

Comparison of Vector Voting Rules and Their Relation to Simple Majority Voting

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Abstract

Introduced here are examples of what we call “*vector voting rules*”: social preference orderings deduced from vectors naturally associated with the group preference matrix. These include *higher-order Borda Rules*, B_p , $p = 1, 2, \dots$, and the *Perron Rule* (P). We study the properties of these transitive rules and compare them with *Simple Majority Voting* (SMV). Even when SMV is transitive, it can yield results different from B_1 , B_2 , ... and P, and through simulation, we compile statistics about how often these differ. We also give a new condition (*2/3+ majorities*) that is (just) sufficient for SMV to be transitive and then quantify the frequency of transitivity for graded failures of this hypothesis.

Keywords: Vector Voting; Borda; Perron; Simple Majority Voting.

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1 Introduction

We consider many voters, each of whom has a strictly transitive preference ordering over several alternatives. These may be summarized in a group preference matrix with nonnegative integer entries. The outcome of most traditional voting rules may be deduced from this matrix, and some (weakly) transitive ones result from vectors linear algebraically calculated from it (“vector voting rules”). For example, the classic Borda rule, B_1 , is just the ordering consistent with its row sum vector. Here, we first introduce some natural additional vector voting rules B_p , consistent with the row sum vector of the p -th power, $p = 2, 3, \dots$, “the higher order Borda rules”, and Perron (P), the positive right eigenvector associated with the spectral radius. The former encode information about individual preferences over alternatives highly ranked by other voters, and P is a limiting case. (See also some work of D. Saari [13][14][15] that may be viewed linear algebraically.) In Section 2, we give some necessary background, formal definitions, and examples. In Section 3, we study some properties of B_p , $p = 1, 2, \dots$, and P and examine how frequently, and at which powers, different B_p and P yield different results from the same data. There is remarkable stability. In Section 4, we give a new numerical condition (2/3+ majorities, first noticed by Johnson some time ago) under which simple majority voting (SMV, also deduced from the group preference matrix, but not a vector voting rule) yields transitive results and quantify the number of group preference matrices that give intransitive results when the condition fails. This condition is quite distinct from single-peaked preferences [2]. In Section 5, SMV is compared with B_1 , B_2 , and P. Surprisingly,

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our simulation shows that SMV, even when transitive, can give different results from B_1 , B_2 , and P.

2 Background

Each of our individuals is assumed to have a transitive individual preference ordering (IPO), without indifference between any two alternatives.

For a set of k alternatives $A=\{a_1, a_2, \dots, a_k\}$, an IPO can be any permutation of A. The order of alternatives in an IPO gives the preference of the individual from the most preferred to the least.

For a set of k alternatives $A=\{a_1, a_2, \dots, a_k\}$, an individual v 's preference matrix on A, denoted as $I_v(A)$, has entries $r_{ij} \in \{0,1\}$, $i \in \{1, \dots, k\}$, $j \in \{1, \dots, k\}$, $r_{ij} + r_{ji} = 1$. r_{ij} takes values:

$$\begin{cases} 1 & \iff a_i P_v a_j, i \neq j \\ 0 & \iff a_j P_v a_i, i \neq j \\ 0, & \text{if } i = j \end{cases}$$

This notation, $a_i P_v a_j$, is read as a_i is preferred to a_j by individual v . $I_v(A)$ contains exactly the same information as v 's IPO. The IPO can be deduced from $I_v(A)$.

This 0,1 individual preference matrix is referred to as the *fuzzy preference relations* in the literature. The fuzzy preference relation was first proposed by Blin (1974) to use fuzzy theory in group decision-making and was further discussed by Tanino (1988), among many others [3][16].

Example 1. $A=\{a_1, a_2, a_3\}$. The IPO for individual v on $A=\begin{pmatrix} a_1 \\ a_3 \\ a_2 \end{pmatrix}$. By definition, $a_1 P_v a_3$,

$$a_1 P_v a_2, \text{ and } a_3 P_v a_2. \text{ So the } I_v(A) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

If there are m individuals, the group preference matrix on A is $G(A)=\sum_{v=1}^m I_v(A)$. It has entries $n_{ij} \in \mathbb{Z}$, indicating the number of voters in this group who prefers a_i to a_j . Given a $G(A)$, we can read the group's preference between any two alternatives right off the matrix. $a_i P a_j \iff (n_{ij} > n_{ji}), i \neq j$. The two statements equivalently mean the group prefers a_i to a_j . $a_i I a_j \iff (n_{ij} = n_{ji} = \frac{m}{2})$, which means the group is indifferent between alternative a_i and a_j .

Definition 1. $G(A)$ is transitive if for $\forall i, j, k$, $(a_i P a_j \text{ and } a_j P a_k)$ implies $(a_i P a_k)$, and for $\forall i, j, k$, $(a_i I a_j \text{ and } a_j I a_k)$ implies $(a_i I a_k)$.

Definition 2. $G(A)$ satisfies hypothesis H_t , $t \in [\frac{1}{2}, 1]$, if $\min_{i,j \in \{1, \dots, k\}, i \neq j} [\frac{\max\{n_{ij}, n_{ji}\}}{m}] = t$. Thus, t is determined by the "minimum majority" presented in $G(A)$.

Minimum majority will be an important parameter in our SMV transitivity condition (section 4).

Below we define voting rules considered in this work. We see them as functions that take a set of IPO's via the group matrix $G(A)$ to a social preference relation.

Voting Rule 1. : Simple Majority Voting (Condorcet Method)

A winner under Simple Majority Voting is an alternative that has a majority of votes under pairwise comparison with each of the other alternatives. That is, with some $G(A)$, a_i is the simple majority winner if $n_{ij} > n_{ji}$, for $\forall j \neq i$.

There are IPO's for which a Simple Majority winner (also may be called a Condorcet winner) does not exist. It is possible that the group has a cyclic preference so that no alternative wins the majority votes under all pairwise comparisons. This situation is referred to as the Condorcet Paradox[5][4].

Even if there is a Condorcet winner, SMV may also yield intransitive results. Transitivity is one of the many focuses of studies of SMV. Black (1948) proposed and used the concept of single-peaked preference as a condition ("Black's condition") for transitivity[2]. This condition was also generalized later by Arrow to the "single-peakedness condition", also referred to as the "Arrow-Black's condition"[1]. Jamison (1975) studied transitivity empirically by surveying students on real-world subjects[9]. Gehrlein (1990) calculated the probability of transitivity given the impartial anonymous culture condition (IAC) with a small number of voters and four alternatives[7]. Raffaelli and Marsili (2005) calculated the probability of transitivity with an infinite number of voters given both non-interacting population and interacting population assumptions[11]. Durand (2003) compared the restrictiveness of three conditions for transitivity in Simple Majority Voting, including Black's condition, single-peakedness condition, and Ward's condition[6]. It concluded that under three alternatives, Ward's condition is less restrictive.

Different from the SMV, the following three voting rules are what we called the vector voting rules. They map $G(A)$ to the social order using magnitudes of the entries of a positive vector deduced in a natural linear algebraic way from $G(A)$.

First, for convenience, we denote the i -th row sum of $G(A)$ as $RS_i = \sum_{j=1}^k n_{ij}$.

Voting Rule 2. : Borda's Rule

For an individual, each alternative is assigned a number of counts equal to the number of alternatives it is preferred to. The winner is the alternative with the largest total counts of points summing over all individuals. That is, a_i is the Borda winner for some $G(A)$ if $RS_i \geq RS_q$, for all $q \in \{1, \dots, k\}$. $a_i P_{Borda} a_j \iff RS_i > RS_j$; $a_i I_{Borda} a_j \iff RS_i = RS_j$. $a_i P_{Borda} a_j$ means, under Borda's Rule, a_i is ranked higher than a_j in the social preference ordering. $a_i I_{Borda} a_j$ means Borda gives a tie between a_i and a_j in the social preference ordering.

In a matrix transformation context, Borda's rule right multiplies $G(A)$ with a vector of ones and uses the entries' magnitudes of the resulting positive vector to determine the social preference ordering.

Voting Rule 3. : p -th Higher-order Borda

First order Borda (B_1) refers to the classic Borda Rule in which the group preference ordering depends on the row sums of $G(A)$. p -th order Borda (B_p) takes the row sums of $(G(A))^p$. That is, a_i is the B_p winner if the row sum of the i -th row of $(G(A))^p$ is the greatest among all row sums of $(G(A))^p$.

B_1 only counts the number of times alternative i is preferred to other alternatives in individual preference orderings. B_2 weights the counts by first-order row sums of the other alternatives, so beating more popular alternatives counts for more.

Now, we extend the previous row sum notation and denote the i -th row sum of the p -th order Borda as RS_i^p . $RS_i^1 = \sum_{j=1}^k n_{ij}$.

By matrix multiplication, $RS_i^2 = \sum_{j=1}^k n_{ij} * RS_j^1$; $RS_i^p = \sum_{j=1}^k n_{ij} * RS_j^{p-1}$.

Similarly, B_p right multiplies $(G(A))^p$ with a vector of ones to produce a positive vector, which entries are used to determine the social order.

Voting Rule 4. : Perron Rule

Under the Perron voting rule, the group preference ordering is determined by the magnitudes of the entries of the right Perron vector of $G(A)$. If the i -th entry of the Perron vector has the greatest magnitude, a_i is the winner. Since $G(A)$ is nonnegative by definition, entries of its Perron vector are all nonnegative.

The row sum vector of $(G(A))^p$, with increasing power p , converges to the Perron vector of $G(A)$. So, the group preference ordering under higher-order Borda converges to that under the Perron voting rule. Increasing order Borda is equivalent to the process of the Power Method used to approximate Perron using $[1]_{k-by-1}$ as the initial approximation [8] [10] [12]. Perron can therefore be considered as the Borda of infinite order.

3 Different order Borda Rules and the Perron Rule

We look at Borda and Perron together because of their intrinsic convergence relationship. We are curious about how frequently B_1 would be the same as Perron, so looking at only B_1 suffices; and if not, the ordering would converge at which power.

In the following simulation, we fix the number of alternatives at five and generate preference orderings for each of the m number of voters. So, in each simulation, we have m zero-one matrices I_A representing m individual preference orderings on $A = [a_1, a_2, a_3, a_4, a_5]$. When we sum them, we have a group preference matrix $G(A)$. We calculate B_1, B_2, \dots, B_{10} , and Perron based on $G(A)$ and observe their group preference orderings sequentially.

A flip at power p refers to an ordering change comparing B_p with $B_{(p-1)}$. The power of the last flip, p_{final} , means that the last change of ordering occurs from $B_{(p_{final}-1)}$ to $B_{p_{final}}$, and B_f ordering is the same ordering as Perron for all $f \geq p_{final}$.

For each m , we ran 1000 simulations each as described above.

Each individual preference is generated as follows:

1. For four of the five alternatives, we draw a number from normal distribution $\mathcal{N}(\mu = 1, \sigma^2 = 4)$.
2. Based on the magnitude of the number sampled for each alternative, we construct the preference ordering of the five alternatives.

This construction allows uncertainty among the alternatives with some intrinsic advantage added to one of them so that individuals wouldn't be indifferent between all alternatives.

3.1 Flips in the order of Borda

For our 99000 simulated cases over varying number of voters from 10 to 990, we observed that flips, if there are any, all start at power=2. This suggests that if B_1 and B_2 yield the same ordering, it is highly likely that B_1, B_2, B_3 , and up to Perron would all yield the same ordering. In this case, looking at only the B_1 and B_2 would be sufficient. Another thing to

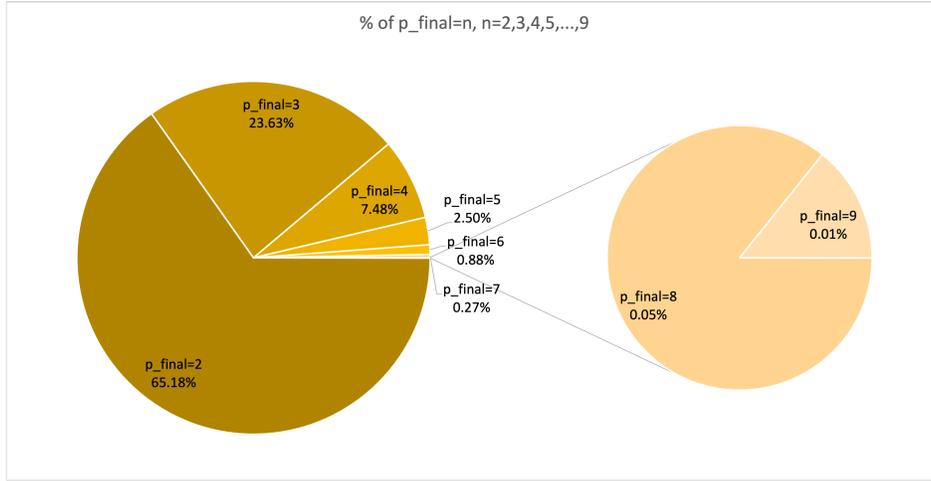


Figure 1: p_{final} distribution over cases with varying number of voters from 10-990

notice is that all flips observed happen consecutively. This suggests that if flips stop at some power p , we can boldly assume that all the following Bq , where $q \geq p$, would all yield the same ordering as the Perron ordering. In other words, the ordering converges to Perron at power p .

When powering, most matrices have B_p ordering converges to Perron ordering fairly quickly. Out of all the cases with $p_{final} \geq 2$, the percentage of cases with $p_{final} = n$ is plotted in Fig. 1. Greater the p_{final} , slower B_p converges to Perron. The $\%p_{final} \geq 7$ is only 0.33%. The observed highest flip occurs at power=9 ($p_{final} = 9$, see example 2).

Example 2. Consider $G(A) = \begin{pmatrix} 0 & 76 & 73 & 68 & 71 \\ 84 & 0 & 71 & 65 & 69 \\ 87 & 89 & 0 & 83 & 78 \\ 92 & 95 & 77 & 0 & 77 \\ 89 & 91 & 82 & 83 & 0 \end{pmatrix}$. Its B_1, \dots, B_{10} , and Perron orderings

are:

$B_1 : [1, 2, 3, 4, 5]$, $B_2 : [2, 1, 3, 4, 5]$, $B_3 : [1, 2, 3, 4, 5]$, $B_4 : [2, 1, 3, 4, 5]$,
 $B_5 : [1, 2, 3, 4, 5]$, $B_6 : [2, 1, 3, 4, 5]$, $B_7 : [1, 2, 3, 4, 5]$, $B_8 : [2, 1, 3, 4, 5]$,
 $B_9 : [1, 2, 3, 4, 5]$, $B_{10} : [1, 2, 3, 4, 5]$, Perron : $[1, 2, 3, 4, 5]$, where "5" indicates the alternative is most preferred, and "1" indicates the alternative is the least preferred by the group. Position i corresponds to alternative i . There are flips in powers $[2, 3, 4, 5, 6, 7, 8, 9]$. The positions of the a_1 and a_2 oscillate.

3.2 Flip Occurrences and the number of voters

Here, we extend the range of the number of voters to 10-99000. It is typical not to have flips when the number of voters (m) is not too small (≥ 20). Starting from $m = 20$, the percentage of having no flips, $\%F_{none}$, is already greater than 60% and continues to increase at a high rate. When m reaches 170, $\%F_{none}$ reaches 88%, while its increase rate drops significantly. Then, the percentage of having no flips converges to 93% with slight fluctuations when m passes 9000. The percentage of having no or only one flip converges to 96%. This result means that with many voters, classic Borda Rule suffices at 93% of the time, and B_2 suffices at 96%, which means B_2 captures almost all the information of the individual preferences.

Since having only one flip is the second most frequent case besides having no flip, we study its change with varying numbers of voters as well.

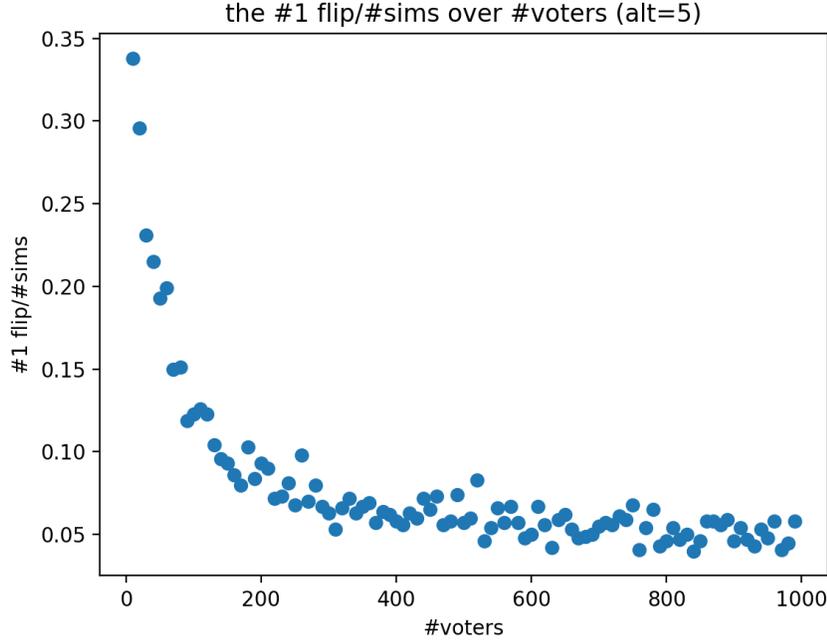


Figure 2: One-flip cases frequency at varying m

Fig. 2 shows the frequency of having only one flip drop exponentially when the number of voters increases. This can be explained. By simulation, the primary reason for having one flip is the presence of tie(s) between two or more row sums in B_1 . Once a tie occurs in B_1 , it is hard to preserve it after powering, thus, causing the ordering to flip from B_1 to B_2 . When the number of voters increases, it is less likely to have identical row sums between rows, making the probability of having one ordering flip drop. Fig. 3 shows the high portion of the one-flip cases caused by tie(s) in B_1 . We can see that when the number of voters is small, the percentage is close to 1, which means row sum ties cause almost all cases of a single flip. When the number of voters increases to 1000, the percentage is still above 0.5.

Another thing to notice is that when the number of voters increases, the number of one-flip cases not due to ties in B_1 increases. However, this increase does not compensate for the decrease in tie occurrence, resulting in the overall decreasing trend of one-flip cases out of all the simulations.

4 Simple Majority Voting (SMV)

The several Borda Rules and the Perron rule use magnitude of entries to determine group preference order, so by design, they naturally yield transitive orderings. However, for Simple Majority Voting, transitivity may not hold.

Thus, we propose a condition under which Simple Majority Voting would yield transitive results.

Theorem 1. For a $G(A) = \sum_{v=1}^m I_v(A)$, such that:

- 1) $G(A)$ satisfies $H_{\frac{2}{3}}$;
- 2) for each triple a_i, a_j, a_l with relationship $n_{ij} \geq \frac{2}{3}m$ and $n_{jl} \geq \frac{2}{3}m$, either $n_{ij} > \frac{2}{3}m$ or $n_{jl} > \frac{2}{3}m$;

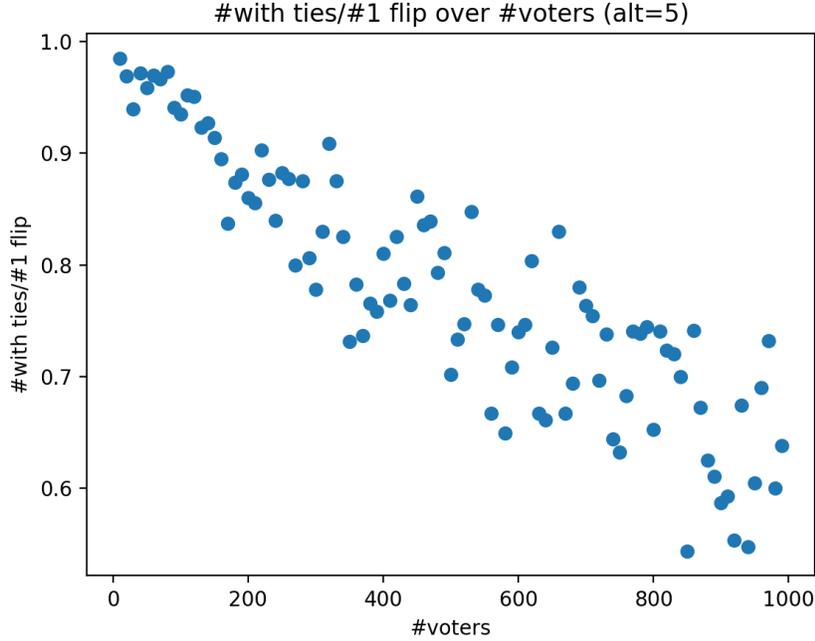


Figure 3: Portion of one-flip cases cause by tie in B_1

3) $I_v(A)$ is transitive, for $\forall v \in \{1, \dots, m\}$;
 $G(A)$ is transitive.

Proof. In the preference relation given by $G(A)$ on A , consider three distinct alternatives a_i, a_j, a_l . Suppose $a_i P a_j$ and $a_j P a_l$. Per hypothesis, $n_{ij} \geq \frac{2}{3}m$ and $n_{jl} \geq \frac{2}{3}m$, with at least one inequality strict. Intersection of these 2 sets of voters implies that $n_{il} > \frac{1}{3}$. But the prevailing hypothesis then insures that $n_{il} \geq \frac{2}{3}m$ on that the relation $G(A)$ is transitive on a_i, a_j , and a_l , since $G(A)$ is transitive on any triple, it follows that $G(A)$ is transitive. \square

Notice that we have proved if the minimum majority of some group preference matrix, $G(A)$, is at least $\frac{2}{3}$, transitivity under SMV follows (assuming other conditions are satisfied). We call this the “ $2/3+$ majorities” condition.

Now we prove the necessity of $\frac{2}{3}$ to ensure transitivity:

We replace the $\frac{2}{3}$ in the statement with x and prove by contradiction.

Consider an arbitrary triple a_i, a_j, a_k with $n_{ij} \geq xm$ and $n_{jk} \geq xm$. $\frac{1}{2} < x < \frac{2}{3}$, so that $n_{ij} > n_{ji}$ and $n_{jk} > n_{kj}$, so $a_i P a_j$ and $a_j P a_k$. Suppose $n_{ij} > xm$, then, more than $(2x - 1)m$ of I_A have $r_{ij} = 1 = r_{jk}$ and $r_{ij} = 0 = r_{kj}$. By transitivity of individual, more than $(2x - 1)m$ of I_A have $r_{ik} > r_{ki}$.

That is, $n_{ik} > (2x - 1)m$, $n_{ki} < m - (2x - 1)m = 2m - 2xm$. We can find a $n_{ki} \in \mathbb{N}^+$, such that $xm \leq n_{ki} < m - (2x - 1)m$. By definition, $xm \in \mathbb{Z}$, and so is $m - (2x - 1)m$. Thus, $m - (2x - 1)m \geq xm + 1$.

MM	[.50,.52)	[.52,.54)	[.54,.56)	[.56,.58)	[.58,.60)	[.62,.64)	[.64,.66)
Transitivity Freq.	0.76530004	0.94875158	0.99401423	0.99965243	1	1	1

Table 1: SMV Transitivity Frequency with Minimum Majority in each interval

That is,

$$m - 2xm + m \geq xm + 1 \quad (1)$$

$$\frac{2}{3} - x \geq \frac{1}{3m} \quad (2)$$

Since $\frac{1}{2} \leq x < \frac{2}{3}$, let $\epsilon = \frac{2}{3} - x, \epsilon > 0$. By Archimedian Property, there exists an $N \in \mathbb{N}$, such that $\frac{1}{N} < \epsilon$. Thus, we can always find such m so that n_{ki} can be greater than xm , and $a_k P a_i$, making $G(A)$ intransitive. \square

Thus, for $\frac{1}{2} \leq x < \frac{2}{3}$, if some m and x satisfy $m - (2x - 1)m \geq xm + 1$, group preference ordering can be intransitive. $\frac{2}{3}$ in the statement is therefore necessary to ensure transitivity in any case of m . If $x = \frac{2}{3}$, the left-hand side of (2) equals zero, and (2) cannot be satisfied for any $m \in \mathbb{N}^+$.

For a fixed pair of m and x , the number of cases of intransitivity for an arbitrary triple depends on how many cases of $n_{ki} \in \mathbb{N}^+$ we can find, such that $xm \leq n_{ki} < m - (2x - 1)m$.

Set $k \in \mathbb{N}^+$,

$$xm + k = m - (2x - 1)m \quad (3)$$

$$k = 2m - 3xm \quad (4)$$

Thus, there are $(2m - 3xm)$ possible intransitive cases for an arbitrary triple. There are $\binom{k}{3}$ triples in total. Therefore, given m and x , we can find $\binom{k}{3} \times (2m - 3xm)$ $G(A)$'s that give intransitive group orderings.

With simulation, we found that the frequency of SMV yielding transitive results increases with the minimum majority (MM) when fixing the number of voters m . This result is in line with the previous representation of the number of possible intransitive cases.

We consider five alternatives, each assigned to a uniform distribution $\mathcal{U}(0+i, 10+i)$. We sample from each distribution and order alternatives based on the magnitude of sampled values. Minimum majority levels are controlled by changing the values of i . An alternative with a bigger i would have a higher probability to be the choice of majority. A greater difference in i between alternatives would result in a higher MM.

Table 1 gives results for $m = 51$. Odd numbers avoid ties between two alternatives. When MM falls in the interval $[\cdot 50, \cdot 52)$, the transitivity frequency is 0.765. This number increases to 1 when MM gets closer to $\frac{2}{3}$.

5 Comparison of B_p , Perron, and SMV

As mentioned above, transitivity is an intrinsic property of B_p 's and Perron but not of SMV. So, we are curious, under cases where SMV yields transitive ordering, whether this order would be the same as that under B_p 's or the Perron.

It is not frequent that SMV yields transitive outcomes based on previous simulation setups where we generate individual preferences and sum them into a $G(A)$. Thus, We directly generate $G(A)$ here for simulation efficiency.

We fix the number of alternatives at 5. Each $n_{ij}, j > i$, is sampled from a uniform distribution $\mathcal{U}(\text{lower bound} \times \#\text{voters}, \#\text{voters})$. Here, we introduce a parameter - majority lower bound (lb) - for the uniform distribution. Each $n_{ji} = \#\text{voters} - n_{ij}$. For $1 > \text{lb} > 0.5$, each

entry in the upper triangle of $G(A)$ would be greater than its corresponding entry in the lower triangle. By definition, the SMV ordering would be $a_1Pa_2Pa_3Pa_4Pa_5$, which is transitive. This ordering is equivalent to the other possible orderings, so we do not lose generality by fixing the ordering to $a_1Pa_2Pa_3Pa_4Pa_5$ and simulating in percentage (%).

Fig.4 graphs the frequency of occurrence of each event over increasing lb from 0.5 to 0.98. For each lb, we run 10^5 trials. We observe that even if SMV gives a transitive result, the result may still differ from B_1 , B_2 , or Perron. A lower bound around 0.68 seems to be a turning point for all the curves to be flatter.

The percentage of (%) B_1 , B_2 , and Perron all different from SMV is zero over all simulation trials, so at least one of the three agrees with SMV. However, B_1 , Perron, and transitive SMV can all yield different results from each other.

With an increasing majority lower bound, Perron, B_1 , and SMV gradually converge to yield the same result. However, B_2 seems to be different at 20% from all the others, even when lb passes 0.9. The inclusion of row sums RS^1 as weights in B_2 is what leads to this divergence (see example 3).

Example 3. Consider $G(A) = \begin{pmatrix} 0 & 91 & 99 & 94 & 91 \\ 9 & 0 & 93 & 97 & 91 \\ 1 & 7 & 0 & 93 & 90 \\ 6 & 3 & 7 & 0 & 98 \\ 9 & 9 & 10 & 2 & 0 \end{pmatrix}$. Its Perron vector is $\begin{pmatrix} 0.81 \\ 0.49 \\ 0.26 \\ 0.17 \\ 0.12 \end{pmatrix}$.

B_1 vector = $\begin{pmatrix} 375 \\ 290 \\ 191 \\ 114 \\ 30 \end{pmatrix}$. B_2 vector = $\begin{pmatrix} 58745 \\ 34926 \\ 15707 \\ 7397 \\ 8123 \end{pmatrix}$. The minimum majority is 91. There is a clear

preference of $a_1Pa_2Pa_3Pa_4Pa_5$ under SMV, B_1 , and Perron. However, under B_2 , a_5Pa_4 . The lower triangle entries in the 5th row have greater values than those in the 4th. With the other alternatives having large majority values, alternative 5 beats 4 via significant weights even when 98 out of 100 voters prefer alternative 4 to 5.

This example suggests we must be cautious in choosing B_2 as a voting rule when the minimum majority value is large.

6 Conclusion

We expand the classic Borda Rule into a series of vector voting rules. In such a series, by running experiments, we found that B_p ordering converges to the limit–Perron ordering– very quickly. We also notice that all changes in ordering happen consecutively and start at B_2 . This provides guidance for how many B_p 's we need to look at to know the orderings of the entire series of Borda. When the number of voters is large, the series converges at the start 93% of the time and converges no later than B_2 at 96% of the time. In other words, when there are many voters, looking at the classic Borda (B_1) and the B_2 is generally enough. We then compared Simple Majority Voting (SMV) with B_1 , B_2 , and Perron. When SMV is forced to be transitive (via 2/3+ majority), it could still give different orderings and winners from B_1 , B_2 , or Perron. We also noticed that when there is a significant majority (minimum majority > 0.92), B_1 , Perron, and SMV will agree. However, B_2 would still disagree around 20% of the time.

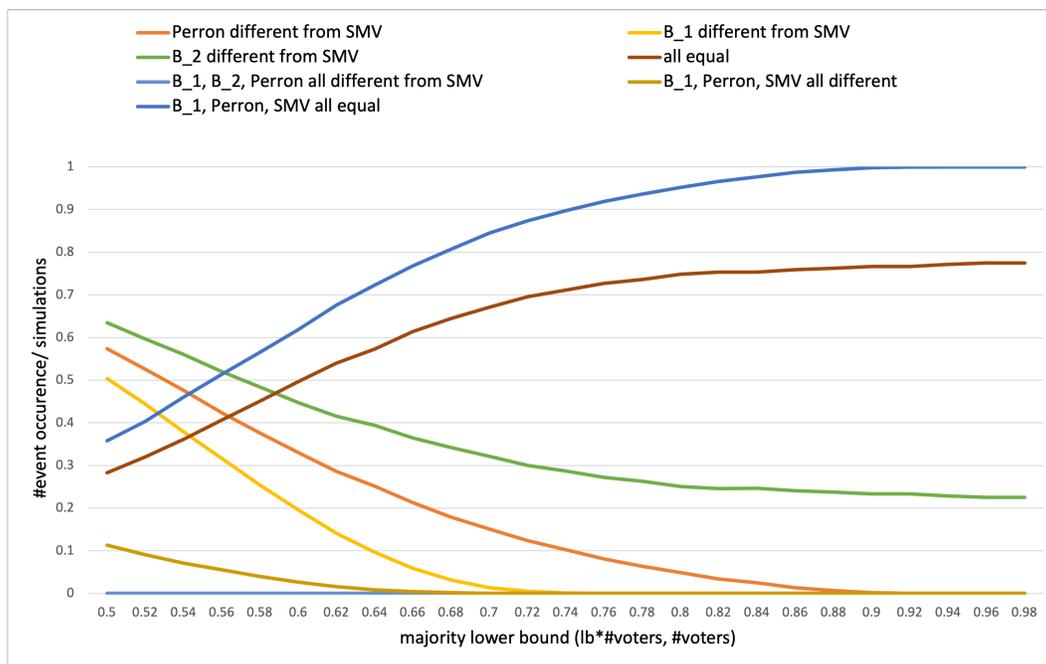


Figure 4: Compare B_1 , B_2 , Perron to SMV

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