

## Honeycombs, Archimedes, and Soccer Teams

By John Gunnar Carlsson

A fundamental concern in the analysis of logistical systems is the trade-off between localized, independent provision of goods and services versus provision along a centralized infrastructure, such as a backbone network. On one hand, service executed at a local level features the obvious benefits of proximity and specialization, as people and communities obtain things from locations that are close to them. Conversely, aggregating network flows via a backbone network allows individuals and

communities to reap the benefits of economies of scale, economies of agglomeration, and economies of density. This compromise phenomenon is also present in many other fields, such as biology (e.g. the design of circulatory or nervous systems), management science (e.g. a centralized or decentralized authority structure in a company), and sports, as described by the highly influential soccer coach, Marcelo Bielsa (translated from Spanish):<sup>1</sup>

<sup>1</sup> <http://metodobielsa.blogspot.com/p/teoria-bielsista.html>

“[T]his is the great difficulty. To summarize, the more you spread out to shake off your attackers, the more difficult it is to reorder your lineup. And if you don’t spread out and shake off enough, then you don’t let the ball move around fluidly. Do you know what happens then? The players are configured too tightly, they don’t shake anyone off because they all want to be in their defensive positions. While struggling to reorder your lineup, you compromise your goal; but if you don’t risk it, you lose the ball too fast and give it to the rival, who then attacks you.”

One way to model this trade-off is by formulating an optimization problem with two cost components that counteract one another. One might seek to place a collection of landmark points in a region, with one cost component measuring how “spread out” those points are within that region and the other measuring the cost of a backbone network that connects the points together. It is not hard to formulate many simple instances of such problems, and we shall do so presently.

Imagine a large set of landmark points,  $P = \{p_1, \dots, p_n\}$ , that are to be placed in a square  $\mathcal{S}$  of area 1. One way to spread the points out is to minimize the average (Euclidean) distance between a uniformly-sampled point in the square and its nearest landmark point, which we’ll call  $\mathcal{D}_{\text{avg}}(P; \mathcal{S})$ :

$$\text{minimize } \mathcal{D}_{\text{avg}}(P; \mathcal{S}) := \iint_{\mathcal{S}} \min_i \|x - p_i\| dx.$$

It is well known [1] that the configuration of points  $P$  that minimizes this (for large  $n$ ) is a hexagonal lattice, which geog-

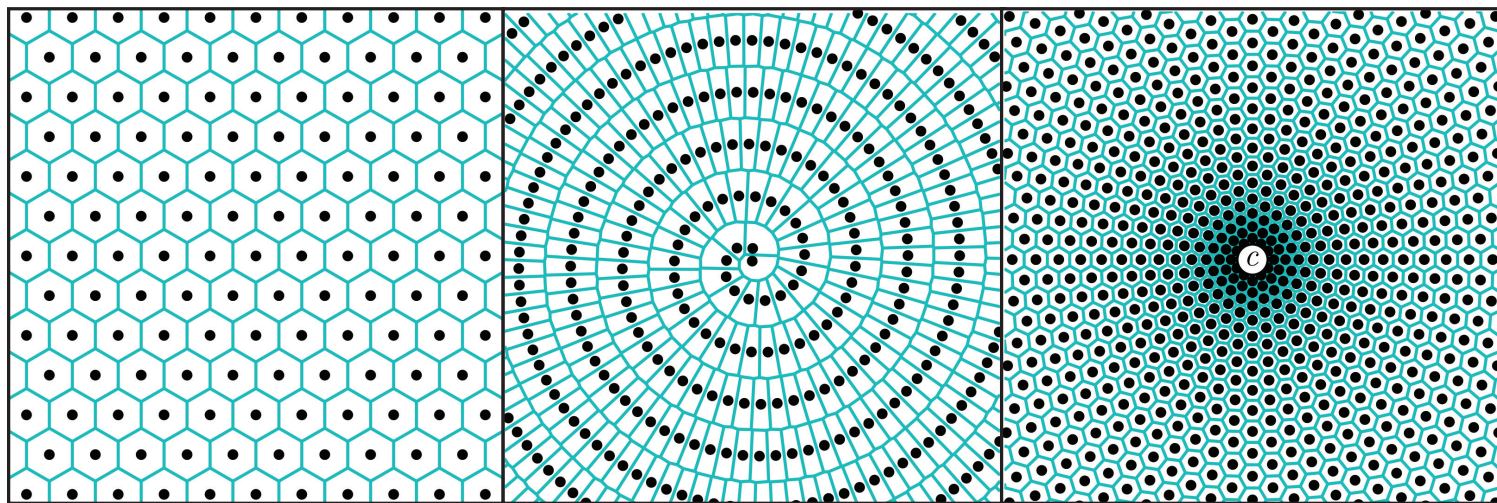


Figure 1a. Honeycomb heuristic.

1b. Archimedes heuristic.

1c. Contracted honeycomb heuristic.

See Honeycombs on page 3

## Mood Maths: Moving Toward a Dynamical Understanding of Bipolar Disorder

By Michael Bonsall

Bipolar disorder (BD) is among the most chronic and severe types of mental illnesses. The disease is a complex neuropsychiatric condition characterized by infrequent but extreme episodes of low (depressed) and elevated (manic) moods. When compared to other mental illnesses, the global health burden of BD is massive: 1-4% of all adults live with this condition, equating to over 17.5 million years lived with the disability (YLDs). Data also indicates that among all mental illness patients, those with BD have the highest rate of suicide.

Treatment for BD has focused principally on the use of lithium, which has been shown to reduce hyperactivity, efficacy of neuron action potential firings, and rescue mitochondria dysfunction. Nevertheless, a more thorough examination of the dynamics of BD is necessary if researchers hope to gain a better understanding of the nuances of this disease.

A rich array of mathematical approaches model the spread of communicable diseases such as influenza, malaria, and dengue fever. However, there is little comprehension of the dynamics of mental illnesses such as bipolar disorder, or whether understanding dynamics might help develop targets for treatment. Given the present interest in dynamics, clearly a role for a deeper mathematical comprehension of the dynamics of BD exists. Our group at the University of Oxford has been using applied maths to develop this idea.

Teaming up with psychiatrists and clinical psychologists, our aim has been to link the observed dynamics of BD, based on noisy patient mood profile data, with mathematics from statistics and dynamical systems theory—colloquially known as ‘mood maths’—to achieve clinically-relevant predictions and a more comprehensive understanding of the disease.

Patients are able to self-report their moods through well-proven and stan-

dardized psychological scoring systems. These types of scales include the Becks Depression Inventory, the Altman Self-Rating Mania Scale, and the Quick-Inventory for Depressive Symptomology (QIDS); each measures different aspects of a patient’s risk of BD. Using data from the QIDS scale, which patients regularly report via smartphone or internet-based technologies, we developed autoregressive time series approaches as novel descriptors of patient mood.

Autoregressive time series models use mood scores at previous points in time as predictors to explain current mood. Mathematically, these time series models necessitate the construction of likelihoods, statistical descriptors relating the hypothesis that model  $M$  generated our observed data  $D$ . Developing these likelihoods required approaches to manage missing values and errors that are not normally distributed. Our likelihoods take the following form:

See Bipolar Disorder on page 3

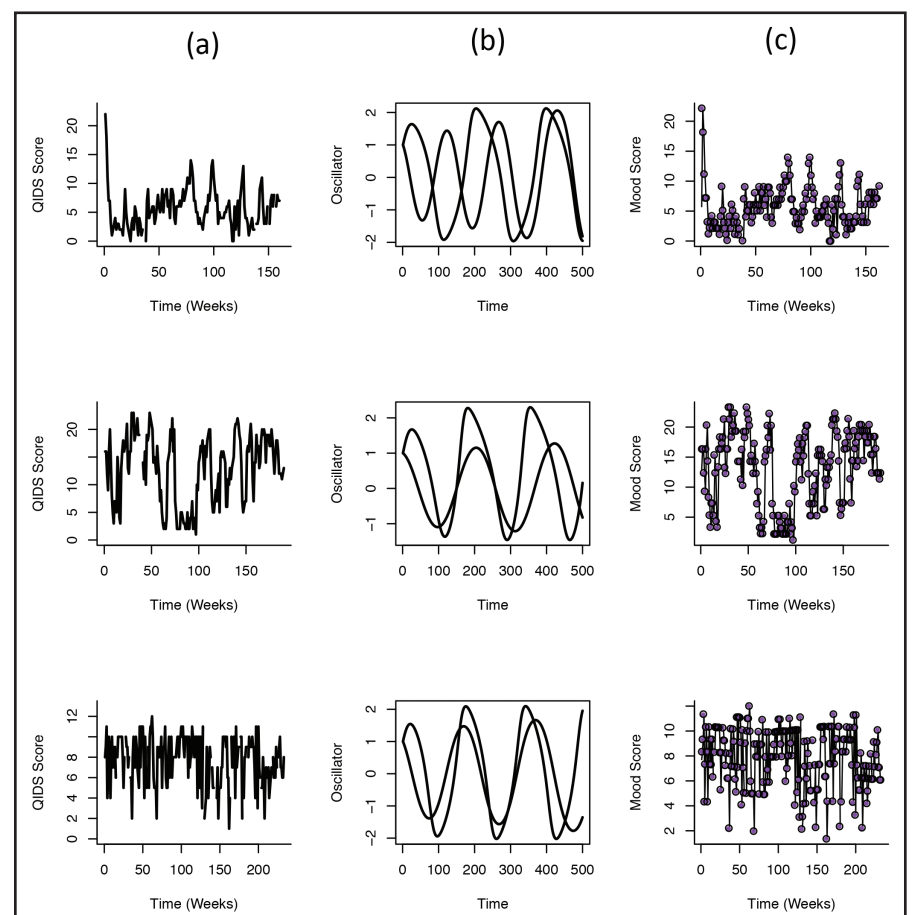


Figure 1. Illustration of predicted relaxation oscillator dynamics for bipolar disorder dynamics. (a) Time series of observed mood scores (based on QIDS scores), (b) predicted independent relaxation oscillator dynamics based on parameters derived from model fit to time series, (c) predicted dynamics based on total derivative (including baseline mood and contribution of oscillator). Adapted from [1].

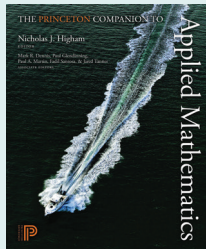
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**4 NSF-IPAM Workshop Tackles Workforce Issues**  
Rachel Levy recaps the Mathematical Sciences Internship Workshop, which called for more internship opportunities to prepare students for a variety of career paths.

**4 SIAM News Again Links Owners of a Copy of Green's 1828 Essay**

**5 Your Companion for All Things Applied Math**  
James Case reviews *The Princeton Companion to Applied Mathematics*. The comprehensive volume features 186 self-contained articles presenting major ideas and branches of applied mathematics.



**6 Pushing the Boundaries of Predictions with Data Analytics**

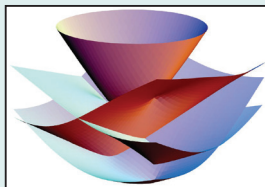
In honor of Math Awareness Month and the "Future of Prediction," Randy Paffenroth discusses how big data, algorithmic predictions, and deep learning are revolutionizing predictive analytics.

**7 Seven Decades of Mathematics and Mechanics**

Recapping a workshop in honor of Jerry Ericksen's 90th birthday, Maria-Carme Calderer and Richard James review his impact on the fields of materials science and nonlinear elasticity, and his work with liquid crystals.

**8 Applied Algebra and Geometry: A SIAGA of Seven Pictures**

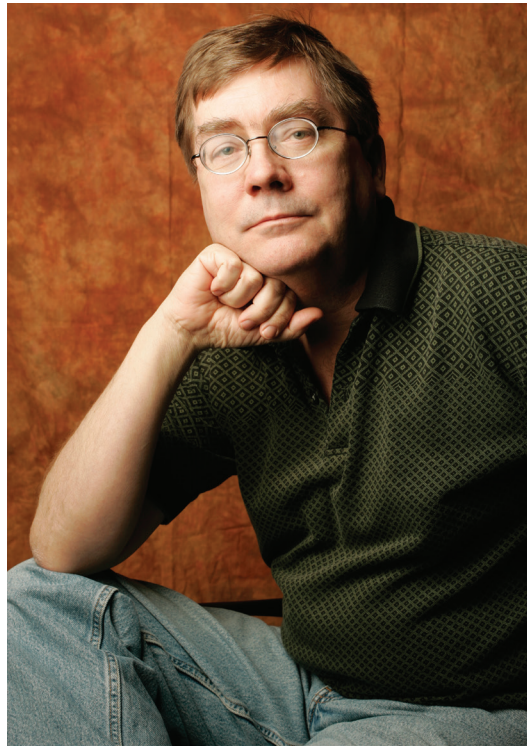
In the first of three installments recognizing the new *SIAM Journal on Applied Algebra and Geometry*, Anna Seigal analyzes two images featured on the journal's cover representing polynomial optimization and robotics.



**7 Professional Opportunities**

## Obituaries

Gerald Farin, beloved mentor and renowned researcher in geometric modelling, passed away on January 14, 2016, at age 62. He was a young person in the 1970s when entering the field of Computer Aided Geometric Design (CAGD), which had been christened in 1974, meaning that he and the field developed simultaneously. His book, *Curves and Surfaces for Computer Aided Geometric Design*—published in five editions—helped establish the subject.



Gerald Farin, 1953-2016. Photo credit: College of Engineering, Arizona State University.

Gerald was born in Angermünde, Germany, on March 20, 1953. His family escaped East Germany just before the Berlin Wall was constructed. With half of the family staying in the East and half moving to the West, Gerald had an exciting family history.

He first encountered applied mathematics while working towards his Ph.D. at the Technical University Braunschweig under the guidance of Professor Wolfgang Böhm, who assigned Gerald to read and report on a paper by Paul de Casteljau, a mathematician with the French automaker Citroën. After earning his degree, Gerald relocated to where the action was in applied mathematics and the new field of computer graphics: the University of Utah. While in Utah in 1978, he and I began a 40-year friendship. We spent a year in the late 1970s in England with Dr. John Gregory, another early pioneer in CAGD. During both this time and a prior year in Utah, Gerald conducted his very important breakthrough research on developing Bernstein-Bezier polynomial patches over arbitrarily-shaped triangles. Smooth sets of triangular patches are essential for interpolation and approximation of real-world data in CAGD and elsewhere, such as with the finite element method for engineering structures.

I'd like to share a little story from that year in England. Gerald bought an ancient Saab automobile, distinguished by the fact that it did not work much of the time. Thus, it was no accident that he went from England back to Germany to work for Mercedes-Benz. He wanted a car that ran!

In 1984, Böhm and I created a journal, *Computed Aided Geometric Design*, published by North-Holland. Gerald became what Böhm called the "undercover editor-in-chief" because he did much of the editorial work. The inaugural issue contains a defining article of CAGD by Gerald, Böhm, and Jurgen Kahmann<sup>1</sup>.

Also in 1984, Gerald met Dianne Hansford, who became his wife in 1992. Dianne was in the same field as Gerald, meaning the two worked and played together. The latter included gourmet cooking, fine wine, running, and traveling.

Gerald played a key role in several CAGD research meetings from 1982 to 1999 at the Oberwolfach Research Institute for Mathematics in Germany. In 1986, he moved with the Mathematics CAGD Group from Utah to Arizona State University (ASU), where he was based for the next 29 years. Soon after, he helped form the SIAM Activity Group in Geometric Design (SIAG/GD), whose first three meetings were held in Tempe, Arizona (near ASU) starting in 1987. Gerald was secretary of SIAG/GD from 1993-1998 and chair from 2002 to 2005. He had been a member of SIAM since 1987.

Gerald also collaborated with Professor Hans Hagen of the University of Kaiserslautern to establish meetings at the Dagstuhl Computer Science Center in Germany. The meetings began in 1991 and continue to the present day.

<sup>1</sup> Böhm, W., Farin, G., & Kahmann, J. (1984). A survey of curve and surface methods. *Computer Aided Geometric Design*, 1(1), 1-60.



(From left to right) Robert Barnhill, Wolfgang Böhm, and Gerald Farin at the third meeting of the SIAG/GD at the Tempe Mission Palms near Arizona State University in 1991. Photo credit: Marigold Linton.

In 1994, Gerald officially became the co-editor-in-chief of the CAGD journal, a post that he held for 20 years. Hartmut Prautzsch, his co-editor during that time, says, "Gerald with his books and long service for the journal has been a personification of the field of CAGD. He had a unique ability to present things simply with a distinct sense of aesthetics, a clear opinion and a certain humor. His e-mails were beautifully minimalistic and to the point."

Gerald loved being a professor. He fully embraced every aspect of this position, with his quality accomplishments in research, student mentoring, and editorial work with journals and meetings.

He is probably the most published person in CAGD and related fields; his record is very impressive with at least 30 books, 100 papers, and many research grants to his name. Several of the books, and many other professional publications, are joint work with his wife Dianne.

Gerald's emphasis on quality showed in his communications, both written and spoken, as well as in his work with students. He supervised some 60 student theses and dissertations, and advised and taught many additional students worldwide. His ability to bring high-level mathematics to a practical level for non-mathematicians was unparalleled. A major example is the ASU Partnership for Research in Spatial Modeling (PRISM), for which Gerald served as co-director. The multidisciplinary consortium includes people from mathematics, computer science, engineering, social science, fine arts, medicine, and other fields.

Gerald was always an exceedingly gracious person. From visiting his parents, Erich and Christel, in Lingen, Germany, his inheritance in grace as well as his inheritance that extended to his perfect bilingual ability in two continents is apparent.

Gerald was a tremendous friend and colleague. He will be missed, but those who knew him and his work will be inspired to even greater achievements. All who knew him salute his many accomplishments, personal and professional, during a life of

unparalleled quality.

Web links to the ASU obituary, video, slide show, and scholarship fund are all available at [www.farinhansford.com/gerald](http://www.farinhansford.com/gerald).

A minisymposium in Gerald's honor will take place at SIAM's 2017 Conference on Industrial and Applied Geometry, which will be held in conjunction with the SIAM Annual Meeting in Pittsburgh, PA, July 10-14, 2017.

*Robert Barnhill is an emeritus professor of computer science at Arizona State University and an emeritus professor of mathematics at the University of Kansas. He is currently Vice President for Science Policy & Strategic Initiatives of the Society for the Advancement of Chicanos & Native Americans in Science. Barnhill is also one of the founders of Computer Aided Geometric Design, having coined the term along with Rich Riesenfeld.*

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## Bipolar Disorder

Continued from page 1

$$L(\mathbf{D}|\mathbf{M}) = \frac{Y_1^{r-1} \left(\frac{r}{\mu_1}\right)^r \exp\left(-\left(\frac{r}{\mu_1}\right)Y_1\right)}{\Gamma(r)}$$

$$\prod_{i=2}^N \frac{Y_i^{r-1} \left(\frac{r}{\mu_i}\right)^r \exp\left(-\left(\frac{r}{\mu_i}\right)Y_i\right)}{\Gamma(r)}$$

where  $r$  is a parameter associated with the underlying probability distribution of the errors between model predictions ( $\mu$ ) and mood observations ( $Y$ ).  $\Gamma(r)$  is the gamma function ( $\Gamma(r) = (r-1)!$ ). Time series likelihoods are based on conditional probabilities: the value now ( $V_t$ ) is dependent on a value at some previous time point ( $V_{t-\tau}$ ). This type of dependency introduces time lag correlations and also requires the development of ways to handle the first observation in a time series. Historically these observations have been excluded from likelihood calculations, but here we condition this observation on the mean of the model predictions.

Based on clinical assessments, patients were grouped into high and low risk (of extreme mood episodes). Across these groups, our time series models highlight different time lag correlation structures in the patients' time series. More correlation lags above and below a threshold were needed to predict and describe the observed dynamics in the 'high-risk' group [2].

But this is only a descriptive approach to BD dynamics, and we really want a more mechanistic link between neurophysiological processes and mood dynamics. Thus, we recently used a 'relaxation oscillator' approach to open up this problem (see Figure 1, on page 1) [1]. As a set of ODEs, relaxation oscillators take the form:

$$\frac{dx}{dt} = y(t) - f(x)$$

$$\frac{dy}{dt} = \frac{-x - a}{b}$$

where  $a$  and  $b$  are parameters and  $f(x) = -x - (x^3/3)$ . This system of equations is based on a van der Pol oscillator, a well-known descriptor for stable oscillating dynamics in electrical engineering. A relaxation oscillator is perfect for linking BD and underlying fluctuating processes (like neuron firing patterns). It has well-known properties, including the characterization of dynamics by periods of low and high states with rapid 'relaxation' between these states. The dynamics can also be stable under certain conditions.

A single relaxation oscillator might be thought to describe the up and down dynamics of low and elevated moods associated with BD. However, our approach has been more nuanced than this; we ask how oscillators (which capture high or low moods) 'relax' from a state of high (or low) mood to a state of average or baseline mood and best capture BD dynamics.

Another aspect of dynamics that is important when modeling BD is the possibility of so-called 'noise-induced instabilities'. Essentially, these are oscillations that occur in a relaxation oscillator and are driven by noise, rather than deterministic changes. This distinction is important because mood patterns in BD patients might simply fluctuate around a baseline (the 'steady-state'), making the relaxation oscillator dynamical repertoire ideally suited for investigating BD dynamics.

We use a total derivative of mood changes through time when linking these relaxation oscillators to mood dynamics. This derivative partitions baseline mood and contributions to mood dynamics from oscillators:

$$\frac{dM}{dt} = \alpha + \beta \frac{dX}{dt}$$

where  $\alpha$  is baseline mood and  $\beta$  is a scalar relating the relaxation oscillator ( $dX/dt$ ) to mood. Using time series methods, we scrutinize the credibility of this model to predict mood dynamics. Specifically, we test many different models where oscillators are coupled, independent, noisy (or not) and conclude that independent oscillators best predict BD mood dynamics across different patients. The departures from the fit of the model to the mood observations are also quite informative. Equally important to our understanding of the dynamics are the levels of baseline mood ( $\alpha$ ) and endogenous noise, and their contributions to changes in mood dynamics. Endogenous noise is identified as stochasticity associated with parameters in our mood model, and the uncertainty between model fit and the data; each individual patient has patterns of this variability.

More recently, we have used simple stochastic processes to understand transitions in mood dynamics before and after non-invasive psychological treatment [3]. Patients recruited to this project were asked to report mood (using the QIDS scoring system) each day for 28 days before and after treatment. The descriptive time series analysis showed differences in the temporal correlations before and after treatment. Before treatment, patients often experience complex mood profiles above and below an average level. After treatment, our time series models predict mood profiles with significantly-reduced volatility.

Categorizing QIDS-based mood scores into three groups (low, medium, and high) for each individual patient allows us to obtain probability transitions (from one day to the next) with simple Markov chains. Using these stochastic process models, we can then solve for the long-term probabili-

ties (for each of the three groupings) before and after treatment.

This simple metric based on the probability of certain 'mood states' (low, medium, or high) has potential for clinical application, particularly if it helps patients understand their mood trajectories more clearly. With greater advances in molecular biology, those involved in 21st century bioscience aspire towards the development of individualized treatments and medicines. Mathematics has a crucial role to play in this; perhaps developing more mathematical approaches to better understand the dynamics of diseases such as BD can set us on a pathway towards this goal.

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Michael Bonsall is a professor of mathematical biology and fellow of St Peter's College at the University of Oxford. He also leads the Mathematical Ecology Research Group in the Department of Zoology. Further details on research in the group are available at: <http://merg.zoo.ox.ac.uk>.

## Honeycombs

Continued from page 1

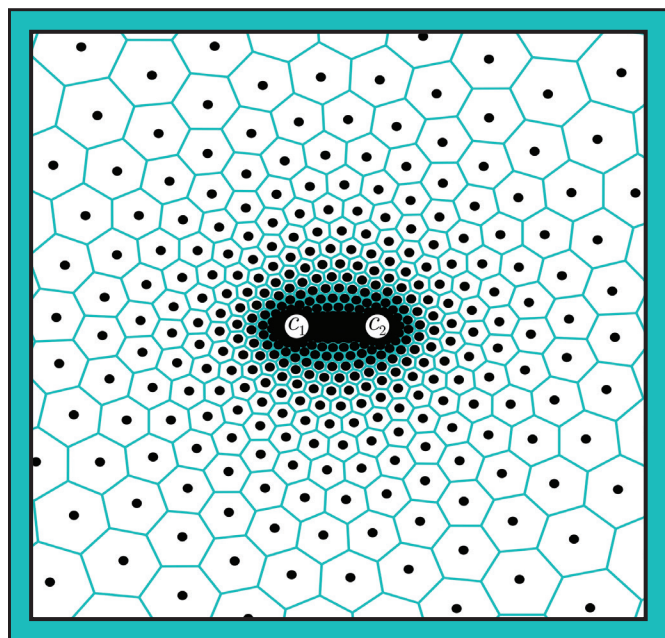


Figure 2. Contracted honeycomb heuristic with two centers.

rappers call the "honeycomb heuristic," as shown in Figure 1a (on page 1).

Let's add a second cost term representing a backbone network; we might try taking a traveling salesman path (TSP) of the landmark points  $P_i$ . This path corresponds to a scenario in which a single vehicle must visit each one of the landmark points (say, to re-supply them). The problem would then look like

$$\text{minimize}_P D_{\text{avg}}(P; S) + \phi \text{TSP}(P),$$

where  $\text{TSP}(P)$  represents the length of the shortest path through the points  $P$  and  $\phi$  is a cost-per-unit-length constant. This problem has many optimal configurations for large  $n$ , and placing the points along an Archimedean spiral, as shown in Figure 1b (on page 1), renders a particularly nice solution. This configuration is also optimal if we take a minimum spanning tree or a Steiner tree of the landmark points instead of a TSP tour.

As another backbone network, maybe we'd like the points  $P$  to be close to a "distribution hub" located in the center  $c$  of the square. So we have to pay a cost of  $\phi \|p_i - c\|$  for each landmark point  $P_i$ , and our problem looks like

$$\text{minimize}_P D_{\text{avg}}(P; S) + \phi \sum_{i=1}^n \|p_i - c\|.$$

The optimal solution takes the form shown in Figure 1c (on page 1), called the "contracted honeycomb." The density of landmark points is proportional to the distance to  $c$  raised to the power of  $-2/3$ .

This "distribution hub" can be generalized in a number of ways. If we have multiple distribution hubs

$c_1, \dots, c_m$  that must be connected to the landmark points, our problem then becomes

$$\text{minimize}_P D_{\text{avg}}(P; S) + \phi \sum_{i=1}^n \sum_{j=1}^m \|p_i - c_j\|,$$

and the optimal density of landmark points is proportional to the sum of the distances to the centers raised to the power of  $-2/3$ . For example, since there are two centers  $c_1$  and  $c_2$  in Figure 2, the density of landmarks is constant along a collection of ellipses. As a final example, we can even consider a hierarchical layout problem in which we have multiple "levels" of landmark points, as suggested in Figure 3a; the optimal layout for network topologies of this kind is shown in Figure 3b.

This kind of framework is extremely general because there are many simple functions that correspond to the two aforementioned costs. For example, one might replace the function  $D_{\text{avg}}(P; S)$  with the maximum distance to a landmark point,  $\max_{x \in S} \|x - p_i\|$ , or maximize the minimum distance between the points,  $\min_{i \neq j} \|p_i - p_j\|$ ; these costs

arise in the  $n$ -centers and  $n$ -dispersion problems respectively. Other choices of backbone networks include complete graphs, Gabriel graphs, Delaunay triangulations, or capacitated vehicle routing problem (VRP) tours. One could even disregard the finite size of  $P$  and consider the problem of constructing a "path" that covers the region efficiently; this is commonly seen in "line-like" location problems and trajectory optimization [2].

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John Gunnar Carlsson is an assistant professor of industrial and systems engineering at the University of Southern California. His research interests include optimization, computational geometry, and transportation, and are supported by grants from NSF, AFOSR, ONR, and DARPA.

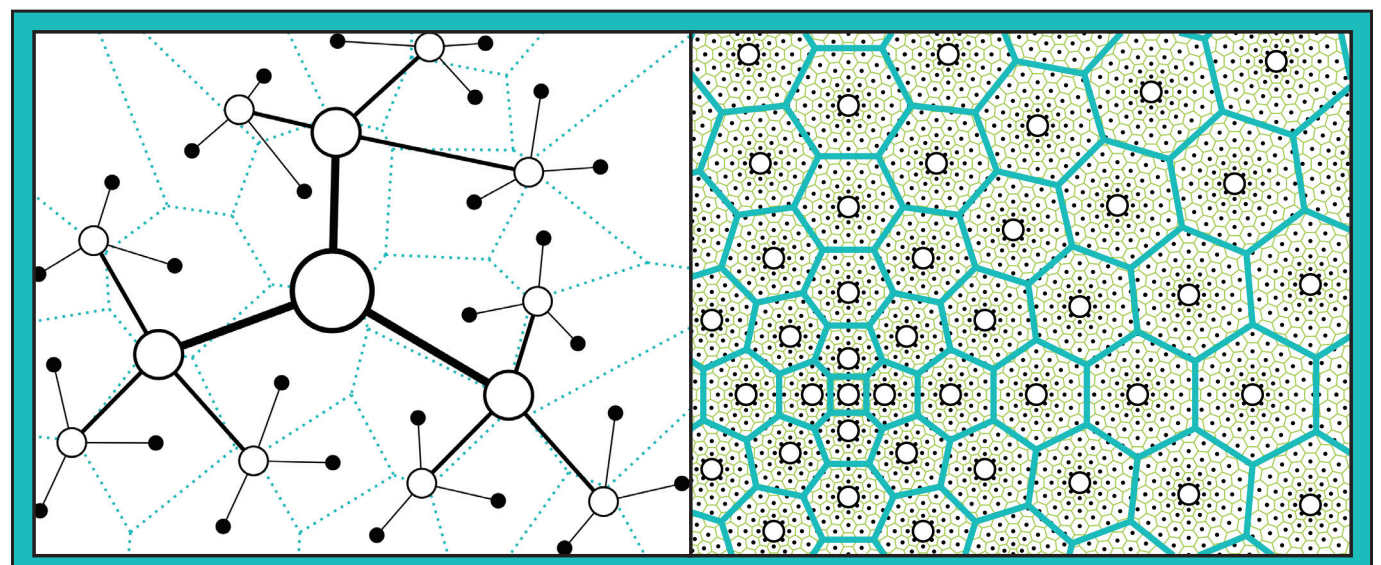


Figure 3a. Multiple "levels" of landmark points.

3b. Hierarchical honeycomb heuristic.

# NSF-IPAM Workshop Tackles Workforce Issues

By Rachel Levy

In the United States, the number of new Ph.D.s in the mathematical sciences (including math, statistics, and biostatistics) has roughly doubled from 1,116 to 1,926 over the past 10 years.<sup>1</sup> Currently, most mathematics Ph.D. programs primarily train students for tenure-track academic positions, though many such positions are no longer tenure-track. At the same time, opportunities in business, industry, and government (BIG) for those trained in the mathematical sciences are growing. As noted by Ram Charan in the February 2015 issue of *Fortune* magazine, “To some degree, every company will have to become a math house. This will require more than hiring new kinds of expertise and grafting new skills onto the existing organization. Many companies will need to substantially change the way they are organized, managed, and led.”<sup>2</sup>

Therefore, the mathematical community faces a significant workforce issue: we need to prepare mathematical sciences students for a variety of career paths, but many

faculty do not have experience in BIG. Internships provide a viable way for students to interact with mentors, gain work experiences, and engage with interesting mathematics problems while pursuing their Ph.D.s. These internships can be regular components of a Ph.D. program, or part of a summer work experience.

Internships were the focus of the NSF-IPAM Mathematical Sciences Internship Workshop held at the Institute for Pure and Applied Mathematics (IPAM) at the University of California, Los Angeles (UCLA) in September 2015. The workshop was organized by Russel Caflisch (UCLA), Alan Lee (Advanced Micro Devices), James L. Rosenberger (Penn State), and Rachel Levy (facilitator, Harvey Mudd College). The diverse group of participants brought perspectives from academic (college/university, public/private), business (large/small), and governmental institutions as well as many areas of the mathematical sciences.

Working groups focused on support, training, logistics, recruiting, and culture. The workshop aimed to accomplish the following: develop recommendations for infrastructure and programs to increase the number of internships targeting mathematical sciences students; open the internship

pipeline to a diverse group of students; offer assistance with timing and logistics for undergraduates, graduate students, and postdocs in pure and applied mathematics; provide training to prepare mathematical sciences students for internships; and develop viable models of how internships best work for mathematical sciences students, postdocs, faculty, and personnel in BIG.

To address the multi-faceted training needs of students, workshop groups recommended a *distributed network internship initiative* with national, regional, and local components.

**National level:** Create a national network to increase internship information exchange, data collection, access, and opportunities:

- Design and implement a data-gathering project to develop the mathematical sciences internship landscape and provide baseline data for new initiatives

- Offer communication and coordination of best practices, training materials and opportunities, models for local programs, and media to aid regional and local outreach efforts

- Build a national network of individuals, companies, government labs, academic institutions, math societies, and mathematical sciences institutes to exchange informa-

tion and increase and advertise internship opportunities

- Develop funding mechanisms and pursue funding for mathematical sciences internship stipends (seed money), internship training, and internship development

**Regional level:** Establish regional internship centers to build internship contacts and organize training opportunities:

- Build internship contacts and opportunities in the region

- Offer centralized training (that could be replicated locally), such as short courses in programming, soft skills, and data

- Hire internship development staff to serve as liaisons between local institutions and potential internship sites and to promote mathematical sciences internships in BIG by communicating how mathematical sciences students make contributions

**Local academic level:** Encourage and enable student participation in internships in mathematical sciences departments:

- Encourage students to pursue training and internships

- Disseminate information from national and regional organizations

- Identify the department chair, director of graduate study, or an interested faculty member to build local institutional mechanisms for internships

To receive information on upcoming internship initiatives, please send your name, institution, and email address to [bigmathnetwork@siam.org](mailto:bigmathnetwork@siam.org). The full NSF-IPAM Mathematical Sciences Internship Report is available on the IPAM website.<sup>3</sup>

Rachel Levy is SIAM VP for Education, as well as an associate professor in the department of mathematics and Associate Dean of Faculty Development at Harvey Mudd College.

<sup>3</sup> <https://www.ipam.ucla.edu/reports/2015-nsf-ipam-mathematical-sciences-internship-workshop-report/>

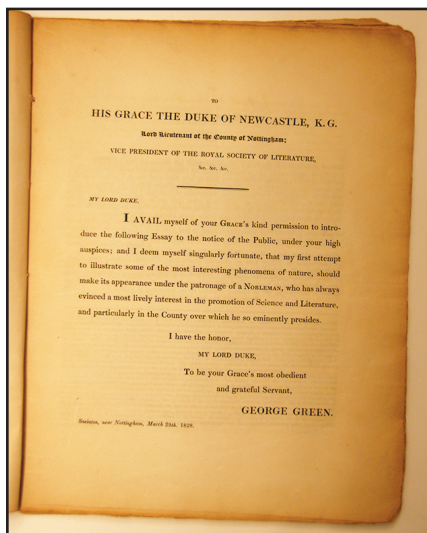


Attendees gather at the NSF-IPAM Mathematical Sciences Internship Workshop.

## SIAM News Again Links Owners of a Copy of Green's 1828 Essay

By Bob O'Malley

George Green was a self-educated miller in Nottingham, England, who wrote *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in 1828. It was published for the author by T. Wheelhouse with 51 local subscribers. The initial number of printed copies is unknown, but is likely no more than 100. Upon the urging of Lord Kelvin, the text was reprinted in *Crelle's Journal* in the 1850s. It contains what we now call Green's Theorem.

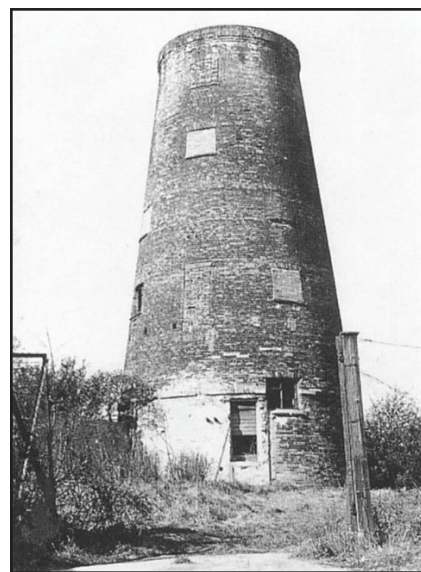


Opening page of George Green's *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Photo credit: Alan Perry.

Mary Cannell, an educator at the then-Nottingham College of Education, spent her retirement learning much about George Green. Cannell wrote a splendid biography entitled *George Green, Mathematician & Physicist 1793-1841*<sup>1</sup> for his bicentennial, which was celebrated at Westminster Abbey. After meeting her, I urged SIAM to publish a second expanded edition (they did in 2001) and wrote a *SIAM News* article<sup>2</sup> in 2000 encouraging applied mathematicians to buy the book.

Three years later, a lawyer—and lover of math and physics books—from North Bergen, NJ, found that article online and emailed me to say that he had a copy of Green's essay. I visited his book-filled home and office and looked through that rare manuscript. He had purchased it along with 1,000 other books in 1978 for \$1,000 from a bookstore in Queens (half the library of a German engineer who died without relatives). I bought the essay from the lawyer in 2007 and wrote a second article<sup>3</sup> for *SIAM News* about my good luck.

Then, in early January of this year, I was contacted by a retired pure mathematician and book collector from London, who had been seeking a copy of Green's essay for 30 years. He had handled one of Kelvin's



George Green's mill in 1975, prior to its restoration. Photo credit: D.M. Cannell, *George Green: Mathematician & Physicist 1793-1841*, SIAM, Philadelphia, 2001.

copies, sold to a London book dealer as part of the Turner Collection at the University of Keele in 1998. He read my 2007 *SIAM News* article and asked to buy my copy. The charming man then flew to Seattle, and left as the owner of the essay.

We can all continue to wonder how often an available copy of Green's essay will turn up, and what it should be worth.

Bob O'Malley, a past president of SIAM, is professor emeritus of applied mathematics at the University of Washington.

<sup>1</sup> London: Athlone Press, 1993.

<sup>2</sup> <https://www.siam.org/news/news.php?id=666>

<sup>3</sup> <https://www.siam.org/news/news.php?id=1236>

## Conference on Topological Data Analysis

The NSF-CBMS 2016 Regional Conference on Topological Data Analysis will be held from May 31 to June 4 at the University of Texas at Austin.

Topological data analysis (TDA) is an emerging field that has generated interest across the disciplines of mathematics, statistics, computer science, machine learning, and electrical engineering. With applications in image analysis, neuroscience, networks analysis, morphology, genetics, and cancer research among other areas, TDA is a heterogeneous field. One of the goals of this workshop is to bring the various disciplines together.

With tutorial and overview talks on TDA, the workshop will provide hands-on data analysis sessions allowing graduate students and junior researchers to obtain exposure to the field as well as opportunities to foster collaborations.

Featuring Duke University's Sayan Mukherjee as the principal lecturer, the conference will have five additional featured speakers including Rabi Bhattacharya, Susan Holmes, Ann Lee, Lek-Heng Lim, and Yusu Wang.

Registration for the conference is open until April 29, 2016. For further details, including a conference program, please visit <https://stat.utexas.edu/training/cbms-2016>.



# Your Companion for All Things Applied Math

**The Princeton Companion to Applied Mathematics.** Ed. Nicholas J. Higham. Princeton University Press, Princeton, NJ, 2015, 1016 pages, \$99.50.

In 2007, Princeton University Press published—to wide acclaim—*The Princeton Companion to Mathematics (PCM)*, edited by Timothy Gowers. The term “companion” indicates that the volume, though broad in its coverage, is less than encyclopedic. The same may be said of *The Princeton Companion to Applied Mathematics (PCAM)*, edited by Nicholas Higham and published in 2015.

PCAM’s 186 self-contained articles—many of them contributed by long-time SIAM members—cover subjects ranging from the obligatory PDEs, ODEs, and numerical analysis to such arcana as “Benford’s Law” and “Modeling a Pregnancy Testing Kit.”

Gil Strang has contributed an elegant discussion of the KKT matrix

$$M = \begin{pmatrix} C^{-1} & A \\ A^T & 0 \end{pmatrix}$$

in which  $A$  is  $m \times n$  and  $C$  is positive-definite symmetric. Such matrices occur in (constrained) numerical optimization problems, finite element analysis, and graph theory, among other such uses. Strang also describes an infinite dimensional analogue in which  $A$  and  $C$  are (differential or integral) operators. Applications invariably exploit the fact that  $K = A^T C A$  is positive-definite symmetric/self-adjoint.

Moving on, Jack Dongarra offers a state-of-the-art overview of high performance computing. His point of departure is a plot, seen in Figure 1, depicting the (very nearly) exponential growth of computing power since the dawn of the computer age. Each black dot represents the performance, in flops per second, of a generation of supercomputers; speed has doubled 46 times since 1950, or about once every 1.4 years!

Petaflop computing ( $10^{15}$  flops per second) was achieved in June 2008, and exascale computing ( $10^{18}$  flops per second) is anticipated in the near future, perhaps as soon as 2020. Such speeds require new standards of fault tolerance; unlike modern personal computers, which may run for weeks or months without rebooting, the current generation of supercomputers must be rebooted every day or two. And since future supercomputers will generate an almost continuous flow of faults, it will no longer be feasible to allow every detected failure to halt whatever applications are running on the affected resources before returning to the most recent checkpoint. Additionally, the current checkpoint/restart technique will not scale to the massively parallel systems of the future because new faults will occur before an application even has a chance to restart. This will throw the machine into a continuous halt/restart cycle. “New fault-tolerant paradigms need to be integrated into both system’s software and user applications,” writes Dongarra.

Unforeseen factors can cause difficulties in supercomputing. In one case, a pseudorandom number generator is used to fill the (lengthy) columns of a square matrix  $A$  and vector  $b$  in the linear system  $Ax = b$ , to be solved by Gaussian elimination. After a run of some 20 hours, the resulting  $x$  was found to be incorrect, due apparently to the non-singularity of  $A$ . Because the number of random elements exceeded the period of the generator, which happened to be a multiple of the dimension of  $A$ , the latter contained identical col-

umns leading to a fatal “zero-pivot,” despite round-off error.

Because modern supercomputers use enormous quantities of energy, their operation can cost upwards of a million dollars per year. Minimizing power consumption is therefore an increasingly important goal of algorithm design, and flops per watt an increasingly popular measure of computer performance. Though none of the recognized problems appear insurmountable, all require attention.

PCAM also features contributions from Barbara Lee Keyfitz, who offers articles on conservation laws and shock waves. The former is particularly interesting, as it explains how conservation laws arise and treats them as an important subclass of quasi-linear PDEs. Keyfitz warns that analysts must strive to identify “function spaces

inclusive enough to admit weak solutions for general classes of conservation laws, but regular enough that solutions and their approximations can be analyzed.”

At present, no fully satisfactory theory exists for systems of conservation laws involving more than a single space variable. Nevertheless, applications abound. The process of paper chromatography (a common method of separating the chemical components of a liquid) is well-modeled by an equation of the form

$$c_x + (f(c))_t = 0,$$

in which  $c(x) = (c_1(x), \dots, c_n(x))$  is an unknown vector of altitude-dependent chemical concentrations, presumably expressed in moles per liter, and  $f$  is an empirically-determined vector-valued function of  $c$ . The equation

$$u_t + (q(u))_x = 0,$$

which expresses a “law of conservation of automobiles,” provides a useful description of single-file traffic flow. Here  $u$  represents the linear density of cars on the road, while  $q(u) = q(u(x))$  represents the flux across a transversal located at position  $x$ .

Irene Fonseca’s article on the calculus of variations is likewise worthy of mention. Following a brief discussion of the isoperimetric problem, the brachistochrone problem, and geodesic lines on a curved surface, Fonseca proceeds directly to the extremal surfaces and hypersurfaces that lie at the forefront of current research. To that end, she formulates the problem

$$\min_{u \in U} F(u) := \int_{\Omega} f(x, u(x), \nabla u(x)) dx, \quad (1)$$

where  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $U$  is a space of functions  $u$  mapping  $\Omega$  into  $\mathbb{R}^d$ ,  $N$  and  $d$  are fixed positive integers,  $\nabla u(x)$  denotes the Shwartzian derivative (gradient) of  $u$ , and  $f = f(x, u, \xi)$  maps

$\Omega \times \mathbb{R}^d \times \mathbb{R}^{N \times d}$  into  $\mathbb{R}$ , before explaining how special cases are currently attacked.

If, for instance,  $f(x, u, \xi) = \sqrt{1 + |\xi|^2}$ , (1) becomes a  $d$ -dimensional minimal surface problem, of the sort solved during the 1930s by Douglas and (independently) by Rado for  $d = 3$ . Or if  $\Omega$  is a compact surface  $S$  embedded in three-dimensional space while  $F = W(S) := \int_S \frac{1}{4} (k_1 + k_2)^2 d\sigma$ , where  $k_1$  and  $k_2$  are the principal curvatures of  $S$ , then  $F$  represents the Willmore (or bending) energy of  $S$ . Willmore energy is minimized by spheres, for which  $W(S) = 4\pi$ . If  $T$  is a torus in 3-space, then  $W(T) \geq 2\pi^2$ . The modern calculus of variations figures prominently in the theories of elastic shells, the surface structure of crystals, and superconductivity, among many other areas of application.

Generally speaking, the articles on such standard topics as complex analysis, solid and fluid mechanics, and Markov chains tend to skim lightly over the basics, in order to bring the reader within striking distance of current research. In contrast, those devoted to the knotting and linking of molecules, mathematical neuroscience, and other less familiar topics tend to be more tutorial in nature.

Finally, a few words about what goes unmentioned would seem appropriate. There are two articles on interval analysis. The first describes the method itself, and the second exposes the techniques by which one may, on occasion, assert not only that a given system  $F(x) = 0$  of nonlinear equations has one of more solutions in each of the vector intervals  $u_i \leq x \leq v_i$ ;  $i = 1(1)n$ , but also that no other solutions exist. Yet neither article mentions that the second of Smale’s 1997 challenge problems—the one concerning the low-dimensional attractor of solutions for the famously chaotic Lorenz equations—was solved in 2002 by an interval-analytic technique applicable to ODEs.

More significantly, PCAM contains no biographical sketches. PCAM includes nearly a hundred of them, chronicling the lives and deeds of mathematicians ranging from Pythagoras, Euclid, and Archimedes, through Fermat, Euler, Gauss, Riemann, Poincaré, and Hilbert, to Gödel, Von Neumann, and Bourbaki.

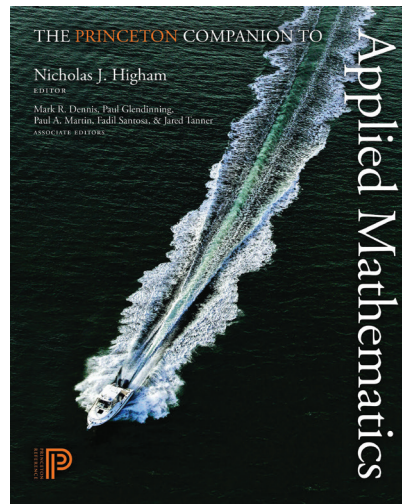
Perhaps surprisingly, PCAM has no systematic discussion of game theory. In an article on his life, PCAM mentions that Von Neumann proved the minimax and perfect information theorems, while PCAM references the subject only in passing. One would never guess that machines now defeat human champions in most board games, including chess, Monopoly, and Scrabble, or that many-player game theory guided the design of the radio spectrum auctions that (as of 2006) had contributed more than \$40 billion to the U.S. Treasury and raised more than \$100 billion abroad.

Nevertheless, the above are mere quibbles. A “companion” cannot include everything, and somebody’s pet topic is bound to be omitted. On the whole, Higham and his associate editors (Mark R. Dennis, Paul Glendinning, Paul A. Martin, Fadil Santosa, and Jared Tanner) have produced an admirably readable and informative volume, which anyone interested in applied mathematics would be well advised to consult or—better still—to own!

James Case writes from Baltimore, Maryland.

## BOOK REVIEW

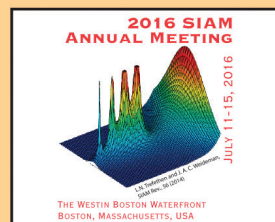
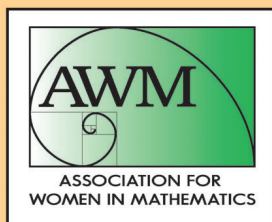
By James Case



*The Princeton Companion to Applied Mathematics.* Edited by Nicholas Higham. Courtesy of Princeton University Press.

## AWM Workshop and Poster Session

Monday, July 11, and Tuesday, July 12, 2016  
at the SIAM Annual Meeting  
Boston, Massachusetts, USA, July 11–15



Talks will focus on Dynamical Systems with Applications to Biology and Medicine

The Poster Session includes all fields

Recent PhDs will join senior women in two minisymposia where they will give 20-minute talks.

Graduate students will present posters in all areas of research.

All mathematicians (female and male) are invited to attend the workshop program, which will also include an informational minisymposium directed at starting a career. Workshop participants will have the opportunity to meet with other women mathematicians at all stages of their careers.

A team of judges will select the top poster. The winner will be offered a prize of a visit to a North American Mathematical Sciences Institute event with basic expenses paid.

For more information, contact the AWM office:  
Phone 703-934-0163 ext. 201 • awm@awm-math.org

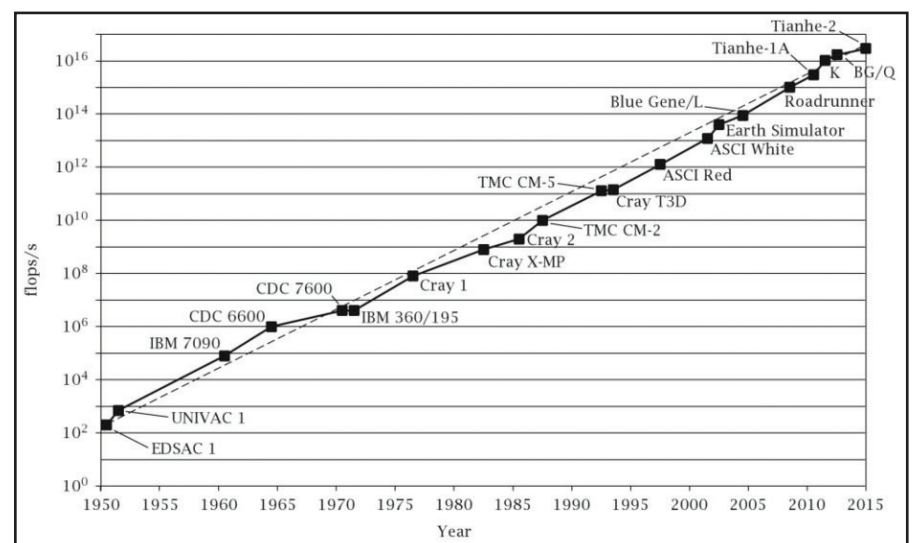


Figure 1. Peak performance of the fastest computer systems over the last six decades (page 840 in PCAM). Courtesy of Princeton University Press.

# A Boat Paradox

Here is a twist on the standard high school boat problem (a topic appropriate for the approaching summer season). In the usual version, a boater walks from stern to bow, causing the boat to slide backward; the question is to determine the sliding distance, ignoring the drag. (The answer is  $\frac{m}{m+M}L$ , where  $L$  is the length of the boat and where  $m, M$  are the masses of the person and of the boat.) In the present “twisted” version, the setting is identical, except that the drag is included and assumed to be linear in the velocity. The drag coefficient  $k$ , as well as the masses  $m, M$ , are given. The boat and the person are initially at rest, and the question remains the same: what is the boat’s eventual displacement from its initial position?

Remarkably, the answer is zero. After the person stops moving, the boat glides, approaching its initial position as  $t \rightarrow \infty$ . The values of  $m, M$  and  $k$  are irrelevant, as long as  $k \neq 0$ .

Here is the proof. Let  $x$  denote the position of the center of mass of the system (boat + person) on the horizontal axis in the figure, and let  $b$  denote the position of the boat, e.g., of its bow. The drag force on the boat is then  $-kb\dot{b}$ , and by Newton’s second law

$$(m+M)\ddot{x} = -kb\dot{b}.$$

In other words, we have

$$\dot{b} = c\ddot{x}, \quad (1)$$

where  $c = (m+M)/k$ , a value that is irrelevant as long as it is finite, i.e. as long as  $k \neq 0$ . Integration by  $t$  gives

$$b \Big|_0^\infty = c\dot{x} \Big|_0^\infty.$$

But  $\dot{x}(0) = \dot{x}(\infty) = 0$  because everything

is at rest before and after, and we conclude that  $b(\infty) - b(0) = 0$ , proving that the boat ends in its initial position, as claimed. And the values of  $m, M$  and  $k$  are immaterial, as long as  $k \neq 0$ .

Figure 1 shows the stages between  $t = 0$  and  $t = \infty$ . The person starts walking forward, causing the boat to start sliding backward. But backward sliding implies the forward drag, and thus the forward

acceleration of the center of mass (this is a bit like starting to walk on ice: the feet slide backward, but the center of mass accelerates forward). To summarize, the boat moves back, but

the center of mass moves forward. Once the person stops, the center of mass—together with the hull—continues to glide forward. In the limit of  $t \rightarrow \infty$ , this gliding exactly restores the earlier backward displacement of the boat.

Mark Levi ([levi@math.psu.edu](mailto:levi@math.psu.edu)) is a professor of mathematics at the Pennsylvania State University.

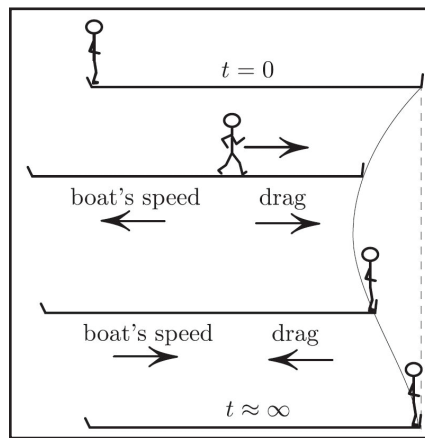


Figure 1. The inevitable return to initial position.

## MATHEMATICAL CURIOSITIES

By Mark Levi

# Pushing the Boundaries of Predictions with Data Analytics In Recognition of Math Awareness Month on the “Future of Prediction”

By Randy C. Paffenroth

From time immemorial, humans have been making predictions. We attempt to predict the weather, elections, disease transmission, market movements, and even the movement of heavenly bodies. A crucial aspect of many, if not all, such endeavors is the observation of the world around us. For example, Danish astronomer Tycho Brahe’s encyclopedic astronomical and planetary observations formed a cornerstone on which the progress of prediction of planetary motion was built. We have come a

long way since Brahe’s time. The amount of data that we gather and store every hour is growing exponentially. Accordingly, the term “big data” has infiltrated many conversations on prediction, and while the phrase is perhaps diluted by its very popularity, the amount of data to which we now have access is truly astounding. In 2010, Google CEO Eric Schmidt observed that we now generate as much data in two days as we did from the start of human history through 2003.<sup>1</sup>

In many ways, the availability of data has outstripped our capacity to make effective predictions based on it. However, in the past decade, much progress in the field of data science has pushed the boundaries of what

was once thought possible. For instance, as I am writing this text, a machine learning-based program called AlphaGo is competing with humans at the highest levels in the strategic board game Go, a feat long thought beyond the reach of computers.<sup>2</sup>

Rather than making predictions based on a particular set of rules, statistical and machine learning techniques allow desired predictions to arise as emergent phenomena in response to data provided to the algorithm [3]. In particular, the exact reasons for the algorithm’s given prediction may be unknown, and perhaps even unknowable, to the scientists

that created it. AlphaGo is almost certainly a much better Go player than any of its creators, making moves that they themselves would not have anticipated. Similarly, many of the data-based predictions made by modern algorithms may be equally opaque to the scientists who designed them. In fact, one feature by which algorithms are judged is the interpretability of their decision-making process. It is interesting to consider whether we can or should trust predictions when the underlying reasons for those predictions are difficult, if not impossible, to understand.

Fortunately, ever since people have been making predictions from data, scientists have directed much effort towards under-

standing the trustworthiness of algorithmic predictions, even without fully comprehending the manner in which they are made. In particular, considerations of “overfitting” and “generalization” play key roles in modern algorithmic predictions [1]. Using such ideas, scientists strive to understand how and why algorithms trained on data we already possess can make credible predictions for observations of the world that haven’t yet been made.

One particularly exciting realm of modern data analysis is that of *feature extraction*. The key idea is that data in its raw form can be quite noisy. Perhaps even more importantly, the features that make for effective predictions are entangled with vast quantities of data that are uninformative or even counterproductive. Consider detecting faces in pictures. The pixel values represent

the sum total of all of the information the picture has to offer, but the prediction of interest (is it a face?) would be greatly facilitated if high-level features (is there a nose?) were available. We can even think of such features in a hierarchical fashion, where low-level features, which are more easily inferred from the raw data, give rise to higher-level features. For example, two dark patches might be nostrils that make up a nose, and a nose can be combined with two ears and two eyes to represent a face. The idea of hierarchical feature extraction from raw data is one of the key elements of *deep learning* [4].

Deep learning is an exciting and active area of research that revolves around the extraction of high-level and, quite often,

See *Data Analytics* on page 8



Interior of a data center. The amount of data that we gather and store is growing exponentially.

<sup>1</sup> <http://techcrunch.com/2010/08/04/schmidt-data/>

<sup>2</sup> [http://www.nytimes.com/2016/03/10/world/asia/google-alpha-go-lee-se-dol.html?\\_r=0](http://www.nytimes.com/2016/03/10/world/asia/google-alpha-go-lee-se-dol.html?_r=0)

The John von Neumann Lecture  
Tuesday, July 12 / 2:30–3:30 P.M.  
Westin Boston Waterfront Grand Ballroom  
2016 SIAM Annual Meeting / Boston, Massachusetts  
— OPEN TO REGISTERED ATTENDEES —

## Satisfiability and Combinatorics

By Donald E. Knuth  
Stanford University, USA



The Satisfiability Problem, which asks whether or not a given Boolean formula can be satisfied for some values of its variables, has long been thought to be computationally hopeless. Indeed, SAT is the well-known “Poster Child” for NP-complete problems.



But algorithmic breakthroughs have made it possible for many important special cases of the problem to be solved efficiently. Industrial-strength “SAT solvers” have become a billion-dollar industry, and they play a vital part in the design of contemporary computers. The speaker will explain how the new SAT technology also helps us solve a wide variety of problems that belong to combinatorial mathematics.



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# Seven Decades of Mathematics and Mechanics

By Maria-Carme Calderer and Richard D. James

On the occasion of Jerald LaVerne Ericksen's 90th birthday, a group of mathematicians and scientists came together in Eugene, Oregon, last October to celebrate accomplishments in the field of mathematics and mechanics. The workshop, entitled "Mathematics and Mechanics in the 22nd Century: Seven Decades and Counting," spanned the seven decades of Ericksen's research career and offered a view of future research prospects.<sup>1</sup>

In the late 1940s, researchers recognized both the inadequacy in existing theory of the description of complex materials and the need for new mathematical concepts and tools. This led to the emergence of modern applied mathematics, in which new concepts in geometry, analysis, partial differential equations, and numerical, computational and stochastic mathematics develop synergistically with molecular and continuum mechanics. The concurrent rise of the fields of materials science and polymer chemistry also contributed to this synergy.



Maria-Carme Calderer, Jerry Ericksen, and Irene Fonseca at the "Mathematics and Mechanics in the 22nd Century: Seven Decades and Counting" workshop held last October.

Due to the works of Ericksen and Ronald Rivlin in the 1950s and 1960s, nonlinear elasticity attracted the attention of the mathematics community. Ericksen's research offered a deep understanding of how invariance principles served to identify special solutions. Two landmark papers from his work during this period deliver explicit classifications of all deformations possible in every incompressible and compressible nonlinear elastic material, regardless of the specific form of the free energy function [5, 7]. Ericksen's work established a fertile mathematical link between the calculus of variations and the stability of materials. But researchers soon realized that the available theory of the direct method at that time was inadequate.

Researchers such as John Ball<sup>2</sup> and Stuart Antman<sup>3</sup> closed this gap in the 1970s, which led to a resurgence of work in the calculus of variations that continues to this day. The study of multiscale problems called for better qualitative methods, and Ennio De Giorgi's [3, 4] method of  $\Gamma$ -convergence assumed a central role. Due to this body of research, we now have a rigorous understanding of the formerly disparate theories of mechanics; this insight increasingly underlies the design and discovery of both new materials and new device concepts [13]. Additionally, research on time-dependent problems led to a deeper understanding of nonlinear hyperbolic partial differential equations, notably represented by the contri-

butions of Constantine Dafermos. Both frequent cross fertilization with closely-related lines of research and a major effort to extend multiscale methods to the atomic scale led to the body of work now associated with the mathematics of materials science.

A vital branch of this research seeks to develop rigorous methods for atomic-to-continuum scales, and encompasses rigorous spatial or temporal averaging, improvement or potential replacement of density functional theory, better understanding of statistical mechanics for solid crystals, and inroads into non-equilibrium statistical mechanics. Given the profound successes of continuum-to-continuum methods, nobody seemed to realize how difficult these atomic-to-continuum problems would be. Nevertheless, they are incredibly important, as they fundamentally underlie the relation between the composition of the material and its properties.

Until the late 1950s, when Ericksen began working on liquid crystals, research in the field was largely confined to a small subset of the chemistry community. At about the same time, the emergence of the Orsay Group, led by Pierre Giles de Gennes (1991 Nobel Prize in Physics) marked the beginning of the study of "soft matter" as a unified field. Ericksen came across liquid crystals while detecting deficiencies in the theories of anisotropic fluids [1]. His article on anisotropic fluids [6] initiated two decades of work by himself, Frank Leslie, and their collaborators, giving rise to the renowned flow theory of liquid crystals.

The equilibrium theory of liquid crystals, formulated decades earlier by Zocher [14], Oseen [11], and Frank [8], prompted an independent path of research within the framework of the direct methods of calculus of variations. Ericksen's arrival at the University of Minnesota forged a link between the analysis and liquid crystal research communities. Ericksen started a graduate course on liquid crystals, through which he met senior mathematician David Kinderlehrer, who became involved in the research himself. Mathematical research on the Ericksen-Leslie equations of liquid crystal flow reemerged, with many challenging problems remaining open to this day. These equations naturally inherit the analytic difficulties of the Navier-Stokes equations and those of the Oseen-Frank energy, in connection with the presence of defects.

A main paradigm in applications of liquid crystals is the ability to control macroscopic size regions of uniform molecular alignment. In display devices, electromagnetic fields achieve this alignment, with the Oseen-Frank energy as the industrial workhorse. The current and next generation of liquid crystal research aims toward the 'topological control' of the systems, with liquid crystal colloids playing a prominent role [2, 12]. These colloids, either in the form of dispersed particles in a liquid crystal or as chromonic phases where the liquid crystal is embedded in an isotropic matrix, offer amazingly rich and novel defect structures. Applying electromagnetic fields to such systems yields new mechanisms to control particle motion and aggregation, with relevant consequences in biological research [10]. Likewise, development of composite materials via patterned liquid crystal substrates is creating a new path in metamaterials research. Furthermore, the discovery of new types of liquid crystals with efficient conversion factors from mechanical to electrical energy currently has a major impact on organic electronics research, including semiconductors and solar cells [9]. These advances open up new synergistic opportunities for the mathematical community to drive technological invention.

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Lev Truskinovskiy (Ecole Polytechnique), Cesare Davini (University of Udine), Richard James (University of Minnesota), Jerry Ericksen (University of Minnesota, emeritus), Kaushik Bhattacharya (California Institute of Technology), and John Ball (University of Oxford).

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Maria-Carme Calderer is a professor of mathematics at the University of Minnesota. Her research is in the areas of applied analysis, liquid crystals, and nonlinear elasticity and gels. Richard James is Distinguished McKnight University Professor in the Department of Aerospace Engineering and Mechanics at the University of Minnesota. He studies phase transformations, multiferroic materials, and an alternative way to think about the structure of matter termed "objective structures."

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<sup>2</sup> <https://people.maths.ox.ac.uk/ball/>

<sup>3</sup> <http://www.math.umd.edu/~ssa/>

# Applied Algebra and Geometry: A SIAGA of Seven Pictures

By Anna Seigal

SIAM's brand-new journal, the *SIAM Journal on Applied Algebra and Geometry*, will feature exceptional research on the development of algebraic, geometric, and topological methods with strong connections to applications. The cover of the new journal shows seven pictures. By describing these pictures and discussing the topics they represent, we hope to give readers a glimpse into the world of algebraic, geometric, and topological problems of interest to applied mathematicians.

This article is part I of a three-part series. Stay tuned for subsequent parts in our next two issues.

## 1. Polynomial Optimization

### The Context

Optimization is a central pillar of applied mathematics, with applications in fields ranging from biology to engineering to finance. Polynomial optimization involves maximizing or minimizing a polynomial function subject to constraints given by polynomial equations. The feasible region, or the set of points in space that satisfy the required constraints, is our geometric object of interest. When considering particular structure on the constraints of an optimization problem, we have corresponding information on the kind of shape the feasible region can take. The boundary of the feasible region is of particular importance, since this is where the optimizing solution will often be found.

The geometry of a feasible region elucidates important aspects of the optimization problem. For example, it may indicate the type of algorithm best suited to maximize a function on that shape. In return, the family of geometrical shapes naturally associated with an optimization problem connects it to the expertise of other areas of mathematics.

Semi-definite optimization is a generalization of linear optimization. Here, we extend from constraints given by positivity conditions on the entries of a vector to the notion of positivity for a matrix. The constraints take the form of positive semi-definiteness of real symmetric matrices. Feasible regions take the form of spectrahedra, which result from intersecting the space of positive definite matrices with a linear space. Semi-definite optimization problems are often valuable stepping-stones in understanding more complicated problems, and can be used as *relaxations* of the harder problems. For more on semi-definite optimization and its connection to algebraic geometry, see [2].

### The Figure

A naturally-occurring feasible region with a nonlinear boundary is the circle

$$(x - u_1)^2 + (y - v_1)^2 = d^2,$$

the collection of points a fixed distance away from the center.

Perhaps instead we want to be a fixed distance away from two points – close to both the train station and the ferry terminal, for example. The collection of points whose

sum of distances from two points is a constant describes an ellipse

$$\left\{ (x, y) \in \mathbb{R}^2 : \sum_{i=1}^2 \sqrt{(x - u_i)^2 + (y - v_i)^2} = d \right\}.$$

For problems involving, say, three factories, five train stations, and a ferry terminal, we want to generalize this notion to the so-called  $k$ -ellipse. This is the set of points whose sum of distances from  $k$  given points is equal to some constant.

We fix the locations of  $k$  focal points  $(u_i, v_i)$  and consider the distance  $d$  to be an unknown. Figure 1 shows the surface

$$\left\{ (x, y, d) \in \mathbb{R}^3 : \sum_{i=1}^k \sqrt{(x - u_i)^2 + (y - v_i)^2} = d \right\}$$

of points for which the sum of the distances from  $(x, y)$  to the focal points is  $d$ . A polynomial equation can describe this constraint, which is the nonlinear boundary of the feasible region in a particular semi-definite optimization problem.

The central convex part of Figure 1 holds the solution to this minimization problem.

Its lowest point is the Fermat-Weber point, which is often sought. If the distance of interest,  $d$ , is fixed, then we take a horizontal slice through the picture and optimize on this slice. The external components are also algebraic solutions to the polynomial constraints.

Figure 1 first appeared in [4], and this version of the image is due to Cynthia Vinzant. It appeared as cover art for the May 2014 issue of the American Mathematical Society's *Notices*.

## 2. Robotics

### The Context

Robotics—the design of mechanical machines to perform complex tasks—is a burgeoning field of study. The range and precision of robots' motion are limited by the mechanics of their constituent parts, which traditionally consist of rigid pieces connected by joints. The interplay between robotics, algebra, and geometry arises naturally from the kinematics of these pieces working together. In fact, numerical algebraic geometry largely arose from these applications. For an introduction, see Chapter 6 of [3].

A rigid part floating unconstrained in three-dimensional space has six degrees of freedom, but the joints of a mechanism restrict its motion. For most joints used in robotics, polynomial equations can describe these restrictions. With multiple

pieces working together, one task is to find the location of a selected terminating part. For example, for fixed locations and angles of your shoulder, elbow, wrist, and carpometacarpal joint, what is the location of your thumb? This is known as *forward kinematics*. Furthermore, given a desired

location of your thumb, what are the possible angles of your arm and hand that would achieve that location? This is *inverse kinematics*.

The numerical algebraic geometry tool of *homotopy continuation* can help answer these questions. In homotopy continuation, we first solve an easier but related set of polynomial equations. Then we deform the easier system to the more challenging problem of interest via a homotopy map. We use the solutions of the easier problem to obtain those of the harder problem by numerically tracking the paths of the original solutions as they deform under the homotopy. For more, see [1].

### The Figure

Figure 2 depicts a special combination of rigid pieces and joints called a *Griffis-Duffy Type 1 Platform*. It consists of two equilateral triangles, one fixed at the base and the other held above it by six rigid legs. Each leg connects a vertex of one triangle to a midpoint of an edge on the other. Although the lengths of each leg are fixed, the angles at each joint are free to move.

The geometry of this problem yields a system of polynomial equations that describes its kinematics: if we fix the point shown on top of the upper triangle, the collection of positions it can reach is the red curve, which is an algebraic set of degree 40.

Figure 2 was created by Charles Wampler of General Motors and Douglas Arnold of the University of Minnesota. It appeared

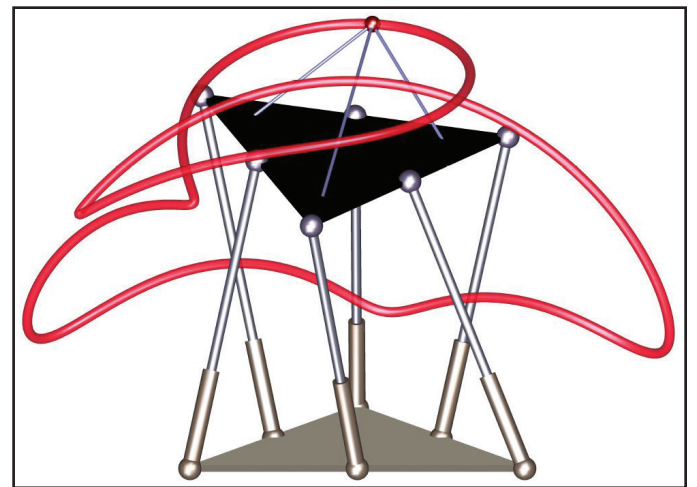


Figure 2. Polynomial movement in robotics.

on the poster for the IMA Thematic Year on Applications of Algebraic Geometry in 2006-07, in which significant progress was made connecting the use of algebraic geometry tools to industrial and applied mathematics.

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Anna Seigal is a graduate student at the University of California, Berkeley working in applied algebra. Her interests lie in tensors and applications to biological systems. She has a particular penchant for writing about mathematics in terms of pictures, and blogs on this subject at <https://picturethis-maths.wordpress.com/>. A version of this article appeared on her blog.

## Data Analytics

Continued from page 6

hierarchical features. "Feature engineering" by human experts has a long history in data analytics, but deep learning and other similar concepts have demonstrated considerable success in automating the unearthing of effective high-level features given vast quantities of raw data. Voice recognition in smartphones, image recognition in web searches, and self-driving cars are all examples of things that affect our daily lives now—or may in the near future—and are supported by advanced predictive techniques that leverage vast quantities of data.

Of course, many open questions regarding the future of prediction from big data remain. Even beyond a myriad of technical queries about the accuracy of predictions and computational efficiency, there are other important points to address [2]. Who owns the vast quantity of data produced every day? Do we really want predictions (accurate or not) of personal information? How will the ability to make predictions based upon new data sources and new algorithms affect society as a whole? These, and many other questions, must be addressed by scientists, policy makers, and the public as we enter a new era of prediction. It is an interesting time to be making predictions from data!

For more on the Future of Prediction, visit the Math Awareness Month website: <http://www.mathaware.org/>.

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Randy Paffenroth is an associate professor of mathematical sciences and associate professor of computer science at Worcester Polytechnic Institute, where he researches large scale dimension reduction problems and compressed sensing. Paffenroth is core faculty in the WPI Data Science Program, and his main area of application is the detection of anomalies in computer networks.

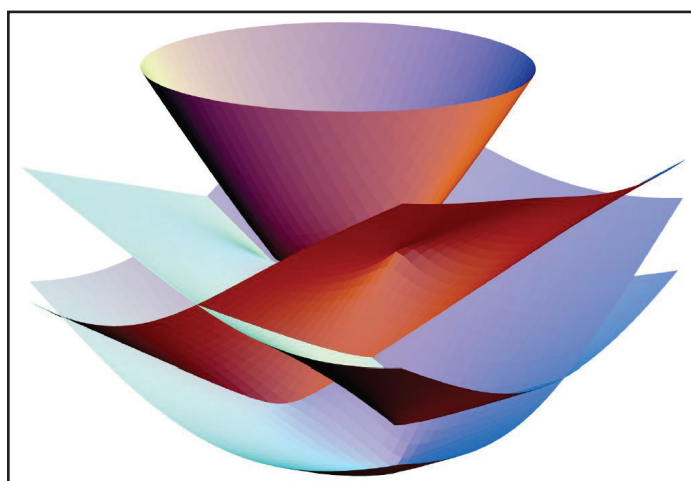


Figure 1. Finding the solution to a polynomial optimization problem.