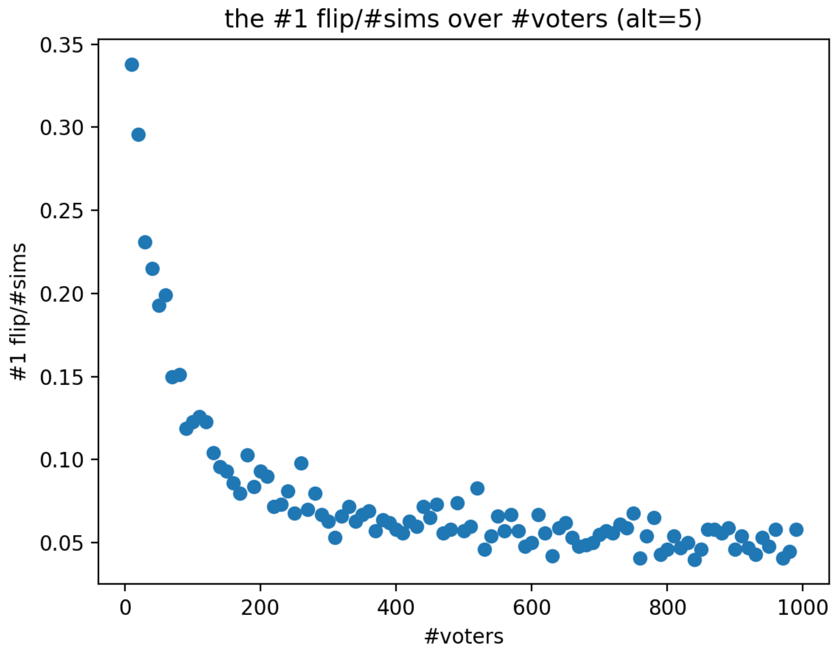

Figure 2: One-flip cases frequency at varying m

Fig. 2 shows the frequency of having only one flip drop exponentially when the number of voters increases. This can be explained. By simulation, the primary reason for having one flip is the presence of tie(s) between two or more row sums in B_1. Once a tie occurs in B_1, it is hard to preserve it after powering, thus, causing the ordering to flip from B_1 to B_2. When the number of voters increases, it is less likely to have identical row sums between rows, making the probability of having one ordering flip drop. Fig. 3 shows the high portion of the one-flip cases caused by tie(s) in B_1. We can see that when the number of voters is small, the percentage is close to 1, which means row sum ties cause almost all cases of a single flip. When the number of voters increases to 1000, the percentage is still above 0.5.

Another thing to notice is that when the number of voters increases, the number of one-flip cases not due to ties in B_1 increases. However, this increase does not compensate for the decrease in tie occurrence, resulting in the overall decreasing trend of one-flip cases out of all the simulations.

4 Simple Majority Voting (SMV)

The several Borda Rules and the Perron rule use magnitude of entries to determine group preference order, so by design, they naturally yield transitive orderings. However, for Simple Majority Voting, transitivity may not hold.

Thus, we propose a condition under which Simple Majority Voting would yield transitive results.

Theorem 1. *For a* $G(A)=\sum_{v=1}^{m} I_v(A)$*, such that:*

1) $G(A)$ *satisfies* $H_{\frac{2}{3}}$*;*

2) for each triple a_i*,* a_j*,* a_l *with relationship* $n_{ij} \geq \frac{2}{3}m$ *and* $n_{jl} \geq \frac{2}{3}m$*, either* $n_{ij} > \frac{2}{3}m$ *or* $n_{jl} > \frac{2}{3}m$*;*

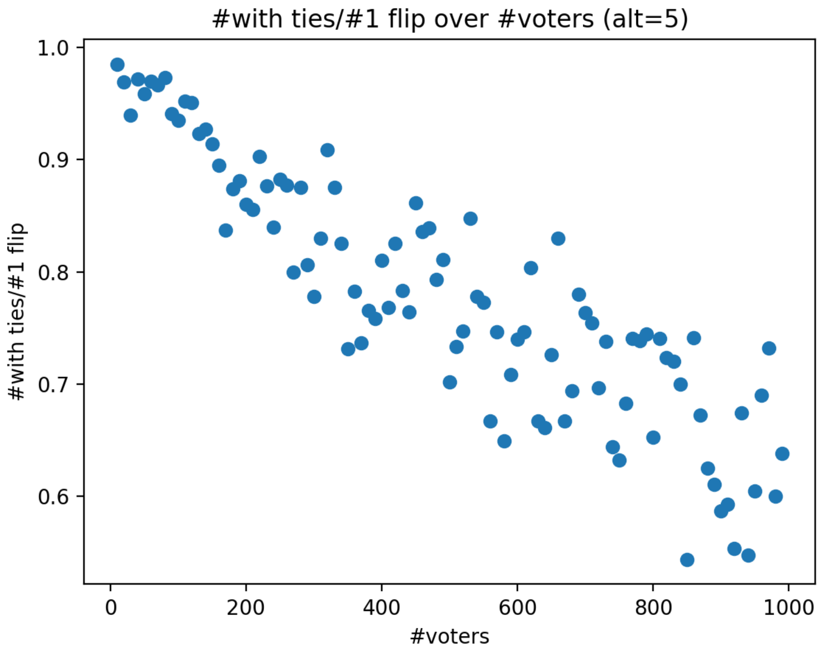

Figure 3: Portion of one-flip cases cause by tie in B_1

3) $I_v(A)$ *is transitive, for* $\forall\ v \in \{1, ..., m\}$*;*
$G(A)$ *is transitive.*

Proof. In the preference relation given by $G(A)$ on A, consider three distinct alternatives a_i,a_j,a_l. Suppose a_iPa_j and a_jPa_l. Per hypothesis, $n_{ij} \geq \frac{2}{3}m$ and $n_{jl} \geq \frac{2}{3}m$, with at least one inequality strict. Intersection of these 2 sets of voters implies that $n_{il} > \frac{1}{3}$. But the prevailing hypothesis then insures that $n_{il} \geq \frac{2}{3}m$ on that the relation $G(A)$ is transitive on a_i,a_j, and a_l, since $G(A)$ is transitive on any triple, it follows that $G(A)$ is transitive. □

Notice that we have proved if the minimum majority of some group preference matrix, $G(A)$, is at least $\frac{2}{3}$, transitivity under SMV follows (assuming other conditions are satisfied). We call this the "2/3+ majorities" condition.

Now we prove the necessity of $\frac{2}{3}$ to ensure transitivity:

We replace the $\frac{2}{3}$ in the statement with x and prove by contradiction.

Consider an arbitrary triple a_i, a_j, a_k with $n_{ij} \geq xm$ and $n_{jk} \geq xm$. $\frac{1}{2} < x < \frac{2}{3}$, so that $n_{ij} > n_{ji}$ and $n_{jk} > n_{kj}$, so a_iPa_j and a_jPa_k. Suppose $n_{ij} > xm$, then, more than $(2x-1)m$ of I_A have $r_{ij} = 1 = r_{jk}$ and $r_{ij} = 0 = r_{kj}$. By transitivity of individual, more than $(2x-1)m$ of I_A have $r_{ik} > r_{ki}$.

That is, $n_{ik} > (2x-1)m$, $n_{ki} < m - (2x-1)m = 2m - 2xm$. We can find a $n_{ki} \in \mathbb{N}^+$, such that $xm \leq n_{ki} < m-(2x-1)m$. By definition, $xm \in \mathbb{Z}$, and so is $m-(2x-1)m$. Thus, $m - (2x-1)m \geq xm + 1$.

MM	[.50,.52)	[.52,.54)	[.54,.56)	[.56,.58)	[.58,.60)	[.62,.64)	[.64,.66)
Transitivity Freq.	0.76530004	0.94875158	0.99401423	0.99965243	1	1	1

Table 1: SMV Transitivity Frequency with Minimum Majority in each interval

That is,

$$m - 2xm + m \geq xm + 1 \tag{1}$$

$$\frac{2}{3} - x \geq \frac{1}{3m} \tag{2}$$

Since $\frac{1}{2} \leq x < \frac{2}{3}$, let $\epsilon = \frac{2}{3} - x, \epsilon > 0$. By Archimedian Property, there exists an $N \in \mathbb{N}$, such that $\frac{1}{N} < \epsilon$. Thus, we can always find such m so that n_{ki} can be greater than xm, and $a_k P a_i$, making $G(A)$ intransitive. □

Thus, for $\frac{1}{2} \leq x < \frac{2}{3}$, if some m and x satisfy $m - (2x-1)m \geq xm + 1$, group preference ordering can intransitive. $\frac{2}{3}$ in the statement is therefore necessary to ensure transitivity in any case of m. If x= $\frac{2}{3}$, the left-hand side of (2) equals zero, and (2) cannot be satisfied for any $m \in \mathbb{N}^+$.

For a fixed pair of m and x, the number of cases of intransitivity for an arbitrary triple depends on how many cases of $n_{ki} \in \mathbb{N}^+$ we can find, such that $xm \leq n_{ki} < m - (2x-1)m$.

Set $k \in \mathbb{N}^+$,

$$xm + k = m - (2x-1)m \tag{3}$$

$$k = 2m - 3xm \tag{4}$$

Thus, there are $(2m - 3xm)$ possible intransitive cases for an arbitrary triple. There are $\binom{k}{3}$ triples in total. Therefore, given m and x, we can find $\binom{k}{3} \times (2m - 3xm)$ $G(A)$'s that give intransitive group orderings.

With simulation, we found that the frequency of SMV yielding transitive results increases with the minimum majority (MM) when fixing the number of voters m. This result is in line with the previous representation of the number of possible intransitive cases.

We consider five alternatives, each assigned to a uniform distribution $\mathcal{U}(0+i,10+i)$. We sample from each distribution and order alternatives based on the magnitude of sampled values. Minimum majority levels are controlled by changing the values of i. An alternative with a bigger i would have a higher probability to be the choice of majority. A greater difference in i between alternatives would result in a higher MM.

Table 1 gives results for $m = 51$. Odd numbers avoid ties between two alternatives. When MM falls in the interval [.50, .52), the transitivity frequency is 0.765. This number increases to 1 when MM gets closer to $\frac{2}{3}$.

5 Comparison of B_p, Perron, and SMV

As mentioned above, transitivity is an intrinsic property of B_p's and Perron but not of SMV. So, we are curious, under cases where SMV yields transitive ordering, whether this order would be the same as that under B_p's or the Perron.

It is not frequent that SMV yields transitive outcomes based on previous simulation setups where we generate individual preferences and sum them into a $G(A)$. Thus, We directly generate $G(A)$ here for simulation efficiency.

We fix the number of alternatives at 5. Each n_{ij}, j > i, is sampled from a uniform distribution $\mathcal{U}$(lower bound × #voters, #voters). Here, we introduce a parameter - majority lower bound (lb) - for the uniform distribution. Each n_{ji} = #voters-n_{ij}. For 1 > lb > 0.5, each entry in the upper triangle of $G(A)$ would be greater than its corresponding entry in the lower triangle. By definition, the SMV ordering would be $a_1Pa_2Pa_3Pa_4Pa_5$, which is transitive. This ordering is equivalent to the other possible orderings, so we do not lose generality by fixing the ordering to $a_1Pa_2Pa_3Pa_4Pa_5$ and simulating in percentage (%).

Fig.4 graphs the frequency of occurrence of each event over increasing lb from 0.5 to 0.98. For each lb, we run 10^5 trials. We observe that even if SMV gives a transitive result, the result may still differ from B_1, B_2, or Perron. A lower bound around 0.68 seems to be a turning point for all the curves to be flatter.

The percentage of (%) B_1, B_2, and Perron all different from SMV is zero over all simulation trials, so at least one of the three agrees with SMV. However, B_1, Perron, and transitive SMV can all yield different results from each other.

With an increasing majority lower bound, Perron, B_1, and SMV gradually converge to yield the same result. However, B_2 seems to be different at 20% from all the others, even when lb passes 0.9. The inclusion of row sums RS^1 as weights in B_2 is what leads to this divergence (see example 3).

Example 3. *Consider* $G(A)=\begin{pmatrix} 0 & 91 & 99 & 94 & 91 \\ 9 & 0 & 93 & 97 & 91 \\ 1 & 7 & 0 & 93 & 90 \\ 6 & 3 & 7 & 0 & 98 \\ 9 & 9 & 10 & 2 & 0 \end{pmatrix}$. *Its Perron vector is* $\begin{pmatrix} 0.81 \\ 0.49 \\ 0.26 \\ 0.17 \\ 0.12 \end{pmatrix}$.

B_1 *vector*$=\begin{pmatrix} 375 \\ 290 \\ 191 \\ 114 \\ 30 \end{pmatrix}$. B_2 *vector*$=\begin{pmatrix} 58745 \\ 34926 \\ 15707 \\ 7397 \\ 8123 \end{pmatrix}$. *The minimum majority is 91. There is a clear preference of* $a_1Pa_2Pa_3Pa_4Pa_5$ *under SMV,* B_1*, and Perron. However, under* B_2*,* a_5Pa_4*. The lower triangle entries in the 5th row have greater values than those in the 4th. With the other alternatives having large majority values, alternative 5 beats 4 via significant weights even when 98 out of 100 voters prefer alternative 4 to 5.*

This example suggests we must be cautious in choosing B_2 as a voting rule when the minimum majority value is large.

6 Conclusion

We expand the classic Borda Rule into a series of vector voting rules. In such a series, by running experiments, we found that B_p ordering converges to the limit–Perron ordering– very quickly. We also notice that all changes in ordering happen consecutively and start at B_2. This provides guidance for how many B_p's we need to look at to know the orderings of the entire series of Borda. When the number of voters is large, the series converges at the start 93% of the time and converges no later than B_2 at 96% of the time. In other words, when there are many voters, looking at the classic Borda (B_1) and the B_2 is generally enough. We then compared Simple Majority Voting (SMV) with B_1, B_2, and Perron. When SMV is forced to be transitive (via 2/3+ majority), it could still give different orderings and winners from B_1, B_2, or Perron. We also noticed that when there is a significant majority (minimum majority > 0.92), B_1, Perron, and SMV will agree. However, B_2 would still disagree around 20% of the time.

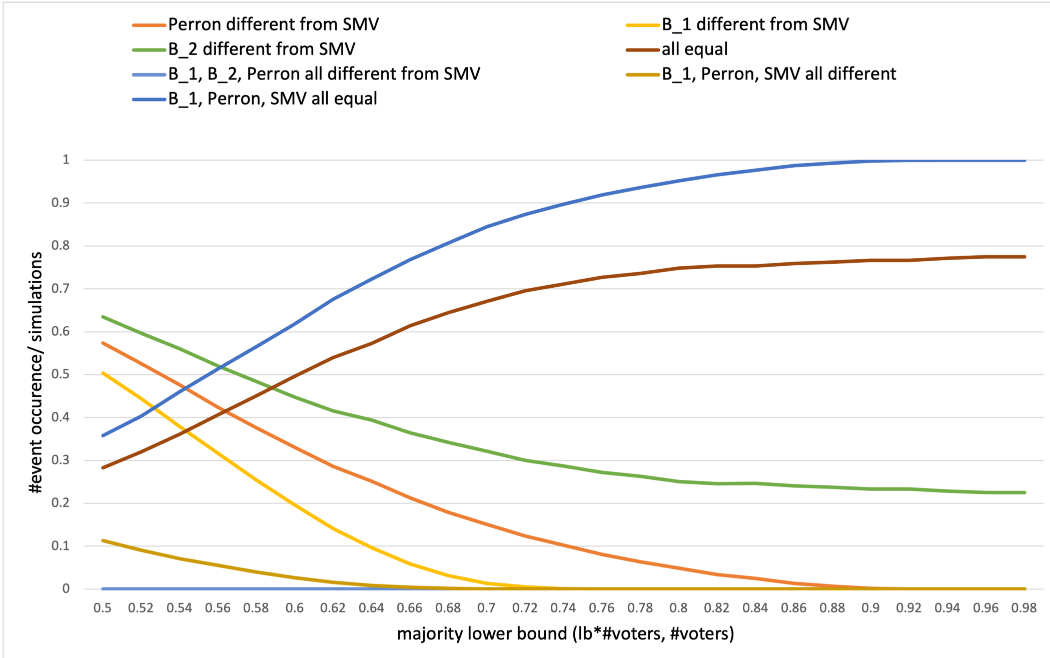

Figure 4: Compare B_1, B_2, Perron to SMV

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