# First Occurrence and Frequency of Invisible Lattice Point Patterns

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**Abstract.** Consider a "forest" of infinitely thin trees arranged on the lattice $\mathbb{Z} \times \mathbb{Z}$. If you are standing at the origin, $(0, 0)$, not all trees are visible despite the fact that they are infinitely thin. In particular, of the trees all lying on a line through $(0, 0)$, only one such point is visible. In this article we conclusively classify all closest occurring invisible rectangular $n \times m$ blocks of points for $1 \leq n, m \leq 4$. This (partially) resolves a question posed by Goins-Harris-Kubik-Mbirika. Furthermore, we compile statistics for all occurring arrangements up to size $4 \times 4$ and discuss interesting patterns that appear in that data.

## 1. Introduction.

For infinitely thin trees located on integer lattice points of a coordinate grid, with trees labeled by their lattice coordinates, $(x, y)$, the trees visible from the origin are exactly the coordinates with $\gcd(x, y) = 1$ (Theorem 2.1). The main focus in this area of research is determining the density and patterns which occur in the invisible trees. As this problem is entirely symmetric we will only consider what happens in the first quadrant.

This problem was first addressed in the 19th century by a number of people with Cesàro credited as the first to pose this version of the problem in 1881 [2]. In particular, Cesàro proved that any given point $(x, y)$ has probability approaching $\frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.608$ of being visible, where $\zeta(s)$ is the Riemann zeta function. In 1971, Herzog and Stewart characterized patterns of visible and invisible points, which remains the main motivation for current work in this area [5]. In 1976, Apostol [1, Theorem 5.29, p. 119] showed that there can be arbitrarily large square arrays of invisible points. In 1990, Schumer [7] used the Chinese Remainder Theorem to find $3 \times 3$ blocks of invisible points (quite far from the origin) and questioned whether utilizing similar methods would be possible for $4 \times 4$ blocks due to the complexity of his calculations. Goodrich-Mbirika-Nielsen [4] took up the challenge using similar methods to find a $4 \times 4$ and even a $5 \times 5$ invisible block, albeit both quite far from the origin. Goins-Harris-Kubik-Mbirika [3] pose the question of finding the nearest invisible forest of dimension $n \times m$; this last question is resolved in this article for $1 \leq m, n \leq 4$.

A quick computer search can find the closest occurrence of all $n \times m$ invisible blocks for $1 \leq n, m \leq 3$ since we need only search up to the first $3 \times 3$ invisible block occurring at $(x, y) = (1274, 1308)$. This search is sufficient as all smaller invisible blocks will occur before or within a $3 \times 3$ invisible block. However, finding the nearest invisible blocks of large size becomes computationally interesting. Our goal is to conclusively classify all closest occurring invisible rectangular blocks of points for $1 \leq n, m \leq 4$.

The paper is organized as follows. In Section 2, the fundamental mathematical results as

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well as the computational framework is described. This section includes the data table on the closest (radial) occurrence of $n \times m$ invisible rectangles. In Section 3, the mathematical explanations for why certain patterns do not occur are examined. In Section 4, the difference between closest invisible forests measured radially versus measured lexicographically is addressed. This section also includes the table containing all the lexicographically closest $n \times m$ invisible rectangular blocks for $1 \leq n, m \leq 4$. Finally in Section 5, the frequency of occurrence of all the possible invisible patterns up to size $4 \times 4$ is examined empirically. The full data for all $4 \times 4$ invisible patterns is available as an auxilary csv file.

Note that all decimal values are rounded to 6 significant figures.

## 2. Computing/Data Gathering.

This section describes the computational method used to examine all occurring invisible patterns up to size $4 \times 4$ for $0 \leq x, y \leq 24{,}000{,}000$. The main program, available upon request, was written in C and run on the Saint Louis University High Performance Computing Cluster. The program recorded the first occurrence of each of the possible $4 \times 4$ patterns, of which there are $2^{16} = 65536$, both according to radial distance from $(0, 0)$ and lexicographically. Additionally, the frequency of each pattern in the domain was also recorded. The statistics of the data is discussed in Section 5. Note that any size pattern is referenced by the coordinates of its lower left hand corner.

The two main obstacles to overcome are the number of computations to perform and the memory problem due to the amount of data being produced. Heavy use is made of the symmetry of this problem. As Theorem 2.1 proves, whether any given $(x, y)$ is invisible is a greatest common divisor computation. Thus, determining the $4 \times 4$ pattern at any given $(x, y)$ requires 16 gcd calculations. Determining the $4 \times 4$ pattern of a point within that square should make use of the calculations already performed. However, storing the result of the gcd calculation of every single point in the search space is not feasible. Our solution to this memory issue is described in Section 2.2. The number of computations is dominated by calculating the gcd of each pair $(x, y)$. We used a basic Euclidean Algorithm method whose number of operations grows logarithmically with $\max(x, y)$ and did not make an attempt to analyze or optimize these calculations.

It should be noted that having the data for $4 \times 4$ patterns is sufficient to have the data for all $m \times n$ patterns for $1 \leq m, n \leq 4$. Statistics for smaller squares are included in Section 5.

### 2.1. Determining if a point is invisible.

The following theorem is well known in this area and is included for completeness.

Theorem 2.1 ([3, Proposition 3]). *A point in the lattice is visible if and only if the greatest common divisor (gcd) of its $x$ and $y$ coordinates is 1.*

Corollary 2.2. *The only invisible rectangles with $(x, y)$ on the diagonal are size $1 \times 1$.*

*Proof.* If $x = y$, then $\gcd(x, y + 1) = \gcd(x + 1, y) = 1$. ■

### 2.2. Working with patterns/mask values.

The general principle for keeping track of invisible patterns is to assign a 16-bit integer to each lower left hand $(x, y)$ coordinate. Each bit represents whether one of the 16 points in the square is invisible or visible. The bit values are assigned as follows:

(2.1)

| | | | |
|---|---|---|---|
| 8 | 4 | 2 | 1 |
| 128 | 64 | 32 | 16 |
| 2048 | 1024 | 512 | 256 |
| 32768 | 16384 | 8192 | 4096 |

We call the 16-bit integer the *mask* value associated to the pattern. A corresponding mask value and pattern is given in the following example.

Example 2.3. *The pattern*

(2.2)

| | | | |
|---|---|---|---|
| • | | | • |
| | • | • | |
| | • | • | |
| • | | | • |

*is given by the mask value*

$$1 + 8 + 32 + 64 + 512 + 1024 + 4096 + 32768 = 38505.$$

In designing a program to solve this problem, the first challenge is tracking the location of each point in a $4\times 4$ array. Using for loops for both the $x$ and $y$ axes means each new position necessitates knowing values for all surrounding relevant points to keep accurate counts. This requires unreasonably large amounts of memory and high computer performance specifications to calculate and store all of the values for every $4 \times 4$ array at once.

For example, storing 1 'bit' value for each point in a 1 million by 1 million lattice requires 250 GB of memory, and storing each value as an int uses 8 TB. By storing the value as bits, one gcd calculation can contribute to 16 different lower left hand corners without additional calculation.

Additionally, we introduce a wrapping system which only keeps the values of 4 columns at a time. That is, after the completion of the column for loop, the data for column 5 writes to the memory that contains column 1. This ensures the values do not override their neighbors until the full $4 \times 4$ array has been calculated and recorded accordingly and keeping total memory requirements reasonable.

The entire program only saves the first location (both radial distance and lexicographic order) and count of each $4\times 4$ pattern, in order to efficiently utilize storage space. Furthermore, we utilize the symmetry of the problem ($\gcd(x, y) = \gcd(y, x)$) and only work with the $(x, y)$ points on or above the diagonal $y = x$.

The final statistics expand this data under symmetry to the whole first quadrant. To avoid extensive live runtime of our program, we split the search space into separate blocks to run concurrently on the Saint Louis University High Performance Cluster, and compile the separate runs into one overall data file.

Our data encompasses an integer lattice of 24 million by 24 million. The computation of our results took 491 days of CPU time on SLUs High Performance Cluster utilizing 112 cores (further specs on the compute nodes is unavailable). The table below lists the location of the first occurrence of each $m \times n$ forest, as well as its total number of occurrences in our search space. Notice as the size of the forest increases, the occurrence total decreases significantly. Our research also confirms the work of Eric Weisstein in the "Visible Point" entry of the MathWorld website, who posited the first location (with $0 < x < y$) of a $4 \times 4$ forest to be at $(7247643, 10199370)$. We found the first five occurrences by distance from $(0, 0)$ of $4 \times 4$ invisible forests (and the first 10 after diagonal symmetry is considered):

(7247643, 10199370), (6349914, 13125369), (13449225, 13458288),
(3268473, 21374352), (16799913, 22339875).

**Table 2.1**
*Invisible Rectangles*

| Invisible Rectangle (Width × Height) | Pattern Value | (x,y) of Closest Occurrence | Total Count | Prop of Total Rectangles |
|---|---|---|---|---|
| 1×1 | 32768 | (2, 2) | 225833983043489 | 0.392073 |
| 1×2 | 34816 | (2, 6) | 61505212491040 | 0.106780 |
| 1×3 | 34944 | (2, 6) | 31371300360510 | 0.0544641 |
| 1×4 | 34952 | (2, 30) | 7157758947052 | 0.0124267 |
| 2×1 | 49152 | (6, 2) | 61505212491040 | 0.106780 |
| 2×2 | 52224 | (14, 20) | 1237398519088 | 0.00214826 |
| 2×3 | 52416 | (54, 230) | 42011981298 | $7.29375 \cdot 10^{-5}$ |
| 2×4 | 52428 | (174, 825) | 873069048 | $1.51574 \cdot 10^{-6}$ |
| 3×1 | 57344 | (6, 2) | 31371300360510 | 0.0544641 |
| 3×2 | 60928 | (230, 54) | 42011981298 | $7.29375 \cdot 10^{-5}$ |
| 3×3 | 61152 | (1274, 1308) | 989290450 | $1.71752 \cdot 10^{-6}$ |
| 3×4 | 61166 | (47859, 12824) | 394255 | $6.84470 \cdot 10^{-10}$ |
| 4×1 | 61440 | (30, 2) | 7157758947052 | 0.0124267 |
| 4×2 | 65280 | (825, 174) | 873069048 | $1.51574 \cdot 10^{-6}$ |
| 4×3 | 65520 | (47859, 12824) | 394255 | $6.84470 \cdot 10^{-10}$ |
| 4×4 | 65535 | (7247643, 10199370) | 10 | $1.73611 \cdot 10^{-14}$ |

As the data for the $1 \times 1$ invisible rectangles in Table 2.1 indicates, the proportion of total invisible points is 0.392073; thus approximately 40 percent of the points in the lattice are invisible. Conversely, the proportion of total visible points is 0.607927; thus approximately 60 percent of the points in the lattice are visible.

Sequences for the radial minimal $x$ and $y$ coordinates for $2 \times 2$, $3 \times 3$, and $4 \times 4$ invisible forests were added to the On-line Encyclopedia of Integer Sequences as sequences A325602, A325603, A325604, A325605, A325606, and A325607.

## 3. Non-Occurring Patterns.

It is known that any given $n \times m$ invisible rectangle occurs through a simple application of the Chinese Remainder Theorem (see, for example, [6, Theorem 2.4]): choose which distinct primes divide which pairs of coordinates and solve the congruence equations for $x$ and $y$. In this section we look at the possible patterns that do not arise in our data and prove that they do not ever occur. The following table summarizes the proportion of patterns that did not occur.

All of these non-occurring patterns can be explained by examining the possible patterns modulo 2. In particular, the one not occurring $2 \times 2$ pattern is the pattern with all four points

**Table 3.1**
*Non-Occurring Patterns*

| Size | Total Number of Patterns | Number of Patterns That Do Not Occur | Proportion of Total |
|---|---|---|---|
| $2 \times 2$ | $2^4 = 16$ | 1 | 0.0625 |
| $3 \times 3$ | $2^9 = 512$ | 135 | 0.263672 |
| $4 \times 4$ | $2^{16} = 65536$ | 50626 | 0.772491 |

visible. If this $2 \times 2$ pattern is contained within any larger pattern, then it cannot occur.

Lemma 3.1. *No patterns with $2 \times 2$ square(s) of visible points occur.*

*Proof.* Let $(x, y)$ be a point in the first quadrant with $x, y \in \mathbb{Z}$. If the greatest common divisor of at least 1 pair of coordinates in the $2 \times 2$ square

(3.1)

| | |
|---|---|
| $(x, y+1)$ | $(x+1, y+1)$ |
| $(x, y)$ | $(x+1, y)$ |

is greater than 1, then the $2 \times 2$ square is not visible.

There are 4 possible cases:

- Case 1: $x$ is even and $y$ is even. Therefore, $\gcd(x, y) \geq 2$.
- Case 2: $x$ is even and $y$ is odd. Therefore, $y + 1$ is even and $\gcd(x, y + 1) \geq 2$.
- Case 3: $x$ is odd and $y$ is even. Therefore, $x + 1$ is even and $\gcd(x + 1, y) \geq 2$.
- Case 4: $x$ is odd and $y$ is odd. Therefore, $x + 1$ and $y + 1$ are even so that $\gcd(x + 1, y + 1) \geq 2$.

In all possible cases, at least 1 pair of coordinates has a greatest common divisor of at least 2. Therefore, you cannot get a $2 \times 2$ square of visible points. ■

However, this is not the entire story. For example, any $4 \times 4$ pattern containing the following 4 visible points also cannot occur

(3.2)

| | | | |
|---|---|---|---|
| • | | | • |
| | | | |
| | | | |
| • | | | • |

This is because these four locations are the same as a $2 \times 2$ visible rectangle when taking modulo 2 and, as in the proof of the lemma, at least 1 of these 4 must have a gcd of at least 2 and so must be invisible. We formalize this notion with the following notation. Given a $2 \times 2$ block such that $x, y \in \mathbb{Z}$, $x$ and $y$ can be even or odd and form an ordered pair in 1 of 4 combinations:

$$(\text{even, even}), (\text{even, odd}), (\text{odd, even}), (\text{odd, odd})$$

Let the four types be called $A$, $B$, $C$, and $D$ which can be assigned arbitrarily. Then a $4 \times 4$ pattern contains the following types of coordinate pairs (after the possible re-assignment of type names).

(3.3)

| | | | |
|---|---|---|---|
| C | D | C | D |
| A | B | A | B |
| C | D | C | D |
| A | B | A | B |

Corollary 3.2. *Any pattern that contains at least one visible point of every type $A$, $B$, $C$, and $D$ cannot occur.*

There are $4^4 = 256$ possible ways to choose one coordinate pair of each type and these possibilities cannot all exist. Checking all possible patterns that do not occur in our data, every such pattern is explained in this way. This is a special case of the more general theorem that says the only non-occurring patterns are those which form complete residue classes of pairs modulo some prime [5, Theorem 1].

Corollary 3.3. *Any occurring $n \times m$ rectangle must have at least $\lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{m}{2} \rfloor$ invisible points.*

*Proof.* This is a pigeonhole principle argument. An occurring pattern cannot have every type $A, B, C, D$ occur. We must have all invisible $x$ coordinates as even or odd and all invisible $y$ coordinates as even or odd. Even and odd are the two categories which integer values are sorted into. Given three or more integers, i.e., one more than the number of categories present, there must exist at least two even integers or at least two odd integers by the pigeonhole principle.

All (even, even) points are invisible. The fewest possible invisible points then occur when the fewest (even, even) points are present within the chosen rectangle, assuming a worst case where no other pair types are invisible. For consecutive points, the fewest occurrences of even integers in either direction is one half the dimension, rounded down for odd dimensions. Thus, there are $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{m}{2} \rfloor$ minimum possible occurrences, respectively. ■

Proposition 3.4. *Every possible occurring pattern occurs within radial distance*

$$\begin{cases} 25.24 & 2 \times 2 \\ 6688.16 & 3 \times 3 \\ 12512213.14 & 4 \times 4 \end{cases}$$

*from the origin* $(0, 0)$.

*Proof.* Since the data includes the occurrence of all patterns that possibly occur, we take the maximum of the minimal distance of each occurring pattern in our data. ■

Theorem 3.5. *In a $4 \times 4$ square, patterns with less than 4 invisible points cannot exist.*

*Proof.* In a $4 \times 4$ square, each of $A$, $B$, $C$, and $D$ occur exactly 4 times. For any set of invisible points of cardinality less than 4, there remain at least 1 visible point of each type $A$,

$B$, $C$, and $D$, which is impossible. Therefore, in a $4 \times 4$ square there must exist at least 4 invisible points. ■

**4. Radial Distance versus Lexicographic Distance.** So far we have discussed finding the first occurring pattern as measured by minimal distance to $(0, 0)$. We could also consider "minimal" under the lexicographic ordering. In other words, for a given (occurring) pattern, what is the smallest possible $x$ value for which the pattern occurs? This problem is less amenable to computation as there is no simple way to enumerate all points up to some distance under the lexicographic ordering since there are infinitely many coordinates $(x, y)$ for any fixed $x$ value. However, for invisible rectangles, it is possible to prove the smallest occurring pattern location by examining prime divisibility properties. The following theorem summarizes the results.

Theorem 4.1. *The following table provides the first occurrence of each $m \times n$ rectangle under lexicographic ordering $x > y$.*

**Table 4.1**
*First Occurrences of Invisible Rectangles*

| Invisible Rectangle | (x,y) of Closest | Invisible Rectangle | (x,y) of Closest |
|---|---|---|---|
| 1×1 | (2, 2) | 3x1 | (6, 2) |
| 1×2 | (2, 6) | 3x2 | (104, 740) |
| 1×3 | (2, 6) | 3x3 | (104, 6200) |
| 1×4 | (2, 30) | 3x4 | (662, 128930788) |
| 2×1 | (6, 2) | 4x1 | (30, 2) |
| 2×2 | (14, 20) | 4x2 | (230, 7104) |
| 2×3 | (20, 384) | 4x3 | (644, 22984014) |
| 2×4 | (33, 15554) | 4x4 | (8853, 5583967323) |

*Proof.* For each $m \times n$ rectangle we perform the following steps to find the first occurrence.

1. Determine conditions on the number of distinct prime divisors of $x, x + 1, \dots, x + n$.
2. Find the smallest $x$ satisfying those minimal conditions using Sage.
3. Find the smallest $y$ that realizes an $m \times n$ invisible rectangle using Sage.

As the second and third steps are searches, it is only the first step that requires proof. Recall that a point is invisible if and only if $\gcd(x, y) \neq 1$. In particular $x$ needs at least one prime divisor. For $1 \leq n \leq 4$, $x = 2$ satisfies the necessary condition.

For $2 \times n$ rectangles, $x, x + 1, \dots, x + n$ must share a common divisor with $y$ and $y + 1$. Since $\gcd(y, y+1) = 1$, each of $x, x+1, \dots, x+n$ has at least two prime divisors. This results in the smallest $x$ values

$$(n, x) = (1, 6), (2, 14), (3, 20), (4, 33).$$

For $3 \times n$ rectangles, $x, x + 1, \dots, x + n$ must share a common divisor with $y$, $y + 1$, and $y + 2$. So either $x, x + 1, \dots, x + n$ have at least three distinct prime divisors, or the values of $x, x + 1, \dots, x + n$ which are even have at least two prime divisors, one of which is 2, and the values $x, x + 1, \dots, x + n$ which are odd have at least three distinct odd prime divisors. This results is the smallest $x$ values

$$(n, x) = (1, 6), (2, 104), (3, 104), (4, 662).$$

For $4 \times n$ rectangles, $x, x+1, \ldots, x+n$ must share a common divisor with $y$, $y+1$, $y+2$, and $y+3$. So either $x, x+1, \ldots, x+n$ have at least four distinct prime divisors, or less when one of those divisors is 2 or 3. There are six possible residue classes modulo 2 and 3 for $x$ which give conditions on the number of prime divisors. However, the least number of prime divisors needed for each $x, x+1, \ldots, x+n$ is three, and when $n = 4$, at least one of $x, x+1, \ldots, x+n$ must have four distinct prime divisors larger than 3. This results in the smallest $x$ values

$$(n, x) = (1, 30), (2, 230), (3, 644), (4, 8853).$$

∎

**Proposition 4.2.** *For any given $x$ coordinate which satisfies the necessary prime divisor conditions for which an $m \times n$ rectangle may appear, there are infinitely many $y$ coordinates for which $(x, y)$ is the lower left hand corner of an $m \times n$ invisible rectangle.*

*Proof.* Given the factorizations of $x, x+1, x+2, x+3$, we may set-up a system of linear congruences to solve for $y$. This system may be solved via the Chinese Remainder Theorem, which provides an infinite set of solutions. ∎

**Example 4.3.** *Consider $x = 20$ for a $2 \times 3$ rectangle. Then we have $20 = 2^2 \cdot 5$, $21 = 3 \cdot 7$, and $22 = 2 \cdot 11$, so we have the system of congruences*

$$\begin{aligned} y &= 0 \pmod{2 \cdot 3} \\ y &= -1 \pmod{5 \cdot 7 \cdot 11}. \end{aligned}$$

*This results in the solution*

$$y \equiv 384 \pmod{2310}.$$

*Notice that we could also have had the system*

$$\begin{aligned} y &= 0 \pmod{2 \cdot 7} \\ y &= -1 \pmod{3 \cdot 5 \cdot 11}. \end{aligned}$$

*This results in the solution*

$$y \equiv 1484 \pmod{2310}.$$

Another interesting question is, for the coordinates of an $m \times n$ invisible rectangle, what is the smallest number of prime divisors that are possible? As a simple upper bound, if every integer $x, x+1, \ldots, x+n$ has at least $m$ prime divisors, then using the the Chinese Remainder Theorem, a system of congruences can be constructed and solved (for $y$) to find an explicit invisible rectangle. However, this is clearly not optimal as Example 4.3 shows and for the simple reason that every second number is divisible by 2, so not every coordinate needs $m$ prime divisors. An alternate formulation of this problem is to ask, what is the fewest number of primes needed to divide every term in a sequence of consecutive integers? This problem has received some study under the form of the question: what is the longest sequence of consecutive integers divisible by the first $k$ primes (OEIS A058989)? The first answers are $(\{2\},1), (\{2,3\},3), (\{2,3,5\},5), (\{2,3,5,7\},9), (\{2,3,5,7,11\},13)$. This does not quite resolve our problem since the primes that divide our consecutive integers do not need to be from among the first $k$ primes. There is an interesting discussion of this more general problem in Quanta Magazine (https://www.quantamagazine.org/solution-the-prime-rib-problem-20170908/). For example the 13 numbers $24, \ldots, 36$ are all divisible by the primes $\{2,3,5,29,31\}$. We can prove that this is optimal. We can arrange $2,3,5$ to divide 11 of 13 consecutive integers starting at $x$ by setting it up as

$$\begin{aligned} x &\equiv 0 \pmod{2}, \\ x &\equiv 0 \pmod{3}, \\ x+1 &\equiv 0 \pmod{5}. \end{aligned}$$

Since there are only two numbers left $x+5, x+7$, they are odd and must have distinct prime divisors, so we need at least five primes. By working through the possible combinatorics, it can be determined that this arrangement results in the fewest possible number of primes. Note that the five primes $\{2,3,5,7,11\}$ can also solve this problem starting at $x = 114$, but this does not give the smallest solution. When the number of primes becomes seven, not only does the first occurrence differ for smallest primes versus arbitrary primes, but so does the maximal number of consecutive integers divisible by the set. So the problems are truly different. This appears to a rich area of research and warrants study in a future project.

## 5. Frequency of Pattern Occurrence.

In this section we make some empirical observations about the frequency with which invisible patterns occur. An interesting open problem would be to prove these observations conclusively.

### 5.1. $2 \times 2$ patterns.

There are 16 possible $2 \times 2$ patterns and all occur except the pattern with no invisible points. In particular, 93.75% of patterns occur. Interestingly, as seen in Table 5.1, the frequency with which a pattern occurs appears to depend only on the number of invisible points.

**Table 5.1**
*2x2 Patterns*

| Number of Points | Pattern | Frequency |
|---|---|---|
| 1 | 1024 | 0.125487 |
| 1 | 16384 | 0.125487 |
| 1 | 2048 | 0.125487 |
| 1 | 32768 | 0.125487 |
| 2 | 34816 | 0.0716601 |
| 2 | 49152 | 0.0716601 |
| 2 | 17408 | 0.0716601 |
| 2 | 3072 | 0.0716601 |
| 2 | 18432 | 0.0716601 |
| 2 | 33792 | 0.0716601 |
| 3 | 19456 | 0.0164857 |
| 3 | 35840 | 0.0164857 |
| 3 | 50176 | 0.0164857 |
| 3 | 51200 | 0.01648575 |
| 4 | 52224 | 0.00214826 |

**5.2. $3 \times 3$ patterns.** As seen in Table 5.2, the behavior of $3 \times 3$ patterns appears to be more complicated. There are 512 possible $3 \times 3$ patters and 73.63% of them occur. For $3 \times 3$ patterns with six or more points, all possible patterns occur. The non-occurring patterns are completely explained in Corollary 3.2.

In the $3 \times 3$ case, there was an overall pattern that the number of points determined the frequency with which a pattern occurred, but only up to a point. In other words, the frequency of pattern occurrence had a larger variation between number of points in the pattern compared to the variation in frequency within patterns with a fixed number of points. In the following table we compile the frequency of occurrence of patterns groups by type of pattern. The pattern type will be determined by the letters $A$, $B$, $C$, $D$ representing $(x, y)$ coordinates arranged in the following way:

(5.1)

| | | |
|---|---|---|
| A | B | A |
| C | D | C |
| A | B | A |

The following findings use the operations "+" and "or" in addition to the letters $A$, $B$, $C$, $D$. "+" separates two necessary conditions while "or" separates two or more possible conditions (one of which must be chosen). A single letter indicates that one point of that type must be selected while a letter with a coefficient in front of it indicates that two, three, or four points of that type must be selected.

**Table 5.2**

*3x3 Patterns*

| Number of Points | Number of Patterns | Pattern Types | Proportion |
|---|---|---|---|
| 1 | 1 | D | 0.01230689 |
| 2 | 8 | D + (A or B or C) | 0.0152519 |
| 2 | 2 | 2B or 2C | 0.0275588 |
| 3 | 26 | D + (2A or (A+B) or (A+C) or (B+C)) | 0.00329798 |
| 3 | 12 | (2B + (A or C)) or (2C + (A or B)) | 0.0185499 |
| 3 | 2 | D + (2B or 2C) | 0.0218479 |
| 4 | 44 | A + B + C + D | 0.000379761 |
| 4 | 28 | (2A + (2B or 2C)) or (A + ((2B + C) or (B + 2C))) | 0.00367773 |
| 4 | 12 | ((2B + D) + (A or C)) or ((2C + D) + (A or B)) | 0.00405751 |
| 4 | 1 | 2B + 2C | 0.00735549 |
| 4 | 1 | 4A | 0.0683364 |
| 5 | 40 | (2A + B + C + D) or (3A + (B or C) + D) | $2.82946 \cdot 10^{-5}$ |
| 5 | 32 | (2A + B + C) or (3A + (2B or 2C)) | 0.000408056 |
| 5 | 28 | (2B + D + (2A or (A + C)) or (2C + D + (2A or (A + B))) | 0.000436352 |
| 5 | 4 | A + 2B + 2C | 0.000816114 |
| 5 | 1 | 2B + 2C + D | 0.000844407 |
| 5 | 5 | 4A + (B or C or D) | 0.0263134 |
| 6 | 16 | 3A + (B or C) + D | $1.519132 \cdot 10^{-6}$ |
| 6 | 16 | 3A + (2B + C) or (B + 2C)) | $2.98136 \cdot 10^{-5}$ |
| 6 | 32 | (2A + ((2B + C) or (B + 2C) + D) or (3A + (2B or 2C) + D) | $3.13332 \cdot 10^{-5}$ |
| 6 | 6 | 2A + 2B + 2C | $5.96282 \cdot 10^{-5}$ |
| 6 | 4 | A + 2B + 2C + D | $6.11476 \cdot 10^{-5}$ |
| 6 | 10 | 4A + (2B or 2C or (B + C) or (B + D) or (C + D)) | 0.00452367 |
| 7 | 16 | 3A + ((2B + C) or (B + 2C)) + D | $1.64427 \cdot 10^{-6}$ |
| 7 | 4 | 3A + 2B + 2C | $3.16359 \cdot 10^{-6}$ |
| 7 | 6 | 2A + 2B + 2C + D | $3.22603 \cdot 10^{-6}$ |
| 7 | 10 | 4A + ((B + C + D) or (2B + (C or D)) or (2C + (B or D)) | 0.000469330 |
| 8 | 4 | 3A + 2B + 2C + D | $1.31291 \cdot 10^{-7}$ |
| 8 | 5 | 4A + ((2B + C + D) or (B + 2C + D) or (2B + 2C)) | $3.31089 \cdot 10^{-5}$ |
| 9 | 1 | 4A + 2B + 2C + D | $1.71752 \cdot 10^{-6}$ |

We were unable to find any clear patterns in the $3 \times 3$ data, so we did not expand these computations to categorize all of the $4 \times 4$ visible patterns. Computing these in a similar fashion as the $3 \times 3$ patterns would be time-consuming to perform by hand because the number of different proportions for $4 \times 4$ visible patterns with 9 to 14 points becomes large (see the data tables below).

The following tables provide information on invisible $2 \times 2$, $3 \times 3$, and $4 \times 4$ patterns. They indicate that the number of points in a pattern relate to the proportion at which this pattern occurs. For a proportion (proportion1) to be deemed "the same" as another proportion (proportion2), the absolute value of the difference between proportion1 and proportion2 needs to be less than 1 percent of proportion1.

**Table 5.3**
*2x2 Invisible Patterns*

| Number of Points in Pattern | Number of Patterns that Do Occur | Number of Patterns that Do Not Occur | Total Number of Patterns | Proportion of Total Patterns that Do Occur | Number of Different Proportions |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 4 | 0 | 4 | 0.266667 | 1 |
| 2 | 6 | 0 | 6 | 0.4 | 1 |
| 3 | 4 | 0 | 4 | 0.266667 | 1 |
| 4 | 1 | 0 | 1 | 0.0666667 | 1 |
| Total | 15 | 1 | 16 | 1 | 4 |

**Table 5.4**
*3x3 Invisibe Patterns*

| Number of Points in Pattern | Number of Patterns that Do Occur | Number of Patterns that Do Not Occur | Total Number of Patterns | Proportion of Total Patterns that Do Occur | Number of Different Proportions |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 8 | 9 | 0.00265252 | 1 |
| 2 | 10 | 26 | 36 | 0.0265252 | 2 |
| 3 | 40 | 44 | 84 | 0.106101 | 3 |
| 4 | 86 | 40 | 126 | 0.228117 | 5 |
| 5 | 110 | 16 | 126 | 0.291777 | 6 |
| 6 | 84 | 0 | 84 | 0.222812 | 6 |
| 7 | 36 | 0 | 36 | 0.0954907 | 4 |
| 8 | 9 | 0 | 9 | 0.0238727 | 2 |
| 9 | 1 | 0 | 1 | 0.00265252 | 1 |
| Total | 377 | 135 | 512 | 1 | 30 |

**Table 5.5**
*4x4 Invisible Patterns*

| Number of Points in Pattern | Number of Patterns that Do Occur | Number of Patterns that Do Not Occur | Total Number of Patterns | Proportion of Total Patterns that Do Occur | Number of Different Proportions |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 16 | 16 | 0 | 0 |
| 2 | 0 | 120 | 120 | 0 | 0 |
| 3 | 0 | 560 | 560 | 0 | 0 |
| 4 | 4 | 1816 | 1820 | 0.000268258 | 1 |
| 5 | 48 | 4320 | 4368 | 0.0032191 | 2 |
| 6 | 264 | 7744 | 8008 | 0.017705 | 4 |
| 7 | 880 | 10560 | 11440 | 0.0590168 | 7 |
| 8 | 1974 | 10896 | 12870 | 0.132385 | 12 |
| 9 | 3120 | 8320 | 11440 | 0.209242 | 16 |
| 10 | 3528 | 4480 | 8008 | 0.236604 | 20 |
| 11 | 2832 | 1536 | 4368 | 0.189927 | 26 |
| 12 | 1564 | 256 | 1820 | 0.104889 | 42 |
| 13 | 560 | 0 | 560 | 0.0375562 | 53 |
| 14 | 120 | 0 | 120 | 0.00804775 | 30 |
| 15 | 16 | 0 | 16 | 0.00107303 | 7 |
| 16 | 1 | 0 | 1 | $6.70646 \cdot 10^{-5}$ | 1 |
| Total | 14911 | 50625 | 65536 | 1 | 221 |

**6. Conclusion.** In this paper, we find the first occurrence of invisible rectangles of size $m \times n$ for $1 \leq m, n \leq 4$. In addition, we record the invisible patterns and their corresponding frequencies of occurrence up to $(x, y) = (24000000, 24000000)$, and we observe and explain the patterns that do not occur. We gather statistics that demonstrate nice patterns relating the number of points in a pattern to the proportion at which this pattern occurs, and the next step would be attempting to prove whether or not these patterns hold generally. We find the first occurring $4 \times 4$ invisible rectangle computationally, but based on the expected growth rate between previous invisible squares, finding the first occurring $5 \times 5$ invisible rectangle does not seem like a feasible computational problem with the current method. Therefore, we would need to utilize a different approach than brute force to find the first occurrence of this next invisible block. One consideration might be to look where small, distinct prime divisors occur most commonly in invisible rectangles to narrow the search criteria. For example, the $x$ and $y$ coordinates of the lower left point of most invisible rectangles are divisible by 2.

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