)$</td>
</tr>
</tbody>
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- time reversal
- cross correlation
Wave equation and Green’s function (1/3)

- Scalar wave model:
  \[
  \frac{1}{c^2(x)} \frac{\partial^2 u}{\partial t^2} - \Delta_x u = n(t, x)
  \]

  \(n(t, x):\) source.
  \(c(x):\) propagation speed (parameter of the medium), assumed to be constant outside a domain with compact support.

- The time-dependent Green’s function \(G(t, x, y)\) is the solution of
  \[
  \frac{1}{c^2(x)} \frac{\partial^2 G}{\partial t^2} - \Delta_x G = \delta(t)\delta(x - y)
  \]

  Assume that \(G(t, x, y) = 0 \forall t < 0 \implies\) unique solution (causal Green’s function).
  Emission from a point source at \(y\) emitting a Dirac pulse at time 0.

  If the medium is homogeneous \(c(x) \equiv c_0\), then
  \[
  G(t, x, y) = \frac{1}{4\pi|x - y|} \delta\left(t - \frac{|x - y|}{c_0}\right), \quad t > 0
  \]

  \(\leftrightarrow\) spherical wave propagating at speed \(c_0\).
Wave equation and Green’s function (2/3)

• The time-harmonic Green’s function

\[ \hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \int G(t, \mathbf{x}, \mathbf{y}) e^{i\omega t} dt \]

is the solution of the Helmholtz equation

\[ \Delta \hat{G} + \frac{\omega^2}{c^2(\mathbf{x})} \hat{G} = -\delta(\mathbf{x} - \mathbf{y}), \]

with the Sommerfeld radiation condition (\(c(\mathbf{x}) = c_0\) at infinity):

\[ \lim_{|\mathbf{x}| \to \infty} |\mathbf{x}| \left( \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla_{\mathbf{x}} - i\frac{\omega}{c_0} \right) \hat{G}(\omega, \mathbf{x}, \mathbf{y}) = 0 \]

If the medium is homogeneous \(c(\mathbf{x}) \equiv c_0\), then

\[ \hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} e^{i\frac{\omega}{c_0} |\mathbf{x} - \mathbf{y}|} \]
Wave equation and Green’s function (3/3)

- The solution of the wave equation with source $n(t, x)$ has the form:

$$u(t, x) = \int \int G(t - s, x, y)n(s, y)dyds$$

In the Fourier domain:

$$\hat{u}(\omega, x) = \int u(t, x)e^{i\omega t}dt$$

we have

$$\hat{u}(\omega, x) = \int \hat{G}(\omega, x, y)\hat{n}(\omega, y)dy$$
Complement: Acoustic wave equations

Acoustic wave equations:

\[ K^{-1}(x) \partial_t p + \nabla \cdot u = 0, \]
\[ \rho(x) \partial_t u + \nabla p = 0, \]

\( p \): pressure field, \( u \): velocity field.
\( K \): bulk modulus, \( \rho \): density of the medium.

The pressure field satisfies:

\[ K^{-1}(x) \partial_t^2 p - \nabla \cdot (\rho^{-1}(x) \nabla p) = 0 \]

If \( \rho(x) \equiv \rho_0 \):

\[ c^{-2}(x) \partial_t^2 p - \Delta p = 0 \]

with

\[ c^2(x) = \frac{K(x)}{\rho_0} \]
Complement: Sommerfeld radiation condition (1/2)

• Consider the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$:

$$\Delta_x \hat{G}(\omega, x, y) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y) = -\delta(x - y)$$

This equation has an infinite number of solutions.

• A solution is called radiating if it satisfies the Sommerfeld radiation condition:

$$\hat{G}(\omega, x, y) = O_{|x| \to \infty} \left( \frac{1}{|x|} \right)$$

and

$$\left( \frac{x}{|x|} \cdot \nabla_x - i \frac{\omega}{c_0} \right) \hat{G}(\omega, x, y) = o_{|x| \to \infty} \left( \frac{1}{|x|} \right)$$

uniformly in all directions ($c(x) = c_0$ at infinity).

• Example: $c(x) \equiv c_0$. There exist infinitely many solutions, in particular

$$\hat{G}_a(\omega, x, y) = \frac{1 - a}{4\pi|x - y|} \exp \left( i \frac{\omega}{c_0} |x - y| \right) + \frac{a}{4\pi|x - y|} \exp \left( - i \frac{\omega}{c_0} |x - y| \right)$$

for some constant $a$. Only the solution with $a = 0$ satisfies the Sommerfeld radiation condition. It corresponds to a field radiating from $y$. The other solutions are “unphysical”. For example, the solution with $a = 1$ can be interpreted as energy coming from infinity and sinking at $y$. 
Complement: Sommerfeld radiation condition (2/2)

- Consider the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$:

$$\Delta_x \hat{G}(\omega, x, y) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y) = -\delta(x - y)$$

A solution is called radiating if it satisfies the Sommerfeld radiation condition

$$\lim_{|x| \rightarrow \infty} |x| \left( \frac{x}{|x|} \cdot \nabla_x - i \frac{\omega}{c_0} \right) \hat{G}(\omega, x, y) = 0$$

uniformly in all directions ($c(x) = c_0$ at infinity).

- Theorem: The Helmholtz equation (with $c$ bounded and constant outside a compact) has a unique radiating solution.

R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Chap. IV, Sec. 5.

Reciprocity

\[ \hat{G}(\omega, x, y) = \hat{G}(\omega, y, x) \]
Proof of reciprocity (1/2)

We consider the equations satisfied by the Green's function with the source at \( y_2 \) and with the source at \( y_1 \) \((y_2 \neq y_1)\):

\[
\Delta_x \hat{G}(\omega, x, y_2) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y_2) = -\delta(x - y_2)
\]

\[
\Delta_x \hat{G}(\omega, x, y_1) + \frac{\omega^2}{c^2(x)} \hat{G}(\omega, x, y_1) = -\delta(x - y_1)
\]

We multiply the first equation by \( \hat{G}(\omega, x, y_1) \) and subtract the second equation multiplied by \( \hat{G}(\omega, x, y_2) \):

\[
\nabla_x \cdot \left[ \hat{G}(\omega, x, y_1) \nabla_x \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \nabla_x \hat{G}(\omega, x, y_1) \right] = \hat{G}(\omega, x, y_2) \delta(x - y_1) - \hat{G}(\omega, x, y_1) \delta(x - y_2)
\]

We next integrate over the ball \( B(0, L) \) which contains both \( y_1 \) and \( y_2 \) and use the divergence theorem:

\[
\int_{\partial B(0, L)} \mathbf{n} \cdot \left[ \hat{G}(\omega, x, y_1) \nabla_x \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \nabla_x \hat{G}(\omega, x, y_1) \right] dS(x) + o(1)
\]

\[
= \hat{G}(\omega, y_1, y_2) - \hat{G}(\omega, y_2, y_1)
\]

where \( \mathbf{n} \) is the unit outward normal to the ball \( B(0, L) \), which is \( \mathbf{n} = x/|x| \).
Proof of reciprocity (2/2)

\[
\int_{\partial B(0, L)} n \cdot \left[ \hat{G}(\omega, x, y_1) \nabla_x \hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2) \nabla_x \hat{G}(\omega, x, y_1) \right] dS(x)
\]

\[
= \hat{G}(\omega, y_1, y_2) - \hat{G}(\omega, y_2, y_1)
\]

where \( n = x/|x| \).

If \( x \in \partial B(0, L) \) and \( L \to \infty \), then by the Sommerfeld radiation condition:

\[
n \cdot \nabla_x \hat{G}(\omega, x, y) = i \frac{\omega}{c_0} \hat{G}(\omega, x, y) + o\left(\frac{1}{L}\right)
\]

Therefore, for \( L \to \infty \),

\[
\hat{G}(\omega, y_1, y_2) - \hat{G}(\omega, y_2, y_1)
\]

\[
= i \frac{\omega}{c_0} \int_{\partial B(0, L)} \hat{G}(\omega, x, y_1)\hat{G}(\omega, x, y_2) - \hat{G}(\omega, x, y_2)\hat{G}(\omega, x, y_1) dS(x)
\]

\[
= 0
\]
If the medium is homogeneous (velocity $c_0$) outside $B(0, D)$, then $\forall x_1, x_2 \in B(0, D)$ we have for $L \gg D$:

$$
\hat{G}(\omega, x_1, x_2) - \hat{G}(\omega, x_1, x_2) = \frac{2i\omega}{c_0} \int_{\partial B(0, L)} dS(y) \hat{G}(\omega, x_1, y) \hat{G}(\omega, x_2, y)
$$

Proof: second Green’s identity and Sommerfeld radiation condition.

Useful for: scattering theory, time reversal experiment, and cross correlation.
Proof of Helmholtz-Kirchhoff theorem (1/2)

Consider

\[ \Delta_y \hat{G}(\omega, y, x_2) + \frac{\omega^2}{c^2(y)} \hat{G}(\omega, y, x_2) = -\delta(y - x_2) \]

\[ \Delta_y \hat{G}(\omega, y, x_1) + \frac{\omega^2}{c^2(y)} \hat{G}(\omega, y, x_1) = -\delta(y - x_1) \]

Multiply the first equation by \( \hat{G}(\omega, y, x_1) \) and subtract the second equation multiplied by \( \hat{G}(\omega, y, x_2) \):

\[ \nabla_y \cdot \left [ \hat{G}(\omega, y, x_1) \nabla_y \hat{G}(\omega, y, x_2) - \hat{G}(\omega, y, x_2) \nabla_y \hat{G}(\omega, y, x_1) \right ] = \hat{G}(\omega, y, x_2) \delta(y - x_1) - \hat{G}(\omega, y, x_1) \delta(y - x_2) \]

\[ = \hat{G}(\omega, x_1, x_2) \delta(y - x_1) - \hat{G}(\omega, x_1, x_2) \delta(y - x_2) \]

using the reciprocity property \( \hat{G}(\omega, x_2, x_1) = \hat{G}(\omega, x_1, x_2) \).

Integrate over the ball \( B(0, L) \) and use the divergence theorem:

\[ \int_{\partial B(0,L)} \mathbf{n} \cdot \left [ \hat{G}(\omega, y, x_1) \nabla_y \hat{G}(\omega, y, x_2) - \hat{G}(\omega, y, x_2) \nabla_y \hat{G}(\omega, y, x_1) \right ] dS(y) = \hat{G}(\omega, x_1, x_2) - \hat{G}(\omega, x_1, x_2) \]

where \( \mathbf{n} \) is the unit outward normal to the ball \( B(0, L) \), which is \( \mathbf{n} = y/|y| \).
Proof of Helmholtz-Kirchhoff theorem (2/2)

\[
\int_{\partial B(0, L)} \mathbf{n} \cdot \left[ \hat{G}(\omega, \mathbf{y}, \mathbf{x}_1) \nabla_y \hat{G}(\omega, \mathbf{y}, \mathbf{x}_2) - \hat{G}(\omega, \mathbf{y}, \mathbf{x}_2) \nabla_y \hat{G}(\omega, \mathbf{y}, \mathbf{x}_1) \right] dS(\mathbf{y})
\]
\[
= \hat{G}(\omega, \mathbf{x}_1, \mathbf{x}_2) - \hat{G}(\omega, \mathbf{x}_1, \mathbf{x}_2)
\]

where \( \mathbf{n} = \mathbf{y}/|\mathbf{y}|. \)

This equality can be viewed as an application of the second Green’s identity.

The Green’s function also satisfies the Sommerfeld radiation condition

\[
\lim_{|\mathbf{y}| \to \infty} |\mathbf{y}| \left( \frac{\mathbf{y}}{|\mathbf{y}|} \cdot \nabla_y - \frac{i\omega}{c_0} \right) \hat{G}(\omega, \mathbf{y}, \mathbf{x}_1) = 0
\]

uniformly in all directions \( \mathbf{y}/|\mathbf{y}|. \) Substitute \( i(\omega/c_0)\hat{G}(\omega, \mathbf{y}, \mathbf{x}_2) \) for \( \mathbf{n} \cdot \nabla_y \hat{G}(\omega, \mathbf{y}, \mathbf{x}_2) \) in the surface integral over \( \partial B(0, L) \), and \( -i(\omega/c_0)\hat{G}(\omega, \mathbf{y}, \mathbf{x}_1) \) for \( \mathbf{n} \cdot \nabla_y \hat{G}(\omega, \mathbf{y}, \mathbf{x}_1) \), which gives the desired result:

\[
\hat{G}(\omega, \mathbf{x}_1, \mathbf{x}_2) - \hat{G}(\omega, \mathbf{x}_1, \mathbf{x}_2) = \frac{2i\omega}{c_0} \int_{\partial B(0, L)} dS(\mathbf{y}) \hat{G}(\omega, \mathbf{x}_1, \mathbf{y}) \hat{G}(\omega, \mathbf{x}_2, \mathbf{y})
\]
Complement: Green’s identities

• Divergence theorem:
  \[ \int_B \nabla \cdot F \, dx = \int_{\partial B} n \cdot F \, dS(x) \]

• First Green’s identity: This identity is derived from the divergence theorem applied to \( F = \psi \nabla \phi \).
  \[ \int_B \psi \Delta \phi + \nabla \phi \cdot \nabla \psi \, dx = \int_{\partial B} \psi n \cdot \nabla \phi \, dS(x) \]

• Second Green’s identity:
  \[ \int_B \psi \Delta \phi - \phi \Delta \psi \, dx = \int_{\partial B} \psi n \cdot \nabla \phi - \phi n \cdot \nabla \psi \, dS(x) \]
A quick introduction to geometric optics (1/2)

We look for an approximate expression as $\varepsilon \to 0$ for $\hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right)$ solution of

$$\Delta_x \hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right) + \frac{\omega^2}{c^2(x)\varepsilon^2} \hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right) = -\delta(x - y)$$

Note that, if $c(x) = c_0$, then

$$\hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right) = \frac{1}{4\pi |x - y|} e^{i\frac{\omega}{\varepsilon} \frac{|x - y|}{c_0}}$$

Consider a smoothly varying $c(x)$ and look for an expansion of the form:

$$\hat{G}\left(\frac{\omega}{\varepsilon}, x, y\right) = e^{i\frac{\omega}{\varepsilon} \mathcal{T}(x, y)} \sum_{j=0}^{\infty} \varepsilon^j A_j(x, y) \frac{\omega^j}{\omega^j}$$

Substitute the ansatz into Helmholtz equation and collect the terms with the same powers in $\varepsilon$:

$$O\left(\frac{1}{\varepsilon^2}\right) : \quad |\nabla_x \mathcal{T}|^2 - \frac{1}{c^2(x)} = 0$$

$$O\left(\frac{1}{\varepsilon}\right) : \quad 2\nabla_x \mathcal{T} \cdot \nabla_x A_0 + A_0 \Delta_x \mathcal{T} = 0$$

$\leftrightarrow$ Eikonal equation for the quantity $\mathcal{T}$ (that turns out to be the travel time) + transport equation for the amplitude $A_0$.

Solve by method of characteristics (ray equations).
A quick introduction to geometric optics (2/2)

Geometric optics approximation of the Green’s function:

\[ \hat{G} \left( \frac{\omega}{\varepsilon}, x, y \right) \sim A(x, y) e^{i \frac{\omega}{\varepsilon} T(x, y)} \]

valid when \( \varepsilon \ll 1 \), where the travel time is

\[ T(x, y) = \inf \left\{ T \text{ s.t. } \exists (X_t)_{t \in [0, T]} \in C^1, \ X_0 = x, \ X_T = y, \ |\frac{dX_t}{dt}| = c(X_t) \right\} \]

The curve(s) that minimizes this functional are called ray(s).

Simple geometry hypothesis: \( c(x) \) is smooth and there is a unique ray between any pair of points (in the region of interest).

In the homogeneous case \( c(x) \equiv c_0 \):

\[ \hat{G} \left( \frac{\omega}{\varepsilon}, x, y \right) = A(x, y) e^{i \frac{\omega}{\varepsilon} T(x, y)}, \ \text{with} \ A(x, y) = \frac{1}{4\pi|x-y|}, \quad T(x, y) = \frac{|x-y|}{c_0} \]
Complement: Solving the eikonal equation (1/3)

Let us consider the general nonlinear equation with unknown $u$:

$$F(x, u, \nabla_x u) = 0$$

Let $u(x)$ be a solution. Denote $p = \nabla_x u(x)$.

- Consider an elementary variation $\delta x$, giving rise to variations $\delta u$ and $\delta p$.

Since $F(x, u, \nabla_x u) = 0$ and $F(x + \delta x, u + \delta u, p + \delta p) = 0$:

$$\nabla_x F \cdot \delta x + \partial_u F \delta u + \nabla_p F \cdot \delta p = 0$$

Since $u = u(x)$ and $p = p(x)$:

$$\delta u = \nabla_x u \cdot \delta x = p \cdot \delta x$$

$$\delta p = (\delta x \cdot \nabla_x)p = (\delta x \cdot \nabla_x)(\nabla_x u) = \nabla_x (p \cdot \delta x)$$

We find

$$[\nabla_x F + \partial_u F p + (\nabla_p F \cdot \nabla_x)p] \cdot \delta x = 0$$

valid for any $\delta x \implies (\nabla_p F \cdot \nabla_x)p = -\nabla_x F - \partial_u F p$. 

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Complement: Solving the eikonal equation (2/3)

- Consider a path $x(s)$ with equation

$$\frac{dx}{ds} = \nabla_p F(x, u(x), p(x))$$

Along this path we have (with the notation $u(s) = u(x(s))$ and $p(s) = p(x(s))$):

$$\frac{du}{ds} = \left(\frac{dx}{ds} \cdot \nabla_x\right)u = \nabla_p F \cdot p$$

$$\frac{dp}{ds} = \left(\frac{dx}{ds} \cdot \nabla_x\right)p = (\nabla_p F \cdot \nabla_x)p = -\nabla_x F - \partial_u F p$$

We have seven ODES with seven unknowns $x(s)$, $u(s)$ and $p(s)$ $\implies$ we can calculate $u$ along the path $x(s)$.

- Application with $u(x) = T(x, y)$ (for a fixed $y$) and $F(x, u, p) = \frac{1}{2}(|p|^2 - c^{-2}(x))$.

$$\frac{dx}{ds} = p, \quad \frac{dp}{ds} = \frac{1}{2} \nabla_x (c^{-2})(x), \quad \frac{dT}{ds} = |p|^2 = c^{-2}(x)$$

Ray equations

Hamilton equations:

$$\frac{dx}{ds} = \nabla_p \mathcal{H}(x, p), \quad \frac{dp}{ds} = -\nabla_x \mathcal{H}(x, p) \quad \text{with} \quad \mathcal{H} = \frac{1}{2}(|p|^2 - c^{-2}(x))$$

Classical mechanics for a particle (with mass 1) in the potential $-\frac{1}{2}c^{-2}(x)$.
Complement: Solving the eikonal equation (3/3)

- Fermat’s principle: the path taken between two points by a ray of light is the path that can be traversed in the least time.

According to Fermat’s principle:

\[
T_f(x, y) = \inf \left\{ T \text{ s.t. } \exists (X_t)_{t \in [0, T]} \in C^1, X_0 = x, X_T = y, \left| \frac{dX_t}{dt} \right| = c(X_t) \right\}
\]

\[
= \inf \left\{ \int_0^S \frac{1}{c(X_s)} \left| \frac{dX_s}{ds} \right| ds, X_0 = x, X_S = y \right\}
\]

Action integral with the Lagrangian

\[
\mathcal{L} = |\dot{X}| c^{-1}(X)
\]

The extremum satisfies the Euler-Lagrange equations:

\[
\nabla_x \mathcal{L} - \frac{d}{ds} \nabla_{\dot{X}} \mathcal{L} = 0 \iff \text{ray equations}
\]

and then \( |\dot{X}| c^{-1}(X) = \frac{dT}{ds} \) so that \( T_f(x, y) = T(x, y) \).
Passive array imaging
Passive array imaging: data acquisition

The array sensors are used only as receivers.

The source $y$ emits a pulse.

The sensors $(x_r)_{r=1,...,N}$ record the waves.

The data set is $(P(t,x_r))_{t \in \mathbb{R}, r=1,...,N}$.

Goal: find the source position $y$ (more generally, find the source region).
Passive array imaging: simulation

Configuration

Data set \((P(t, x_r))_{80 \leq t \leq 200, r = 1, \ldots, 270}\)

Simulations carried out by C. Tsogka (University of Crete).
Passive array imaging: imaging functional

Data acquisition

Search region

Goal: find the source point $\mathbf{y}$ (more generally, find the source region).

The data set is $(P(t, x_r))_{t \in \mathbb{R}, r = 1, \ldots, N}$.

Given the data set, build an imaging functional in the search region $\Omega \subset \mathbb{R}^3$:

\[
\mathcal{I} : \Omega \to \mathbb{R}^+ \\
\mathbf{y}^S \mapsto \mathcal{I}(\mathbf{y}^S)
\]

which plots an image of the search region.
Passive array imaging - the linear forward operator

The source term is of the form \( n(t, y) = \rho_{\text{real}}(y)\delta(t) \).

Goal: find the source function \( \rho_{\text{real}} \).

Here: the Green’s function is known.

The data set is \( \hat{P}(\omega) = (\hat{P}(\omega, x_r))_{r=1,...,N} \) with \( \hat{P}(\omega, x_r) = \int \hat{G}(\omega, x_r, y)\rho_{\text{real}}(y)dy \).

We define \( \hat{A}(\omega) = \int \hat{G}(\omega, x_r, y)\rho(y)dy \).

\( \hat{A}(\omega) \) is the frequency-dependent, linear operator that maps the source function to the array data \( \hat{P}(\omega) \):

\[ \hat{P}(\omega) = \hat{A}(\omega)\rho_{\text{real}} \]
Passive array imaging - the adjoint operator

The least squares inverse problem is to minimize $J[\rho] + \alpha \|\rho\|_{\text{REG}}^2$ where

$$ J[\rho] = \int d\omega \sum_r |\hat{P}(\omega, x_r) - [\hat{A}(\omega)\rho](x_r)|^2 $$

Solution:

$$ \rho_{\text{LS}} = \left( \int \hat{A}^H(\omega)\hat{A}(\omega)d\omega \right)^{-1} \left( \int \hat{A}^H(\omega)\hat{P}(\omega)d\omega \right) $$

The adjoint operator is

$$ [\hat{A}^H(\omega)\hat{P}(\omega)](y) = \sum_r \hat{G}(\omega, y, x_r)\hat{P}(\omega, x_r) $$

Remember: complex conjugation = time reversal.

Adjoint operator = Backpropagation to the test point $y$. 
Complement : Least square inversion (1/2)

For a fixed frequency. The data set is the vector \( \hat{P} = (\hat{P}(x_r))_{r=1,...,N} \).

It is related to the unknown function \( \rho_{\text{real}} = (\rho_{\text{real}}(y))_{y\in\Omega} \) through the linear relation \( \hat{P} = \hat{A}\rho_{\text{real}} \), where

\[
[\hat{A}\rho](x_r) = \int_\Omega \hat{G}(\omega, x_r, y)\rho(y)dy
\]

Discretize by introducing a regular grid \( (y_j)_{j=1,...,M} \) of the search domain \( \Omega \):

\[
[\hat{A}\rho](x_r) = \sum_{j=1}^M \hat{G}(\omega, x_r, y_j)\rho(y_j)(\delta y)^3
\]

The problem is reduced to: find the vector \( \rho = (\rho(y_j))_{j=1,...,M} \) solution of \( \hat{P} = A\rho \), with the matrix \( A_{r,j} = \hat{G}(\omega, x_r, y_j)(\delta y)^3 \)

\( A \) of size \( N \times M \), not invertible in general.

Least square inversion: Find the vector \( \rho \) minimizing the error (misfit function):

\[
\mathcal{E} = \frac{1}{2} \| \hat{P} - A\rho \|^2 = \frac{1}{2} \sum_{r=1}^N |(\hat{P} - A\rho)_r|^2
\]
Complement: Least square inversion (2/2)

Minimization of the quadratic misfit function:

\[ \mathcal{E} = \frac{1}{2} \| \hat{P} - A\rho \|^2 \]

An extremal point satisfies the constraints:

\[ 0 = \frac{\partial \mathcal{E}}{\partial \rho_j}, \quad 0 = \frac{\partial \mathcal{E}}{\partial \rho_j} \]

\[ 0 = -\sum_{r=1}^{N} [A_{rj}(\hat{P} - A\rho)_r] = -[A^H(\hat{P} - A\rho)]_j, \quad j = 1, \ldots, N \]

or

\[ A^H(\hat{P} - A\rho) = 0 \iff A^H A\rho = A^H \hat{P} \iff \rho = (A^H A)^{-1} A^H \hat{P} \]

\( A^H A \) is a nonnegative matrix, may be well or ill-conditioned.

Tykhonov regularization: minimize \((\alpha > 0)\)

\[ \mathcal{E} = \frac{1}{2} \| \hat{P} - A\rho \|^2 + \frac{1}{2} \alpha \| \rho \|^2 \]

\[ \rho = (A^H A + \alpha I)^{-1} A^H \hat{P} \]
Passive array imaging - the reverse-time imaging functional

- Least-squares imaging functional:

\[ \mathcal{I}_{LS}(y^S) = \left[ \left( \int \hat{A}^H(\omega)\hat{A}(\omega)d\omega \right)^{-1} \left( \int \hat{A}^H(\omega)\hat{P}(\omega)d\omega \right) \right](y^S) \]

with

\[ \left[ \hat{A}^H(\omega)\hat{P}(\omega) \right](y) = \sum_r \hat{G}(\omega, y, x_r)\hat{P}(\omega, x_r) \]

- The operator \( \int \hat{A}^H(\omega)\hat{A}(\omega)d\omega \) is proportional to the identity operator \( \rightarrow \) drop the normalizing factor in the LS functional.

\( \leftrightarrow \) Reverse Time imaging functional for the search point \( y^S \):

\[ \mathcal{I}_{RT}(y^S) = \int d\omega \left[ \hat{A}^H(\omega)\hat{P}(\omega) \right](y^S) = \int d\omega \sum_r \hat{G}(\omega, y^S, x_r)\hat{P}(\omega, x_r) \]
Kirchhoff Migration (or travel-time migration) for passive array imaging

- Reverse Time imaging functional for the search point \( y^S \):

  \[
  I_{RT}(y^S) = \int d\omega \sum_r \hat{G}(\omega, y^S, x_r) \hat{P}(\omega, x_r)
  \]

- If we take \( \hat{G}(\omega, x, y) \simeq \exp[i\omega T(x, y)] \), where \( T(x, y) \) is the travel time from \( x \) to \( y \), then we get the Kirchhoff Migration imaging functional:

  \[
  I_{KM}(y^S) = \int d\omega \sum_r \exp[-i\omega T(x_r, y^S)] \hat{P}(\omega, x_r)
  \]

  \[
  = 2\pi \sum_r P(T(x_r, y^S), x_r)
  \]

Kirchhoff Migration (or travel-time migration) has been analyzed in detail (Beylkin, Burridge, Symes, Bleistein, ...) and is used extensively in practice.

Kirchhoff migration for passive array imaging: simulation

Data \( \left( P(t, x_r) \right) \) for \( t \in [40, 150], r = 1, \ldots, 264 \)

KM imaging functional \( I_{KM}(y^S) \)

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Active array imaging
Active array imaging: data acquisition

Active array $\iff$ The array sensors can be used as sources and/or as receivers.

For each $s = 1, \ldots, N$:
- The source $x_s$ emits a pulse.
- The sensors $(x_r)_{r=1,\ldots,N}$ record the waves.

The data set is $(P(t, x_r, x_s))_{t \in \mathbb{R}, r,s=1,\ldots,N}$ (the time-dependent response matrix).

Goal: find the reflector position $y$ (more generally, find the reflectivity function of the medium).
Active array imaging: simulation

Configuration

Data set \((P(t, x_r, x_{135}))_{80 \leq t \leq 200, r=1, \ldots, 270}\) (traces recorded for a central illumination)
Active array imaging: imaging functional

Data acquisition

Search region

Goal: find the reflector position $y$ (more generally, find the reflectivity function of the medium).

The data set is $(P(t, x_r, x_s))_{t \in \mathbb{R}, r, s = 1, \ldots, N}$.

Given the data set, build an imaging functional in the search region $\Omega \subset \mathbb{R}^3$:

$\mathcal{I} : \Omega \rightarrow \mathbb{R}^+$

$y^S \rightarrow \mathcal{I}(y^S)$ which plots an image of the search region.
Passive and active array imaging: comparison

Passive array of sensors
The sensors $(\mathbf{x}_r)_{r=1,...,N}$ record
$\mathbf{y}$ is a source
Data: $\left( P(t, \mathbf{x}_r) \right)_{t \in \mathbb{R}, r=1,...,N}$

Active array of sensors/sources
The sensors $(\mathbf{x}_r)_{r=1,...,N}$ record
$\mathbf{y}$ is a reflector, $\mathbf{x}_s$ is a source
Data: $\left( P(t, \mathbf{x}_r, \mathbf{x}_s) \right)_{t \in \mathbb{R}, r,s=1,...,N}$

Goal: process the data to find $\mathbf{y}$. 
Active array imaging - modeling

Goal: find the propagation speed $c_{\text{real}}(y)$.

The time Fourier transform of the data $\hat{P}(\omega, x_r, x_s)$ is

$$\hat{P}(\omega, x_r, x_s) = \hat{G}(\omega, x_r, x_s, c_{\text{real}}) \hat{f}(\omega)$$

where $\hat{f}(\omega)$ is the Fourier transform of the source pulse $\hat{f}(\omega) = \int f(t) e^{i\omega t} \, dt$.

$\hat{G}(\omega, x, y; c)$ is the Green’s function that solves the Helmholtz equation

$$\Delta \hat{G} + \frac{\omega^2}{c^2(x)} \hat{G} = -\delta(x - y),$$

with the Sommerfeld radiation condition. It depends on the velocity $c(x)$. 

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Active array imaging - nonlinear inversion

Data:
\[
\hat{P}(\omega) = (\hat{P}(\omega, x_r, x_s))_{r,s=1,\ldots,N} = (\hat{G}(\omega, x_r, x_s, c_{\text{real}})\hat{f}(\omega))_{r,s=1,\ldots,N}
\]

Goal: find the propagation speed \( c_{\text{real}}(x) \).

The inverse problem, the array least squares problem, is:

Minimize
\[
J[c] + \alpha \| c \|_{\text{REG}},
\]

where
\[
J[c] = \int d\omega \sum_{r,s} |\hat{P}(\omega, x_r, x_s) - \hat{G}(\omega, x_r, x_s; c)\hat{f}(\omega)|^2
\]

and \( \alpha \) is a strength of regularization parameter.

→ this is a nonlinear problem for the unknown propagation speed \( c(x) \).
Active array imaging - linearization

\[
\frac{1}{c^2(x)} = \frac{1}{c_0^2} \left( n_0^2(x) + \rho(x) \right)
\]

where
- \(c_0\) is a reference speed,
- \(n_0(x)\) is a smooth background index of refraction (known, typically constant),
- \(\rho(x)\) is the target reflectivity (unknown but small).

The Green's function satisfies:
\[
\Delta \hat{G} + \frac{\omega^2}{c_0^2} \left( n_0^2(x) + \rho(x) \right) \hat{G} = -\delta(x - y)
\]

The background Green's function is defined by:
\[
\Delta \hat{G}_0 + \frac{\omega^2}{c_0^2} n_0^2(x) \hat{G}_0 = -\delta(x - y)
\]

Born approximation:
\[
\hat{G}(\omega, x, y) = \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, x, z) \rho(z) \hat{G}_0(\omega, z, y) dz
\]

First term: direct waves.
Second term: single-scattered waves \(y \rightarrow z \rightarrow x\).
Consider
\[ \Delta_z \hat{G}(\omega, z, x) + \frac{\omega^2}{c_0^2} n_0^2(z) \hat{G}(\omega, z, x) = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, z, x) - \delta(z - x) \]
\[ \Delta_z \hat{G}_0(\omega, z, y) + \frac{\omega^2}{c_0^2} n_0^2(z) \hat{G}_0(\omega, z, y) = -\delta(z - y) \]

We multiply the first equation by \( \hat{G}_0(\omega, x, y) \) and subtract the second equation multiplied by \( \hat{G}(\omega, x, z) \):

\[ \nabla_z \cdot \left[ \hat{G}_0(\omega, z, y) \nabla_z \hat{G}(\omega, z, x) - \hat{G}(\omega, z, y) \nabla_z \hat{G}_0(\omega, z, x) \right] \]
\[ = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, z, x) \hat{G}_0(\omega, z, y) - \hat{G}_0(\omega, z, y) \delta(z - x) + \hat{G}(\omega, z, x) \delta(z - y) \]
\[ = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, z, x) \hat{G}_0(\omega, z, y) - \hat{G}_0(\omega, x, y) \delta(z - x) + \hat{G}(\omega, y, x) \delta(z - y) \]

\[ \text{reciprocity} \]
\[ = -\frac{\omega^2}{c_0^2} \rho(z) \hat{G}(\omega, x, z) \hat{G}_0(\omega, z, y) - \hat{G}_0(\omega, x, y) \delta(z - x) + \hat{G}(\omega, x, y) \delta(z - y) \]

We integrate over \( B(0, L) \) (with \( L \) large enough so that it encloses the support of \( \rho \)):

\[ 0 = -\frac{\omega^2}{c_0^2} \int \hat{G}(\omega, x, z) \rho(z) \hat{G}_0(\omega, z, y) dz - \hat{G}_0(\omega, x, y) + \hat{G}(\omega, x, y) \]
Complement: The Born approximation (2/2)

- Lippmann-Schwinger equation (exact):
  \[
  \hat{G}(\omega, x, y) = \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} \int \hat{G}(\omega, x, z) \rho(z) \hat{G}_0(\omega, z, y) dz
  \]

- Born approximation (approximate):
  \[
  \hat{G}(\omega, x, y) \simeq \hat{G}_0(\omega, x, y) + \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, x, z) \rho(z) \hat{G}_0(\omega, z, y) dz
  \]
Active array imaging - the linearized forward operator

The data set is modeled by:

\[ \hat{P}(\omega, x_r, x_s) = \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, x_r, z)\rho(z)\hat{G}_0(\omega, y, x_s)dz \]

(removing \( \hat{G}_0(\omega, x_r, x_s) \)).

We define

\[ [\hat{A}(\omega)\rho](x_r, x_s) = \int \hat{G}_0(\omega, x_r, z)\rho(z)\hat{G}_0(\omega, z, x_s)dz \]

It is the frequency dependent, linear operator that maps the reflectivity function to the array data.
Active array imaging - inversion

The least squares linearized inverse problem is to minimize $J_{LS}[\rho]$ where

$$J_{LS}[\rho] = \int d\omega \sum_{r,s} |\hat{P}(\omega, x_r, x_s) - [\hat{A}(\omega)\rho](x_r, x_s)|^2$$

Solution:

$$\rho_{LS} = \left( \int (\hat{A}^H(\omega)\hat{A}(\omega) d\omega \right)^{-1} \left( \int \hat{A}^H(\omega)\hat{P}(\omega) d\omega \right)$$

The adjoint operator is

$$[\hat{A}^H(\omega)\hat{P}(\omega)](y) = \sum_{r,s} \hat{G}(\omega, y, x_r)\hat{G}(\omega, x_s, y)\hat{P}(\omega, x_r, x_s)$$

Remember: complex conjugation = time reversal.
Adjoint operator = Backpropagation both from $x_r$ and from $x_s$ to the test point $y$. 

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Kirchhoff migration (or travel-time migration) for active array imaging

- Reverse-time migration (using the full Green’s function for migration).

Imaging functional for the search point $y^S$:

$$I_{RT}(y^S) = \int d\omega \sum_{r,s} \hat{G}(\omega, y^S, x_r)\hat{G}(\omega, x_s, y^S)\hat{P}(\omega, x_r, x_s)$$

- Kirchhoff migration (or travel time migration).

If we take $\hat{G}(\omega, x, y) \simeq \exp[i\omega \mathcal{T}(x, y)]$, where $\mathcal{T}(x, y)$ is the travel time from $x$ to $y$, then we get the KM imaging functional:

$$I_{KM}(y^S) = \int d\omega \sum_{r,s} \exp[-i\omega(\mathcal{T}(x_r, y^S) + \mathcal{T}(x_s, y^S))]\hat{P}(\omega, x_r, x_s)$$

$$= 2\pi \sum_{r,s} P(\mathcal{T}(x_r, y^S) + \mathcal{T}(x_s, y^S), x_r, x_s)$$
Kirchhoff migration for active array imaging: simulation

Data \((P(t, x_r, x_{128})) \mid t \in [80, 200], r = 1, \ldots, 264\)

KM imaging functional \(I_{\text{KM}}(y^S)\)
Array imaging: basic resolution analysis
Active array imaging with a finite-aperture mirror

Square array $[-D/2, D/2] \times [-D/2, D/2] \times \{0\}$. Point source $y = (0, 0, L)$.

RT imaging functional:

$$\mathcal{I}_{RT}(y^S) = \int \hat{\mathcal{I}}_{RT}(\omega, y^S) d\omega$$

$$\hat{\mathcal{I}}_{RT}(\omega, y^S) = \sum_{r=1}^{N} \hat{G}(\omega, y^S, x_r) \hat{G}(\omega, x_r, y) f(\omega)$$

- Resolution analysis when the medium is homogeneous

$$\hat{G}(\omega, x, y) = \frac{1}{4\pi|x-y|} e^{i \frac{\omega |x-y|}{c_0}}$$

G2S3  
July, 2011
Consider the array \( x_r = (0, rD/(2N)), r = -N, \ldots, N \), (the diameter of the array is \( D \)), the target \( y = (L, 0) \), and the search point \( y^S = (L + \eta, \xi) \).

Analysis of

\[
E_\omega(\eta, \xi) = \sum_{r=-N}^{N} \exp \left[ ik \left( |x_r - y| - |x_r - y^S| \right) \right], \quad k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}
\]

Assume \( \lambda \ll D \ll L \) (paraxial regime).
Assume \( D/N < \lambda \) (continuous approximation \( x_r = (x, 0), x \in [-D/2, D/2] \)).
Resolution analysis - time-harmonic - finite aperture array (2/3)

\[ E_\omega(\eta, \xi) = \sum_{r=-N}^{N} \exp \left[ i k (|x_r - y| - |x_r - y^S|) \right], \quad k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda} \]

We have (assume \(D^4 \ll L^3 \lambda, \xi^4 \ll L^3 \lambda\))

\[ |x_r - y| = (L^2 + x^2)^{1/2} = L \left(1 + \frac{x^2}{L^2}\right)^{1/2} \simeq L + \frac{x^2}{2L} \]

\[ |x_r - y^S| = ((L + \eta)^2 + (x - \xi)^2)^{1/2} \simeq L + \eta + \frac{(x - \xi)^2}{2(L + \eta)} \]

\[ |x_r - y| - |x_r - y^S| \simeq -\eta - \frac{\xi^2}{2(L + \eta)} + \frac{x\xi}{L + \eta} + \frac{\eta x^2}{2L(L + \eta)} \]

\[ E_\omega(\eta, \xi) = \frac{2N}{D} e^{-ik(\eta + \frac{\xi^2}{2(L + \eta)})} \int_{-D/2}^{D/2} e^{-ik\left(\frac{x\xi}{L + \eta} + \frac{\eta x^2}{2L(L + \eta)}\right)} \, dx \]

\[ |E_\omega(\eta, \xi)| = 2N \left| \int_{-1/2}^{1/2} e^{-i\frac{\pi}{\lambda}\left(\frac{2Ds}{L + \eta} + \frac{D^2\eta}{L(L + \eta)s^2}\right)} \, ds \right| \]

If \(\eta \ll L\):

\[ |E_\omega(\eta, \xi)| = 2N \left| \int_{-1/2}^{1/2} e^{-i\frac{\pi}{\lambda}\left(\frac{2Ds}{L + \eta} + \frac{D^2\eta}{L^2s^2}\right)} \, ds \right| \]
Resolution analysis - time-harmonic - finite aperture array (3/3)

Analysis of the point spread function ($\lambda \ll D \ll L$, $\lambda^2 L^2 \ll D^4 \ll \lambda L^3$):

$$|E_\omega(\eta, \xi)|^2 = 4N^2\Psi\left(\frac{D\xi}{L\lambda}, \frac{D^2\eta}{2\lambda L^2}\right)^2, \quad \Psi(u, v) = \left|\frac{1}{2} \int_{-1}^{1} e^{-i\pi(u s + v s^2)} ds\right|$$

- Transverse radius: $\frac{\lambda L}{D}$ (Rayleigh resolution formula).
- Longitudinal radius: $\frac{\lambda L^2}{D^2}$ (larger than the transverse radius).

- Transverse shape:
  $$\Psi(u, 0) = \text{sinc}(\pi u) := \frac{\sin \pi u}{\pi u}$$

Longitudinal shape:

$$\Psi(0, v) = \left|\int_{0}^{1} e^{-i\frac{\pi}{2} v s^2} ds\right| \quad \text{(Fresnel integral)}$$

$\bullet$ Transverse radius: $\frac{\lambda L}{D}$ (Rayleigh resolution formula).

$\bullet$ Longitudinal radius: $\frac{\lambda L^2}{D^2}$ (larger than the transverse radius).

$\bullet$ Transverse shape:

$$\Psi(u, 0) = \text{sinc}(\pi u) := \frac{\sin \pi u}{\pi u}$$

Longitudinal shape:

$$\Psi(0, v) = \left|\int_{0}^{1} e^{-i\frac{\pi}{2} v s^2} ds\right| \quad \text{(Fresnel integral)}$$

$\times \frac{\theta}{2}$

$z$ in multiples of $\frac{\lambda L}{D}$

$z$ in multiples of $\frac{\lambda L^2}{D^2}$
Resolution analysis - time-dependent - finite aperture array

Assume spectrum supported in $[\omega_0 - B, \omega_0 + B]$, with $B \ll \omega_0$.

Note:

$$\int_{\omega_0 - B}^{\omega_0 + B} e^{-i\frac{\omega}{c_0} \eta} d\omega = 2Be^{-i\frac{\omega_0}{c_0} \eta} \text{sinc}\left(\frac{B}{c_0} \eta\right)$$

Analysis of the point spread function:

$$E(\eta, \xi) = \left| \int_{\omega_0 - B}^{\omega_0 + B} E_\omega(\eta, \xi) d\omega \right|^2 = 4B^2 N^2 \text{sinc}^2\left(\frac{B\eta}{c_0}\right) \Psi\left(\frac{D\xi}{L\lambda_0}, \frac{D^2\eta}{2\lambda_0 L^2}\right)^2,$$

$$\Psi(u, v) = \left| \frac{1}{2} \int_{-1}^{1} e^{-i\pi (us + v \frac{u^2}{2})} ds \right|$$

- Transverse radius: $\frac{\lambda L}{D}$ (Rayleigh resolution formula).

Longitudinal radius: $\min\left(\frac{\lambda_0 L^2}{D^2}, \frac{c_0}{B}\right)$ (longitudinal resolution proportional to the bandwidth).
Resolution analysis - full aperture array (1/2)

What is the resolution of the RT migration in a homogeneous medium when the array \(\partial D (=\partial B(0, L) \text{ for instance})\) encloses the source point?

\[
\hat{I}_{RT}(\omega, y^S) = \sum_{r=-N}^{N} \hat{G}(\omega, y^S, x_r) \hat{P}(\omega, x_r) \sim \int_{\partial D} \hat{G}(\omega, y^S, x) \hat{P}(\omega, x) dS(x)
\]

We will show that the resolution is \(\lambda/2\), which is a well-known result (diffraction limit).
Resolution analysis - full aperture array (2/2)

\[ \hat{I}_{RT}(\omega, y^S) = \int_{\partial D} \hat{G}(\omega, y^S, x)\overline{\hat{G}(\omega, y, x)}dS(x) \]

Using Helmholtz-Kirchhoff identity:

\[ \hat{I}_{RT}(\omega, y^S) = \frac{c_0}{\omega} \text{Im}(\hat{G}(\omega, y^S, y)) \]

which gives with \( \hat{G}(\omega, x, y) = \frac{1}{4\pi|x-y|} e^{\frac{i}{c_0} |x-y|} \):

\[ \hat{I}_{RT}(\omega, y^S) \sim \text{sinc}(k|y - y^S|), \quad \text{sinc}(x) = \frac{\sin x}{x}, \quad k = \frac{2\pi}{\lambda} = \frac{\omega}{c_0} \]

We get a resolution estimate from the first zero (\( \pi \)) of the sinc function. We have 
\[ k|y - y^S| = \pi \text{ or } |y - y^S| = \lambda/2. \]

\[ x, z \text{ in multiples of } \lambda \]
Array imaging with additive noise
Passive array imaging

- In the presence of a point source at $z_{so}$ the response vector is

$$\left( \hat{P}_0(\omega) \right)_r = \hat{G}(\omega, x_r, z_{so}) \sigma_{so}(\omega), \quad r = 1, \ldots, N$$

or equivalently:

$$\hat{P}_0(\omega) = \tau_{so}(\omega) g(\omega, z_{so}), \quad \text{with} \quad \tau_{so}(\omega) = \sigma_{so}(\omega) \left( \sum_{l=1}^{N} |\hat{G}(\omega, z_{so}, x_l)|^2 \right)^{1/2}$$

- $\tau_{so}(\omega)$ is proportional to the emission power,
- $g(\omega, x)$ is the normalized vector of Green’s functions from the array to the point $x$:

$$g(\omega, x) = \frac{1}{\left( \sum_{l=1}^{N} |\hat{G}(\omega, x, x_l)|^2 \right)^{1/2}} \left( \hat{G}(\omega, x, x_j) \right)_{j=1,\ldots,N}$$

- With an additive Gaussian white noise (with variance $a^2$), the data set is

$$P(\omega) = P_0(\omega) + W(\omega)$$

where $(W_r(\omega))_{r=1,\ldots,N}$ are complex-valued, independent and identically distributed Gaussian $\mathcal{N}(0, a^2)$ ($W_r = W_{r1} + iW_{r2}$ with $W_{r1}$ and $W_{r2}$ real-valued, independent and identically distributed Gaussian $\mathcal{N}(0, a^2/2)$).
- We want to estimate $z_{so}$ from the measured vector $P(\omega)$:

$$P(\omega) = P_0(\omega) + W(\omega)$$

- RT imaging functional:

$$\mathcal{I}_{RT}(z^S) = g(\omega, z^S)^T P(\omega)$$

where $g$ is the normalized vector of Green’s functions:

$$g(\omega, x) = \frac{1}{\left( \sum_{l=1}^{N} |\hat{G}(\omega, x, x_l)|^2 \right)^{1/2}} \left( \hat{G}(\omega, x, x_j) \right)_{j=1,\ldots,N}$$

- KM imaging functional:

$$\mathcal{I}_{KM}(z^S) = d(\omega, z^S)^T P(\omega)$$

where

$$d(\omega, z) = \frac{1}{\sqrt{N}} \left( \exp \left( i\omega \mathcal{T}(x_j, z) \right) \right)_{j=1,\ldots,N}$$
Complement: Bayes’ theorem

Bayes’ theorem shows how to determine inverse probabilities: knowing the conditional probability of $B$ given $A$, what is the conditional probability of $A$ given $B$?

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

For probability densities: Assume $(X, Y)$ is a random vector with pdf $p_{X,Y}(x, y)$:

$$p_X(x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{p_Y(y|X = x)p_X(x)}{p_Y(y)}$$

Here:
- $p_{X,Y}(x, y)$ is the joint pdf (probability density function) of $(X, Y)$.
- $p_X(x|Y = y)$ is the posterior pdf of $X$ given $Y = y$.
- $p_X(x)$ and $p_Y(y)$ are the marginal pdfs of $X$ and $Y$ respectively ($p_X(x)$ is the prior pdf of $X$).
Bayesian analysis.

Given $z$, $\tau$, and $a$, the data $P$ has the probability density function

$$p(P \mid z, \tau, a) = \frac{1}{(\pi a^2)^N} \exp \left( - \frac{\|P - \tau g(z)\|^2}{a^2} \right)$$

(with respect to the Lebesgue measure over the space of complex vectors).

Maximum likelihood estimator: We look for $\hat{z}$ that maximizes the pdf $z \rightarrow p(z \mid P)$ given the data $P$.

Using Bayes theorem with the Jeffreys prior for the parameters $a, \tau$ (a non-informative prior distribution proportional to $a^{-1}$) and the uniform distribution in the search domain for $z$, we find that, given the observations $P$, the likelihood function of the parameters $z$, $\tau$, and $a$ is proportional to

$$l_0 (z, \tau, a \mid P) = \frac{1}{a^{2N+1}} \exp \left( - \frac{\|P - \tau g(z)\|^2}{a^2} \right)$$
Complement: Jeffreys prior

In Bayesian analysis, the Jeffreys prior $\pi(\theta)$ is a non-informative prior distribution on parameter space $\Theta$ that is proportional to:

$$\pi(\theta) \propto \sqrt{\det I(\theta)}$$

where $I(\theta)$ is the Fisher information matrix:

$$I_{ji}(\theta) = \int_{\Theta} \partial_{\theta_j} \left[ \ln p(\theta) \right] \partial_{\theta_j} \left[ \ln p(\theta) \right] p(\theta) d\theta$$

It is invariant under reparameterization of the parameter vector $\theta$.

Examples:
the Jeffreys prior for the mean of a Gaussian variable is uniform over the entire real line.
the Jeffreys prior for the variance $a^2$ of a Gaussian variable is $1/a$. 
The maximum likelihood estimate of \( z \) and the nuisance parameters \( a \) and \( \tau \) are found by maximizing the likelihood function with respect to these:

\[
(\hat{z}, \hat{\tau}, \hat{a}) = \arg\max_{z, \tau, a} l_0 (z, \tau, a \mid P)
\]

We first eliminate \( a \) by requiring

\[
\frac{\partial l_0 (z, \tau, a \mid P)}{\partial a} = 0
\]

which gives

\[
\hat{a} = \frac{\| P - \tau g(z) \|}{\sqrt{2N + 1}}
\]

and then the likelihood ratio is proportional to

\[
l_0 (z, \tau, \hat{a} \mid P) \sim \| P - \tau g(z) \|^{-N-1/2}
\]
Since $\|g(z)\| = 1$, we have

$$\hat{\tau} = \arg\min_{\tau} \|P - \tau g(z)\|^2 = \langle g(z), P \rangle$$

where $\langle a, b \rangle = a^T b$.

Therefore the estimate $\hat{z}$ derives from maximizing the function

$$\hat{z} = \arg\min_z \|P - \langle g(z), P \rangle g(z)\|^2$$

We find

$$\|P - \langle g(z), P \rangle g(z)\|^2 = \|P\|^2 - |\langle g(z), P \rangle|^2$$

$$= \|P\|^2 - |I_{RT}(z)|^2$$

The maximum likelihood estimation of the reflector location is

$$\hat{z} = \arg\max_{z} |I_{RT}(z)|$$
Active array imaging

• In the presence of a point reflector at \( z_{\text{ref}} \) and in the Born approximation, the response matrix is

\[
(P_0(\omega))_{rs} = \hat{G}(\omega, x_r, z_{\text{ref}}) \frac{\omega^2}{c_0^2} \sigma_{\text{ref}} l_{\text{ref}}^3 \hat{G}(\omega, z_{\text{ref}}, x_s), \quad r, s = 1, \ldots, N
\]

or equivalently:

\[
P_0(\omega) = \tau_{\text{ref}} g(\omega, z_{\text{ref}}) g(\omega, z_{\text{ref}})^T, \quad \text{with} \quad \tau_{\text{ref}} = \frac{\omega^2}{c_0^2} \sigma_{\text{ref}} l_{\text{ref}}^3 \left( \sum_{l=1}^{N} |\hat{G}(\omega, z_{\text{ref}}, x_l)|^2 \right)
\]

- \( \tau_{\text{ref}} \) is the scattering amplitude of the reflector,
- \( l_{\text{ref}}^3 \) is the volume of the reflector,
- \( g(\omega, x) \) is the normalized vector of Green’s functions from the array to the point \( x \):

\[
g(\omega, x) = \frac{1}{\left( \sum_{l=1}^{N} |\hat{G}(\omega, x, x_l)|^2 \right)^{1/2}} \left( \hat{G}(\omega, x, x_j) \right)_{j=1,\ldots,N}
\]

The matrix \( P_0 \) is symmetric and it has rank 1.

• With an additive Gaussian white noise (with variance \( a^2 \)), the data set is

\[
P(\omega) = P_0(\omega) + W(\omega)
\]
We want to estimate $z_{\text{ref}}$ from the symmetrized matrix $P_s(\omega) = \frac{1}{2}(P(\omega) + P(\omega)^T)$.

- RT imaging functional:
  \[
  I_{\text{RT}}(z^S) = g(\omega, z^S)^T P_s(\omega) g(\omega, z^S)
  \]
  where $g$ is the normalized vector of Green’s functions.

- KM imaging functional:
  \[
  I_{\text{KM}}(z^S) = d(\omega, z^S)^T P_s(\omega) d(\omega, z^S)
  \]
  where
  \[
  d(\omega, z) = \frac{1}{\sqrt{N}} \left( \exp \left( i \omega T(x_j, z) \right) \right)_{j=1,\ldots,N}
  \]

- MUSIC imaging functional:
  \[
  I_{\text{MU}}(z^S) = \frac{1}{|g(\omega, z^S) - \langle v_1(\omega), g(\omega, z^S) \rangle v_1(\omega)|^2}
  \]
  where $v_1(\omega)$ is the first vector of the symmetric SVD of $P_s(\omega) = V(\omega) \Sigma(\omega) V(\omega)^H$. 

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• MUSIC imaging functional:

\[ I_{MU}(z^S) = \frac{1}{|g(\omega, z^S) - \langle v_1(\omega), g(\omega, z^S) \rangle v_1(\omega)|^2} \]

where \( v_1(\omega) \) is the first vector of the symmetric SVD of \( P^s(\omega) = V(\omega)\Sigma(\omega)V(\omega)^H \).

Here:

• the unperturbed matrix \( P_0(\omega) \) has rank one. Its image space is spanned by its first singular vector \( g(\omega, z_{ref}) \).

• \( \langle v_1(\omega), g(\omega, z^S) \rangle v_1(\omega) \) is the projection of \( g(\omega, z^S) \) on the “image space” of \( P^s(\omega) \).

• \( g(\omega, z^S) - \langle v_1(\omega), g(\omega, z^S) \rangle v_1(\omega) \) is the projection of \( g(\omega, z^S) \) on the “noise space” of \( P^s(\omega) \).

If there is no noise, then \( v_1(\omega) = g(\omega, z_{ref}) \) up to a phase term, and the projection on the noise space is zero if \( z^S = z_{ref} \).

Note: in the absence of noise:

\[ I_{MU}(z^S)^{-1} = 1 - |\langle g(\omega, z_{ref}), g(\omega, z^S) \rangle|^2 = 1 - \frac{1}{\tau_{ref}} g(\omega, z^S)^T P_0 g(\omega, z^S) \]

\[ = 1 - \frac{1}{\tau_{ref}} I_{RT}(z^S) \]
• Bayesian analysis.

Given \( z, \tau, \) and \( a, \) the symmetrized response matrix \( P^s \) has the probability density function

\[
p(P^s \mid z, \tau, a) = \frac{1}{2^N \pi^{\frac{N^2 + N}{2}} a^{N^2 + N}} \exp \left( -\frac{\|P^s - \tau g(z)g(z)^T\|^2}{2a^2} \right)
\]

(with respect to the Lebesgue measure over the space of complex symmetric matrices)

Using Bayes theorem with the Jeffreys prior for the parameters \( \tau, a \) (non-informative prior distributions) and the uniform distribution in the search domain for \( z, \) we find that, given the observations \( P^s, \) the likelihood function of the parameters \( z, \tau, \) and \( a \) is proportional to

\[
l_0 (z, \tau, a \mid P^s) = \frac{1}{a^{N^2 + N + 1}} \exp \left( -\frac{\|P^s - \tau g(z)g(z)^T\|^2}{2a^2} \right)
\]
The maximum likelihood estimate of \( z \) and the nuisance parameters \( a \) and \( \tau \) are found by maximizing the likelihood function with respect to these:

\[
(\hat{z}, \hat{\tau}, \hat{a}) = \operatorname{argmax}_{z, \tau, a} l_0 (z, \tau, a \mid P^s)
\]

We first eliminate \( a \) by requiring

\[
\frac{\partial l_0 (z, \tau, a \mid P^s)}{\partial a} = 0
\]

which gives

\[
\hat{a} = \frac{\|P^s - \tau g(z) g(z)^T\|}{\sqrt{N^2 + N + 1}},
\]

and then the likelihood ratio is proportional to

\[
l_0 (z, \tau, \hat{a} \mid P^s) \sim \|P^s - \tau g(z) g(z)^T\|^{-(N^2 + N + 1)/2}
\]

Since \( P^s \) is complex symmetric it admits a symmetric SVD: there exist unitary vectors \( u_j \) and nonnegative numbers \( \sigma_j \) (the singular values) such that

\[
P^s = \sum_{j=1}^{N} \sigma_j u_j u_j^T
\]
We can write
\[ \| P^s - \tau g(z)g(z)^T \|^2 = \| u^{(2)} - \tau g^{(2)}(z) \|^2 \]
with \( u^{(2)} = \sum_{j=1}^N \sigma_j u_j \otimes u_j \) and \( g^{(2)}(z) = g(z) \otimes g(z) \). Since \( \| g(z) \| = 1 \), we have \( \| g^{(2)}(z) \|_2 = 1 \) and we then find that
\[
\hat{\tau} = \text{argmin}_{\tau} \| u^{(2)} - \tau g^{(2)}(z) \|^2 = \langle g^{(2)}(z), u^{(2)} \rangle
\]
Therefore the estimate \( \hat{z} \) derives from maximizing the MUSIC-type function
\[
\hat{z} = \text{argmin}_{z} \| u^{(2)} - \langle g^{(2)}(z), u^{(2)} \rangle g^{(2)}(z) \|^2_2
\]
Note however that \( \hat{z} \) is not the maximizer of the MUSIC functional since all singular vectors (weighted by the singular values) contribute to \( u^{(2)} \). We have in fact
\[
\| u^{(2)} - \langle g^{(2)}(z), u^{(2)} \rangle g^{(2)}(z) \|^2_2 = \| u^{(2)} \|^2_2 - | \langle u^{(2)}, g^{(2)}(z) \rangle |^2 = \| P^s \|^2 - | I_{RT}(z) |^2
\]
The maximum likelihood estimation of the reflector location is
\[
\hat{z} = \text{argmax}_{z} | I_{RT}(z) |
\]
• Conclusion: In the presence of additive noise (measurement noise): Reverse-Time imaging (or Kirchhoff migration) is optimal.

• Warning: Bayesian analysis is powerful but depends on the prior. Here the prior is: there exists a reflector.
Data acquisition for array imaging

- We consider that there are $M$ sources and $N$ receivers.
We want to measure the $N \times M$ response matrix $P_0(\omega)$.
We can do $M$ experiments.
Each source can emit a unit-amplitude time-harmonic signal.
The measures are noisy: the signal measured by a receiver is corrupted by an additive noise (a complex Gaussian random variable with mean zero and variance $a^2$).

- Standard Acquisition
The response matrix is measured during a sequence of $M$ experiments:
- In the $m$th experience, $m = 1, \ldots, M$, the $m$th source generates a time-harmonic signal with unit amplitude.
- The $N$ receivers record the backscattered waves:

$$P_{nm} = P_{0,nm} + W_{nm}, \quad n = 1, \ldots, N$$

$\Rightarrow$ we obtain

$$P = P_0 + W,$$

where $P_0$ is the unperturbed response matrix and $W_{nm}$ are independent complex Gaussian random variables with mean zero and variance $a^2$. 
Optimal Acquisition: Hadamard Technique

Noise reduction technique in the presence of additive noise.

Definition: A real Hadamard matrix \( H \) of order \( M \) is a \( M \times M \) matrix whose elements are \(-1\) or \(+1\) and such that \( H^T H = M I \).

Real Hadamard matrices do not exist for all \( M \). A necessary condition for the existence is that \( M = 1, 2 \) or a multiple of 4. A sufficient condition is that \( M \) is a power of two. Explicit examples are known for all \( M \) multiple of 4 up to \( M = 664 \).

Hadamard conjecture: a Hadamard matrix of order \( 4k \) exists for every integer \( k \).

Definition: A complex Hadamard matrix \( H \) of order \( M \) is a \( M \times M \) matrix whose elements are of modulus one and such that \( H^\dagger H = M I \).

Complex Hadamard matrices exist for all \( M \). For instance the Fourier matrix

\[
H_{nm} = \exp \left[ i 2\pi \frac{(n - 1)(m - 1)}{M} \right], \quad m, n = 1, \ldots, M,
\]

is a complex Hadamard matrix.

Proposition: A Hadamard matrix has maximal determinant among matrices with complex entries in the closed unit disk.

More exactly the determinant of any complex \( M \times M \) matrix \( H \) with \( |H_{mn}| \leq 1 \) satisfies \( |\det H| \leq M^{M/2} \), with equality attained by a complex Hadamard matrix.
Let \( H \) be a complex invertible \( M \times M \) matrix with \( |H_{mn}| \leq 1 \).

Multi-source acquisition scheme:
- In the \( m \)th experience, \( m = 1, \ldots, M \), all sources generate time-harmonic signals, the \( m' \) source generating \( H_{m'm} \) (the amplitude is bounded by one).
- The \( N \) receivers record the backscattered waves:

\[
B_{nm} = \sum_{m'=1}^{M} H_{m'm} P_{0,nm'} + W_{nm} = (P_0 H)_{nm} + W_{nm}, \quad n = 1, \ldots, N,
\]

where \( W_{nm} \) are independent complex Gaussian random variables with mean zero and variance \( a^2 \).
- The measured response matrix \( P \) is obtained by right multiplying by \( H^{-1} \):

\[
P := BH^{-1} = P_0 HH^{-1} + WH^{-1}
\]

\( \leftrightarrow \) we get the unperturbed matrix \( P_0 \) up to a new noise

\[
P = P_0 + \tilde{W}, \quad \tilde{W} = WH^{-1}
\]
The choice of the matrix $H$ should fulfill the property that the new noise matrix $\tilde{W} = WH^{-1}$ has independent complex entries with mean zero and minimal variance. We have

$$
\mathbb{E}[\tilde{W}_{nm}\tilde{W}_{n'm'}] = \sum_{q,q'=1}^{M} (H^{-1})_{qm}(H^{-1})_{q'm'} \mathbb{E}[W_{nq}W_{n'q'}]
$$

$$
= a^2 \sum_{q,q'=1}^{M} (H^{-1})_{qm}(H^{-1})_{q'm'} \mathbf{1}_n(n') \mathbf{1}_q(q')
$$

$$
= a^2 \sum_{q=1}^{M} ((H^{-1})^\dagger)_{mq}(H^{-1})_{qm'} \mathbf{1}_n(n')
$$

$$
= a^2 ((H^{-1})^\dagger H^{-1})_{mm'} \mathbf{1}_n(n'),
$$

This shows that we look for a complex matrix $H$ with entries in the unit disk such that $(H^{-1})^\dagger H^{-1} = cI$ with a minimal $c$. This is equivalent to require that $H$ is unitary and that $|\det H|$ is maximal.

$\leftrightarrow$ The optimal matrix $H$ that minimizes the noise variance should be a Hadamard matrix. For instance, take the Fourier matrix (in the case of a linear array, this corresponds to an illumination in the form of plane waves with regularly sampled angles).

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When the multi-source acquisition scheme is used with a Hadamard technique, we measure

$$P = P_0 + \tilde{W},$$

where the new noise matrix $\tilde{W}$ has independent complex entries with Gaussian statistics, mean zero, and variance $a^2/M$:

$$\mathbb{E}[\tilde{W}_{nm}\tilde{W}_{n'm'}] = \frac{a^2}{M} \mathbb{1}_m(m')\mathbb{1}_n(n').$$

This gain of a factor $\sqrt{M}$ in the signal-to-noise ratio is called the Hadamard advantage.
Time reversal
**Time reversal experiment**

Originally: Not for imaging, but for energy focusing (kidney stone destruction).

A Time-Reversal Mirror (TRM) is used first as a sensor array, then as a source array.

The source $\mathbf{y}$ emits a pulse

The TRM records the signals

The TRM emits the time-reversed signals

A sensor array probes the region around the original source location

→ looks like reverse-time migration for passive array imaging! The difference:

backpropagation is performed physically in a TR experiment and numerically in RT migration.
Comparison reverse-time migration vs time reversal

In both cases the recorded data are back-propagated from the array.

In RT migration, backpropagation is carried out numerically, in a fictitious medium.

In TR, backpropagation is carried out physically, in the real medium.

← No difference when the medium is perfectly known.
Random medium
Random medium

• We need a model for the medium that incorporates
  (a) a background velocity that is known,
  (b) the clutter that we do not know and can only estimate statistically.

• This motivates the model:

\[
\frac{1}{c^2(\mathbf{x})} = \frac{1}{c_0^2} \left( n_0^2(\mathbf{x}) + \mu(\mathbf{x}) \right)
\]

\(c_0\) is a reference speed,
\(n_0(\mathbf{x})\) is a smooth background index of refraction (known),
\(\mu(\mathbf{x})\) is a zero-mean, stationary random process that represents the clutter.
Passive array imaging: simulation with clutter

- Absorbing medium
- Array dimensions: $L = 90 \lambda_0$, $d = 6\lambda_0$
- Color scale for intensity
- Graphs showing time in msec vs. range and cross-range for different scenarios
Random medium

- We need a model for the medium that incorporates
  (a) a background velocity that is known,
  (b) the clutter that we do not know and can only estimate statistically.
- This motivates the model:
  \[
  \frac{1}{c^2(x)} = \frac{1}{c_0^2} \left( n_0^2(x) + \mu(x) \right)
  \]
  
  \(c_0\) is a reference speed,
  \(n_0(x)\) is a smooth background index of refraction (known),
  \(\mu(x)\) is a zero-mean, stationary random process that represents the clutter.
- The background Green’s function (known):
  \[
  \Delta \hat{G}_0 + \frac{\omega^2}{c_0^2} n_0^2(x) \hat{G}_0 = -\delta(x - y)
  \]
  
  The physical Green’s function (random and unknown):
  \[
  \Delta \hat{G} + \frac{\omega^2}{c_0^2} \left( n_0^2(x) + \mu(x) \right) \hat{G} = -\delta(x - y)
  \]
A remark: Cluttered noise is not an additive white noise

The structure of $\hat{G} - \hat{G}_0$ depends on the propagation regime (single scattering, multiple scattering, diffusive, randomly layered media, paraxial, ...)

Example: In the single-scattering regime, the response matrix of the sensor array $(x_j)_{j=1,...,n}$ is $(\hat{G}(\omega, x_i, x_j) - \hat{G}_0(\omega, x_i, x_j))_{i,j=1,...,n}$. It has a Hankel structure (coherence along the antidiagonals) if the array is linear.

Experimental response matrix $(\text{Re} (\hat{G}(\omega, x_i, x_j) - \hat{G}_0(\omega, x_i, x_j)))_{i,j=1,...,64}$ in a scattering medium (in the absence of strong reflector) [Aubry and Derode, PRL 102, 084301 (2009)].
Comparison reverse-time migration vs time reversal

• In RT migration, backpropagation is carried out numerically, in a fictitious medium.
• In TR, backpropagation is carried out physically, in the real medium.

No difference when the medium is perfectly known.

But: dramatic difference when the medium is not known or known partially.

\[
\hat{u}_{TR}(\omega, \mathbf{y}^S) = \sum_r \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{P}(\omega, \mathbf{x}_r) = \sum_r \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \hat{f}(\omega)
\]

\[
\hat{I}_{RT}(\omega, \mathbf{y}^S) = \sum_r \hat{G}_0(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{P}(\omega, \mathbf{x}_r)
\]

\[
\hat{I}_{RT}(\omega, \mathbf{y}^S) = \sum_r \hat{G}_0(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{P}(\omega, \mathbf{x}_r) = \sum_r \hat{G}_0(\omega, \mathbf{y}^S, \mathbf{x}_r) \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \hat{f}(\omega)
\]
Time reversal experiment with a full-aperture mirror (1/3)

First part:
A point source at \( \mathbf{y} \) emits a pulse \( f(t) \).
The waves are recorded at the surface \( \partial B(0, L) \):
\[
\hat{u}(\omega, \mathbf{x}) = \hat{G}(\omega, \mathbf{x}, \mathbf{y}) \hat{f}(\omega), \quad \mathbf{x} \in \partial B(0, L)
\]

Second part:
The recorded signals are time-reversed and sent back into the medium.
The signal received at \( \mathbf{y}^S \) is
\[
\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \int_{\partial B(0, L)} dS(\mathbf{x}) \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}) \overline{\hat{G}(\omega, \mathbf{x}, \mathbf{y})} \hat{f}(\omega)
\]
Time reversal experiment with a full-aperture mirror (2/3)

The signal received at $\mathbf{y}^S$ is (using reciprocity: $\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \hat{G}(\omega, \mathbf{y}, \mathbf{x})$):

$$\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \int_{\partial B(0, L)} dS(\mathbf{x}) \frac{\hat{G}(\omega, \mathbf{y}, \mathbf{x}) \hat{G}(\omega, \mathbf{y}^S, \mathbf{x}) \hat{f}(\omega)}{\hat{G}(\omega, \mathbf{y}^S, \mathbf{x})}$$

By Helmholtz-Kirchhoff theorem:

$$\hat{G}(\omega, \mathbf{y}, \mathbf{y}^S) - \hat{G}(\omega, \mathbf{y}^S, \mathbf{y}^S) = \frac{2i\omega}{c_0} \int_{\partial B(0, L)} dS(\mathbf{x}) \frac{\hat{G}(\omega, \mathbf{y}, \mathbf{x}) \hat{G}(\omega, \mathbf{y}^S, \mathbf{x})}{\hat{G}(\omega, \mathbf{y}, \mathbf{x}) \hat{G}(\omega, \mathbf{y}^S, \mathbf{x})}$$

we get

$$\hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) = \frac{\hat{G}(\omega, \mathbf{y}, \mathbf{y}^S) - \hat{G}(\omega, \mathbf{y}^S, \mathbf{y}^S)}{2i\omega/c_0} \hat{f}(\omega) = \frac{c_0}{\omega} \text{Im} \left( \frac{\hat{G}(\omega, \mathbf{y}, \mathbf{y}^S)}{2i\omega/c_0} \hat{f}(\omega) \right)$$

[Remember: $\mathbf{y}$ is the original source location].
Time reversal experiment with a full-aperture mirror (3/3)

The focal spot is determined by the imaginary part of the Green’s function:

\[ \hat{u}_{TR}(\omega, y^S) = \frac{c_0}{\omega} \text{Im}(\hat{G}(\omega, y, y^S)) \hat{f}(\omega) \]

- In a homogeneous medium:

\[ \hat{G}(\omega, y, y^S) = \frac{1}{4\pi|y - y^S|} e^{i\frac{\omega|y - y^S|}{c_0}} \implies \hat{u}_{TR}(\omega, y^S) = \frac{1}{4\pi} \text{sinc}\left(\frac{\omega|y - y^S|}{c_0}\right) \hat{f}(\omega) \]

\( \rightarrow \) refocusing with a focal spot of diameter \( \lambda/2 \) (diffraction limit).

- In a complex medium: \( \text{Im}(\hat{G}(\omega, y, y^S)) \) can be sharper than in a homogeneous medium

“Super-resolution effect” [Lerosey et al, Science 315 (2007), 1120]: if a micro-structured medium surrounds the original source \( y \), then the focal spot can be smaller than the diffraction limit \( \lambda/2 \)!

Main effect of the micro-structured medium: modify the effective wavelength (homogeneization result) [Ammari et al, SIAP 70 (2009) 1428], [Gomez, SIAM MMS 7 (2009) 1348]).
Time reversal experiment with a finite-aperture mirror (1/3)

Square array $[-D/2, D/2] \times [-D/2, D/2] \times \{0\}$. Point source $y = (0, 0, L)$.

Refocused wave: $\hat{u}_{TR}(\omega, y^S) = \sum_{r=1}^{N} \hat{G}(\omega, y^S, x_r) \overline{\hat{G}(\omega, x_r, y) \hat{f}(\omega)}$

- In the homogeneous paraxial regime ($\lambda \ll D \ll L$), for a dense square array:

  $\hat{u}_{TR}(\omega, y^S = (x_1, x_2, L)) = \text{sinc} \left( \frac{\pi x_1}{r_c} \right) \text{sinc} \left( \frac{\pi x_2}{r_c} \right) \overline{\hat{f}(\omega)}$

  $r_c = \frac{\lambda L}{D}$ (Rayleigh resolution formula).

  $\rightarrow$ The diameter of the focal spot is determined by the aperture $D/L$ of the focusing cone.
Time reversal experiment with a finite-aperture mirror (2/3)

- In the random paraxial regime, for a square TR array with diameter $D$:

$$
\mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega, \mathbf{y}^S = (x_1, x_2, L)) \right] = \text{sinc} \left( \frac{\pi x_1}{r_c} \right) \text{sinc} \left( \frac{\pi x_2}{r_c} \right) \exp \left( -\frac{x_1^2 + x_2^2}{2r_a^2} \right) \hat{f}(\omega)
$$

\[ r_a = \frac{\lambda \sqrt{6}}{\pi \sqrt{\gamma_2 L}} \text{ and } \gamma_2 = \int_{-\infty}^{\infty} \mathbb{E} [\nabla_\perp \mu(0,0) \cdot \nabla_\perp \mu(0,z)] dz \]

Rayleigh resolution formula with an effective array diameter $D_{\text{eff}} \sim \sqrt{D^2 + \gamma_2 L^3}$. The diameter of the focal spot is smaller because the effective focusing cone is larger.

- Statistical stability of the refocused focal spot ensured by frequency decorrelation:

$$
\mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega, \mathbf{y}^S)\hat{u}_{\text{TR}}(\omega', \mathbf{y}^S) \right] \simeq \mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega, \mathbf{y}^S) \right] \mathbb{E} \left[ \hat{u}_{\text{TR}}(\omega', \mathbf{y}^S) \right] \text{ for } |\omega - \omega'| > \Omega_c
$$

where $\Omega_c$ is the decoherence frequency (=frequency gap beyond which the signals are not correlated).

$\rightarrow u_{\text{TR}}(t, \mathbf{y}^S)$ is statistically stable provided its bandwidth is larger than $\Omega_c$:

$$
\frac{\text{Var}(u_{\text{TR}}(t, \mathbf{y}^S))}{\mathbb{E}[u_{\text{TR}}(t, \mathbf{y}^S)]^2} \ll 1
$$

Note: other mechanisms (lateral diversity) can ensure statistical stability.
Time reversal experiment with a finite-aperture mirror (3/3)
One can observe a spatial refocusing at the original source location.

Passive array imaging in a cluttered medium

Square array \([-D/2, D/2] \times [-D/2, D/2] \times \{0\}\). Point source \(\mathbf{y} = (0, 0, L)\).

Reverse-time migration: \(\hat{u}_{RT}(\omega, \mathbf{y}^S) = \sum_{r=1}^{N} \hat{G}_0(\omega, \mathbf{y}^S, \mathbf{x}_r)\hat{G}(\omega, \mathbf{x}_r, \mathbf{y})\hat{f}(\omega)\)

Mean “focal spot” (in the random paraxial regime):
\[
\mathbb{E}\left[\hat{u}_{RT}(\omega, \mathbf{y}^S = (x_1, x_2, L))\right] = \text{sinc} \left(\pi \frac{x_1}{r_c}\right) \text{sinc} \left(\pi \frac{x_2}{r_c}\right) \exp(-d(\omega)) \hat{f}(\omega)
\]

where \(d(\omega) = \frac{\gamma_0 L}{8c_0^2} \omega^2 \) and \(\gamma_0 = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(0, z)]dz\)

No statistical stability (standard deviation larger than the mean):
\[
\frac{\text{Var}(I_{RT}(t, \mathbf{y}^S))}{\mathbb{E}[I_{RT}(t, \mathbf{y}^S)]^2} \gg 1
\]
Passive array imaging in a cluttered medium: simulation
Kirchhoff migration: simulation without and with clutter

Realization 1
Realization 2
How can we improve imaging in a cluttered medium?
Comparison reverse-time migration vs time reversal

- The true Green’s function for the random medium is not known and so cannot be used for imaging, but is used in physical TR.

- Reverse-Time migration (and Kirchhoff migration) does not work well in clutter and it is statistically unstable. The reason is that migration tries to cancel the random phases of the signals arriving at the array with a deterministic phase using travel times.

- Time Reversal works all the better (resolution enhancement) as the clutter is important!

- Precise results in the random medium case are based on the statistical analysis of \( \hat{G}_G \) (useful for imaging, time reversal, and cross correlation imaging).

- Idea for imaging in random media: use and backpropagate cross correlations (Coherent Interferometric Imaging).
Coherent Interferometric Imaging (CINT)
Coherent Interferometric Imaging (CINT): principle

- RT (or KM) imaging in scattering media (passive array): backpropagate the data

\[ I_{RT}(y^S) = \sum_r \hat{G}_0(\omega, y^S, x_r) \hat{P}(\omega, x_r) \]

(remember \( \hat{P}(\omega, x_r) \sim \hat{G}(\omega, x_r, y) \)). The problem is that \( \hat{G}_0 \hat{G} \) is not stable.

- CINT imaging: - cross correlate the data \((\to \hat{G}\hat{G})\),
- backpropagate (with \( \hat{G}_0 \hat{G}_0 \)) the cross correlations of the data.

First guess:

\[ I(y^S) = \sum_{r,r'=1}^{N} \int \hat{P}(\omega, x_r) \hat{P}(\omega, x_{r'}) e^{-i\omega T(x_r, y^S) + i\omega T(x_{r'}, y^S)} d\omega \]

but not so good (pairing of \( \hat{G} \) and \( \hat{G}_0 \) is not correct).

- The key idea is to migrate the cross correlations of the data rather than the data.
- It is important to compute the cross correlations \textit{locally} in time and space, and not over the whole time interval and the whole set of pairs of sensors, in order to pair \( \hat{G}\hat{G} \) and \( \hat{G}_0 \hat{G}_0 \) correctly.
Coherent Interferometric Imaging: implementation (1/2)

- Consider the square of the KM functional:

\[
|\mathcal{I}_{KM}(y^S)|^2 = \sum_{r,r'=1}^{N} \int \int \hat{P}(\omega, x_r) \hat{P}(\omega', x_{r'}) e^{-i\omega T(x_r, y^S) + i\omega' T(x_{r'}, y^S)} d\omega d\omega'
\]

→ pairing (of \(\hat{G}\) and \(\hat{G}_0\)) is wrong.

- Decoherence frequency \(\Omega_c\): frequency gap beyond which the frequency components of the recorded signals are not correlated.

Remark: The reciprocal of the decoherence frequency is the delay spread (duration of the coda).

\[
\mathcal{I}_{CINT}(y^S, \Omega_d) = \sum_{r,r'=1}^{N} \int \int \hat{P}(\omega, x_r) \hat{P}(\omega', x_{r'}) e^{-i\omega T(x_r, y^S) + i\omega' T(x_{r'}, y^S)} d\omega d\omega'
\]

|\omega - \omega'| \leq \Omega_d

Compare with the square of the KM functional: The CINT functional and the square of the KM functional differ only in that the frequencies \(|\omega - \omega'| > \Omega_d\) are eliminated in CINT.
Coherent Interferometric Imaging: implementation (2/2)

- Decoherence length $X_c$: distance between sensors beyond which the signals recorded at them are not correlated.

$$I_{\text{CINT}}(y^S, \Omega_d, X_d) = \sum_{r,r' = 1}^N \int \int \hat{P}(\omega, \bm{x}_r) \hat{P}(\omega', \bm{x}_{r'}) e^{-i \omega \tau(\bm{x}_r, y^S) + i \omega' \tau(\bm{x}_{r'}, y^S)} d\omega d\omega'$$

- The cross range resolution of CINT is of the order of $\lambda_0 L / X_d$ (Rayleigh resolution formula with the effective array diameter $D_{\text{eff}} = X_d$).

- The range resolution of CINT is of the order of $c_0 / \Omega_d$.

- The optimal values for the parameters $\Omega_d$ and $X_d$ can be found by a statistical analysis (depends on the propagation regime).

- An adaptive procedure for estimating optimally the parameters $\Omega_d$ and $X_d$ is based on minimizing a suitable norm of the image to improve its quality, both in terms of resolution and signal-to-noise ratio.
Coherent Interferometric Imaging: simulation

Optimal choice of $X_d, \Omega_d$ based on the image (obtained by minimizing an appropriate objective functional, a suitable norm of the image).

Trade-off between resolution and signal-to-noise ratio.

Simulations carried out by C. Tsogka (University of Crete).