Random paraxial wave equation

Propagation of time-harmonic waves in a homogeneous medium.
We consider a time-harmonic initial condition (with frequency $\omega$) in the plane $z = 0$ whose transverse profile is a Gaussian with radius $r_0$:

$$\phi_0(x) = \exp \left( -\frac{x^2}{r_0^2} \right).$$

Solve the Schrödinger equation from the plane $z = 0$ to the plane $z = L$

$$\partial_z \phi = \frac{i}{2k} \partial_x^2 \phi, \quad \phi(z = 0, x) = \phi_0(x),$$

using the Fourier method (use \texttt{fft}). Compute the transmitted wave profile $\phi_t(x)$ for an initial Gaussian beam with radius $r_0 = 4$, a grid of $2^{10}$ points with a transverse grid step 0.1, $k = \omega = 1$, and $L = 10$.

Check that the transmitted wave profile $\phi_t(x)$ is:

$$\phi_t(x) = r_0 \exp \left( -\frac{x^2}{r_t^2} \right), \quad r_t = r_0 \left( 1 + 2i \frac{L}{kr_0^2} \right)^{1/2}$$

by comparing the square modulus of the numerical transmitted wave with

$$|\phi_t(x)|^2 = \frac{r_0}{R_t} \exp \left( -\frac{2x^2}{R_t^2} \right), \quad R_t = r_0 \left( 1 + \frac{4L^2}{k^2 r_0^4} \right)^{1/2}.$$

Time reversal for time-harmonic waves in a homogeneous medium.
Consider the (compactly supported) time-reversal mirror in the plane $z = L$:

$$\chi_M(x) = \left( 1 - \frac{x^2}{2r_M^2} \right)^2 \mathbf{1}_{[-2r_M, 2r_M]}(x).$$

Perform a time-reversal experiment by backpropagating the time-reversed wave $\phi_t$ (i.e. its complex conjugate) from the time-reversal mirror support at $z = L$ to the plane $z = 0$. Compute the refocused wave $\phi_{tr}(x)$ in the plane $z = 0$ for different values for the radius $r_M$ (from 2 to 20) of the mirror. Note how refocusing becomes poor when $r_M$ becomes small.

Note: if one considers a "Gaussian time-reversal mirror" in the plane $z = L$, with a truncation function of the form

$$\chi_M(x) = \exp \left( -\frac{x^2}{r_M^2} \right),$$

then one can check analytically that the refocused wave profile $\phi_{tr}(x)$ is given by

$$\phi_{tr}(x) = \frac{1}{a_{tr}} \exp \left( -\frac{x^2}{r_{tr}^2} \right), \quad r_{tr}^2 = \left( \frac{1}{r_M^2} + \frac{1}{r_0^2} \right)^{-1} + 2i \frac{L}{k}, \quad a_{tr} = \left( 1 + \frac{4L^2}{k^2 r_0^4} + 2i \frac{L}{k r_M^2} \right)^{1/2}.$$

Check this result numerically.

Propagation of time-harmonic waves in a random medium.
Implement a split-step Fourier method with longitudinal step $h$ to solve the Schrödinger equation

$$\partial_z \phi = \frac{i}{2k} \partial_x^2 \phi + \frac{ik}{2} h(z, x) \phi, \quad \phi(z = 0, x) = \phi_0(x) = \exp \left( -\frac{x^2}{r_0^2} \right).$$
Consider a potential \( \mu \) of the form:

\[
\mu(z, x) = \mu_n(x), \quad \text{if } z \in [nz_c, (n + 1)z_c)
\]

where \( \mu_0(x), \mu_1(x), \ldots, \mu_{[L/z_c]}(x) \) are independent realizations of a Gaussian process with mean zero and covariance function \( \mathbb{E}[\mu_n(x)\mu_n(x')] = \sigma^2 \exp(-(x - x')^2/x_c^2) \).

Take \( h = 1, z_c = 1, x_c = 4, \sigma = 1. \)

Check that the mean transmitted wave profile \( \phi_t(x) = \phi(L, x) \) is:

\[
\mathbb{E}[\phi_t(x)] = \frac{r_0}{r_t} \exp \left( -\frac{x^2}{r_t^2} \right) \exp \left( -\frac{\gamma_0 \omega^2 L}{8} \right)
\]

with \( \gamma_0 = \sigma^2 z_c. \) For the evaluation of the mean take the average over 100 runs with 100 independent realizations of the random medium.

**Time reversal for time-harmonic waves in a random medium.**

Consider a Gaussian time-reversal mirror in the plane \( z = L: \chi_M(x) = \exp(-x^2/r_M^2) \).

Perform a time-reversal experiment by backpropagating the time-reversed wave \( \phi_t \) in the same random medium and compute the refocused wave \( \phi_{tr}^{ir}(z, x) \) in the plane \( z = 0 \). Check that the mean refocused wave profile \( \phi_{tr}^{ir}(x) = \phi_{tr}^{ir}(0, x) \) is given by

\[
\mathbb{E}[\phi_{tr}^{ir}(x)] = \frac{1}{a_{tr}} \exp \left( -\frac{x^2}{r_{tr}^2} \right) \exp \left( -\frac{\gamma_2 \omega^2 L}{8} \right)
\]

with \( r_{tr}^2 = \gamma_2 \omega^2 L/48 \) and \( \gamma_2 = 2\sigma^2 z_c/x_c^2. \)

Try different values for the radius \( r_M \) of the mirror. Note that refocusing becomes significantly better in the random medium case than in the homogeneous medium case when the mirror is relatively small (say \( r_M = 2 \)).

Perform a time-reversal experiment by backpropagating the time-reversed wave \( \phi_t \) in a homogeneous medium and compute the refocused wave \( \phi_{tr}^{ir}(z, x) \) in the plane \( z = 0 \). Check that the mean refocused wave profile \( \phi_{tr}^{ir}(x) = \phi_{tr}^{ir}(0, x) \) is given by

\[
\mathbb{E}[\phi_{tr}^{ir}(x)] = \frac{1}{a_{tr}} \exp \left( -\frac{x^2}{r_{tr}^2} \right) \exp \left( -\frac{\gamma_0 \omega^2 L}{8} \right)
\]

**Time reversal for time-dependent waves in a random medium.**

Here we consider a time-dependent initial condition, whose spectrum is flat over \([\omega_0 - B, \omega_0 + B]\), with \( \omega_0 = 1 \) and \( B = 0.75 \), and whose transverse profile is a Gaussian with radius \( r_0 \).

Perform a time-reversal experiment for this wave: simply sum the frequency components computed in the previous section for a set of regularly sampled frequencies (say, 20 frequencies over \([\omega_0 - B, \omega_0 + B]\)).

Observe the refocused wave profile, compare with

\[
\phi_{tr}^{ir}(x) = \frac{1}{a_{tr}} \exp \left( -\frac{x^2}{r_{tr}^2} \right) \exp \left( -\frac{x^2}{r_a^2} \right)
\]

and observe the statistical stability of the refocused wave (i.e. repeat the experiment with different realizations).