Smoothness of the Area Feature Function

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Abstract

We first explain the necessary background and definitions to understand what the area feature function is, what it means for it to be smooth, and why its smoothness is important. Next we find a counter example where the smoothness breaks in the space of functions we call $\mathcal{X}_2$. Then we prove smoothness for a stripe, followed by multiple stripes, then a spot, followed by multiple spots, and then finally the subset of $\mathcal{X}_2$ of functions with periodic boundary conditions.

1 Introduction

Does the area of a sublevel set of a smooth function vary smoothly as you move the function around? In order to understand this question, we must make it more precise, but first it is important to understand why you would want to ask this question in the first place. This question is motivated from [2]. The paper's objective was to essentially develop a way for people to study zebrafish patterns, although they didn't actually use real zebrafish patterns, instead they used a computational model that used reaction diffusion equations to generate the patterns. I worked with Oscar Avalos at an REU at Brown University during the summer of 2024 under the advising of Professor Björn Sandstede to help solidify some of the questions left after working on this paper, one of which was the smoothness of what we call the area feature function.

But before we get to the area feature function, we need to learn how the patterns were generated. First, we create a parameter space of values that we can plug into a reaction diffusion equation, and

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then solving the equation, we get an output solution function $u \in C^2([0,1]^2, \mathbb{R})$. Then we create our pattern by slicing the function at some fixed $c \in \mathbb{R}$ and plotting all the input values (x, y) of u where $u(x, y) \leq c$.

For example, using the function $u(x, y) = x^2 + 2y^2$ and $c = \frac{1}{8}$ this gives us the following plot and pattern:

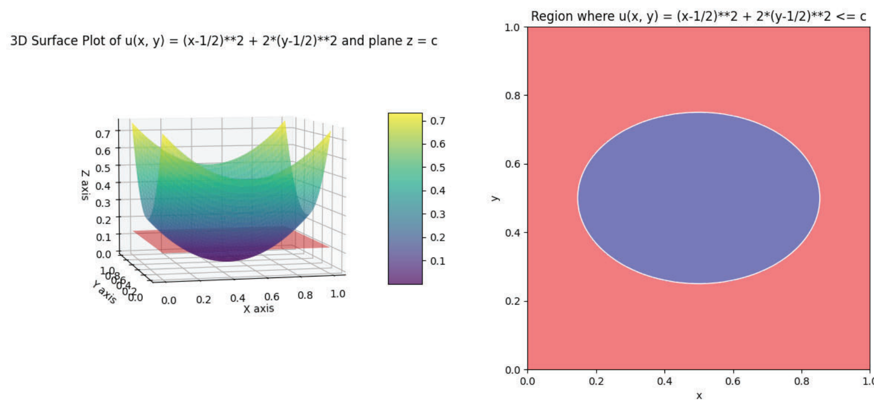


Figure 1 : Example Ellipse Pattern

where the blue region inside the ellipse is the sublevel set i.e. the pattern, which is precisely the subset of $[0,1]^2$ given by $u^{-1}((-\infty, c])$.

This allows us to generate patterns that can model zebrafish patterns. For example, if we use the function $u(x, y) = \sin(6\pi x)\cos(6\pi y) + 1$ and $c = \frac{1}{2}$ we get the following plot and pattern

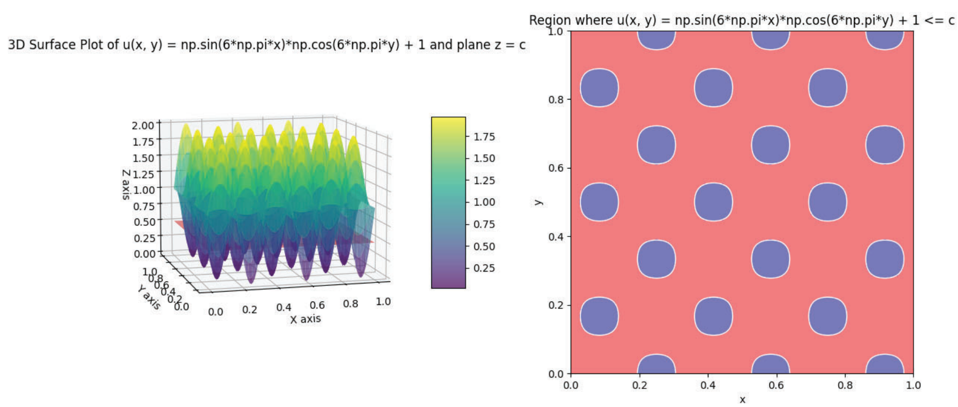


Figure 2 : Example Spots Pattern

which gives spots.

Then if we use the function $u(x, y) = \sin(8\pi x) + 1$ and $c = \frac{1}{2}$ we get the following plot and pattern

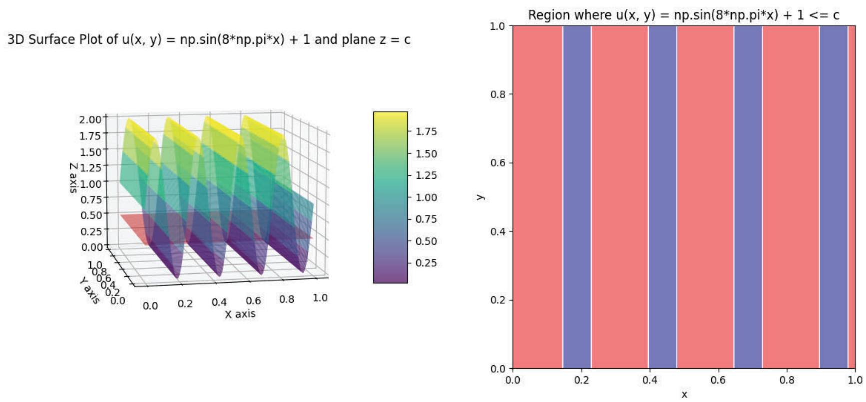


Figure 3 : Example Stripes Pattern

which gives us stripes.

We really only want to get functions $u \in C^2([0,1]^2, \mathbb{R})$ where the resulting pattern gives us a smooth manifold for the boundary of the pattern. This is because spots, stripes, etc, which are the nice patterns we are interested in, will all have this property and functions with this property will also give us nice patterns. So in order for us to guarantee that our function gives us a smooth manifold for the boundary of the pattern, we define the following:

We say that $c \in \mathbb{R}$ is a **regular value** of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ if

$$\boldsymbol{\nabla} u(x) \neq 0 \text{ for all } x \in u^{-1}(\{c\}).$$

We also define

$$\mathcal{X}_2 := \{u \in C^2([0,1]^2, \mathbb{R}) : u^{-1}(\{c\}) \neq \varnothing, \text{ and } c \text{ is a regular value of } u\}.$$

Note that we will keep c fixed throughout this paper, and even though $\mathcal{X}_2$ depends on our choice of c, because it is fixed the entire time, we will not make this dependence explicit in our notation. Also, we want to consider patterns that are not entirely plain, as otherwise the area of the pattern would be locally constant, and thus the smoothness would be guaranteed, so this is why we will only focus on non-uniform patterns where $u^{-1}(\{c\}) \neq \varnothing$.

Then we say that a **pattern** of $u \in \mathcal{X}_2$ is the sublevel set $u^{-1}((-\infty, c])$. By the Regular Level Set Theorem, which is Corollary 5.14 Lee's Smooth Manifolds [1], this will force, at least in the interior of the square, the boundary of the pattern to be a smooth one dimensional submanifold.

Now one of the main purposes of [2] was to compare patterns in order to plot the bifurcation curves that distinguish the qualitative changes between different patterns. See the curves below:

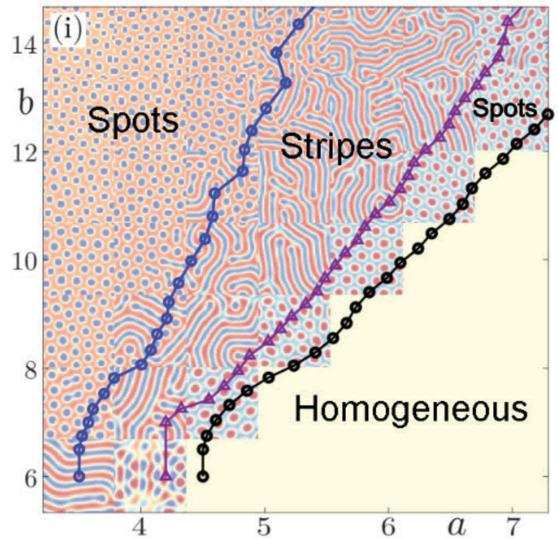


Figure 4 : Bifurcation curve plot (Edited from Figure 7 in [2])

Here they chose to generate the patterns using parameter values $p = (a, b)$ in a square subset of $\mathbb{R}^2$. This graphic was made by putting the patterns generated at certain parameter values on the parameter space near their parameter value to create a square grid that allows us to approximate what the patterns look like for given parameter values. I edited it to add the labels for the regions so it is more clear which is spots and stripes. For example, the homogeneous region is plain patterns, where the function is constant there, and thus we get a pattern that is entirely the same color. To draw the bifurcation curves in this parameter space, they essentially used a Newton's method of stepping and then evaluating the change in the pattern over and over, but in order to do that, they first needed some way to quantify the differences between patterns.

The qualitative nature of different patterns we call features. Examples would be spots, stripes, or spirals. In order to quantify these differences between features, we use what we call feature functions. **Feature functions** act on the functions $u \in \mathcal{X}_2$ used to generate the patterns, and output a mathematical object in a space where we can compare the outputs. An example of a feature function that is actually quite good in spite of its simplicity is counting the number of connected components of a pattern. This allows us to distinguish between spots and stripes, as patterns with stripes often have significantly less connected components than patterns with spots. Note that we call feature functions that act on n-dimensional patterns n-dimensional feature functions.

However, remember that feature functions do not act on patterns directly, instead they act on functions in the space $\mathcal{X}_2$. This means in order to draw the bifurcation curve plot in Figure 4, they had to compose two functions, namely a function from the parameter space into the space of solutions $\mathcal{X}_2$, and then we could act by the feature function. So essentially our composition is given by the diagram below:

$$p \to u \to f(u),$$

where f is what we are denoting as a feature function. Now that they could tell where the patterns differed the most, they would use a Newton's method of evaluating the feature functions locally, seeing where the greatest change occurred in the feature function, and they would step downward in this direction.

However, a Newton's method would only make sense if this composition of functions was smooth.

Standard regularity theory allowed us to conclude the map from the parameter space containing p to our solution function u is a smooth map [2]. However, it was not shown in the paper whether or not certain feature functions they used were smooth. Although counting the connected components was very good, its smoothness didn't make sense globally, as that would imply that the feature function would need to be constant, and so they needed a feature function they could talk about being smooth.

They used what they called a roundness feature function. They would take the area of the pattern, and then divide by its perimeter squared. They also used other feature functions as well, and they conjectured that all of them are smooth because this would align with success of the computational methods used in the paper. However, they did not show whether or not any of the feature functions used were smooth. Thus when we wanted to prove their smoothness in my REU, it made sense to start with what we thought would be the easiest, namely the area of the whole pattern.

Let m_n be the Lebesgue measure on $\mathbb{R}^n$. Precisely, we need the Lebesgue measure feature function, which we will call $f_2 : \mathcal{X}_2 \to \mathbb{R}$ defined by

$$f_2(u) := m_2(u^{-1}((-\infty, c])),$$

to be smooth. From the research we did over the summer, we learned that for one dimensional patterns the length is smooth. However, it was too hard at the time for us to show this in two dimensions, so instead we assumed that f_2 was smooth, and showed that if it was, the roundness feature functions would be smooth, and thus that almost certainly the techniques of drawing these bifurcations was not a fluke but instead is a valid method in this context. Technically more work would need to be done to fully justify smoothness everywhere on the parameter space; however, showing smoothness of the feature functions would allow us to justify that there would be smoothness for most parameter values.

It would be nice if f_2 is smooth on all of $\mathcal{X}_2$, however we will soon learn that it is not smooth on the entire space, but instead a smaller subset of $\mathcal{X}_2$. The exact subset is not presented in this paper; however, we will show in this case that we can guarantee smoothness on the subset of $\mathcal{X}_2$ with periodic boundary conditions, defined on $T = \mathbb{R}^2/\mathbb{Z}^2$ created by identifying the opposite ends of the square $[0,1]^2$. We define this space as

$$\mathcal{X}_T = \{u \in C^2(T, \mathbb{R}) : u^{-1}(\{c\}) \neq \varnothing, \text{ and } c \text{ is a regular value of } u\}.$$

Note that in [2], they use the space $\mathcal{X}_{\text{reg}}$ to denote the space of periodic functions, which is defined on $D = (\mathbb{R}/2\pi\mathbb{Z})^2$, but this is just a rescaled version of the domain of functions T in $\mathcal{X}_T$, and so proving smoothness on $\mathcal{X}_T$ will be equivalent to proving it on $\mathcal{X}_{\text{reg}}$.

In Theorem 3.2, we show smoothness for patterns with a finite number of stripes, and in Theorem 3.3 we show smoothness for patterns with a finite number of spots, which are both strict subsets of $\mathcal{X}_2$. But together, we will learn these are enough to justify smoothness of the feature function f_2 on functions in $\mathcal{X}_T$, which we defined as functions in $\mathcal{X}_2$ with periodic boundary conditions. The key reason that we have smoothness for all functions in $\mathcal{X}_T$ but not all functions in $\mathcal{X}_2$, is because of the role that the boundary of the square $[0,1]^2$ plays in that it allows patterns to vanish or appear without being conserved in $\mathcal{X}_2$, whereas the periodic patterns in $\mathcal{X}_T$ force the pattern to be preserved as it is moved around.

The main structure and results of this paper are as follows: In section 2, definitions are constructed in order to understand what we mean by smoothness. In section 3, we detail the path towards proving smoothness for periodic patterns, while discussing a counterexample in $\mathcal{X}_2$. Precisely, we discuss the counter example when trying to prove smoothness of f_2 on all of $\mathcal{X}_2$ in subsection 3.1, then we prove smoothness for a single stripe in subsection 3.2, followed by multiple stripes in subsection 3.3, then finite spotted patterns in subsection 3.4, and then finally the subset of $\mathcal{X}_2$ of functions with periodic boundary conditions we call $\mathcal{X}_T$ in subsection 3.5.

2 Definitions and Remarks

Let X, Y be Banach spaces. We will use the notation

$$\mathcal{L}(X, Y) := \{L : X \to Y : \text{L is linear}\}.$$

For $A \subseteq \mathbb{R}^n$ let the **characteristic function** $K_A : \mathbb{R}^n \to \mathbb{R}$ be defined

$$K_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

We will use the notation $(-)$ to indicate where an element will be inputted into a function, so for example if we have a function $F : X \times Y \to Z$ then for any $x \in X$ and $y \in Y$ we have that

$$F(-, y)(x) = F(x, -)(y) = F(x, y).$$

We say that the function $F : X \to Y$ is **Fréchet differentiable at $\mathbf{x_0} \in \mathbf{X}$** if there exists a bounded linear transformation $L \in \mathcal{L}(X, Y)$ such that

$$\lim_{x \to x_0} \frac{||F(x) - F(x_0) - L(x - x_0)||_Y}{||x - x_0||_X} = 0,$$

and we define

$$DF(x_0) := L.$$

We say a function $F : X \to Y$ is **Fréchet differentiable** if it is Fréchet differentiable at x_0 for all $x_0 \in X$.

Let $F : X \to Y$ be Fréchet differentiable. Then F is **$\mathbf{C^1}$ or continuously differentiable** if the map

$$DF : X \to \mathcal{L}(X, Y)$$

that takes

$$x_0 \xrightarrow{DF} DF(x_0)$$

is continuous with respect to the corresponding norms for all $x_0 \in X$.

In general, we say that F is **$\mathbf{C^k}$ or k times continuously differentiable** if the map

$$D^k F : X \to \mathcal{L}(X^k, Y)$$

that takes

$$x_0 \xrightarrow{D^k F} D^k F(x_0)$$

is continuous with respect to the corresponding norms for all $x_0 \in X$.

We define the **$\mathbf{C^0}$ norm** for $F \in C^0(X, Y)$ by

$$||F||_{C^0(X,Y)} := \sup_{x \in X} ||F(x)||_Y.$$

We define the **$\mathbf{C^k}$ norm** on $C^k(X, Y)$ for $k \in \mathbb{N}$ to be the complete C^k norm given by

$$||F||_{C^k(X,Y)} := ||F||_{C^0(X,Y)} + \sum_{n=1}^{k} ||D^n F||_{C^0(X, \mathcal{L}(X^n, Y))}.$$

Do note that the definition of C^1 at a point is equivalent to continuity of the partial derivatives, so long as they are continuous in a neighborhood of that point. This is important as we will usually prove a map F is C^1 by showing that the partial derivatives of F are continuous, rather than showing that the Fréchet derivative DF itself is continuous.

3 Smoothness of the Two Dimensional Feature Function

We can now precisely state our goal. We want to show that the area feature function

$$f_2 : \mathcal{X}_{\text{reg}} \to \mathbb{R}$$

is C^1.

It would be nice if f_2 was C^1 on all of $\mathcal{X}_2$; however, this ends up being false as we will see with the following counter example.

3.1 Counter Example for a Particular Function in $\mathcal{X}_2$

After trying to prove f_2 was C^1 on all of $\mathcal{X}_2$ we found the following counter example given by $c = 1$ and

$$u_0(x, y) = x.$$

We then see that the shift operator $\sigma : \mathbb{R} \to \mathcal{X}_2$ defined by

$$\sigma(t)(x, y) = u_0(x, y) + t$$

is C^1 because it is affine, and so we see looking at negative and positive t values for $\sigma(t)$ that we get the following plots and patterns:

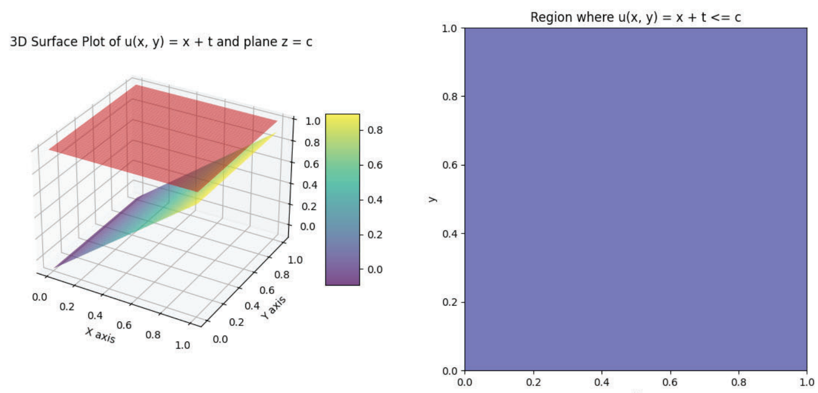


Figure 5 : The graph of the shift $\sigma(t)$ and its pattern for $t < 0$

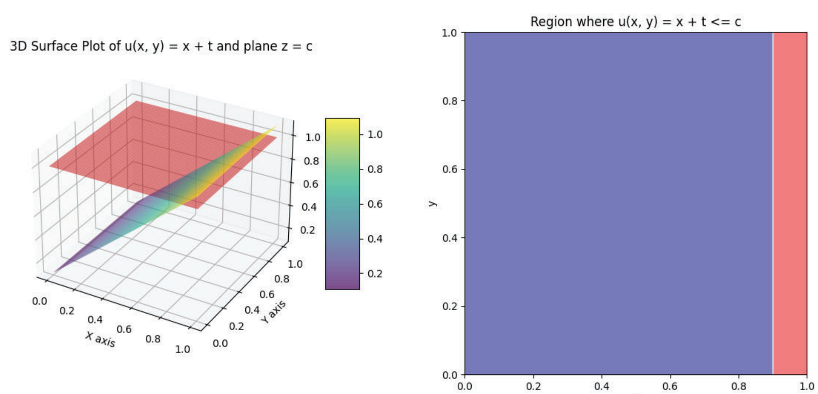


Figure 6 : The graph of the shift $\sigma(t)$ and its pattern for $t > 0$

and thus we can conclude that the composition $f_2 \circ \sigma$ satisfies the following piecewise behavior

$$f_2(\sigma(t)) = \begin{cases} 1 & \text{if } t < 0 \\ 1 - t & \text{if } t > 0 \end{cases}.$$

This then implies that $f_2 \circ \sigma$ is not C^1 at $t = 0$, but because σ is C^1 at $t = 0$, we can conclude that f_2 must not be C^1 at u_0.

Lemma 3.1:
Let $\widetilde{X}$ and Y be Banach spaces and $X \subseteq \widetilde{X}$ be locally convex. We then have that the evaluation map

$$\mathrm{ev}_{X,Y} : X \times C^1(X, Y) \to Y$$

given by

$$\mathrm{ev}_{X,Y}(x, h) = h(x)$$

is C^1.

Proof: Note that we will use the sup norm on $X \times Y$ and by finite dimensional norm equivalence, where we are treating $X \times Y$ as two dimensional, this will imply that $\mathrm{ev}_{X,Y}$ is C^1 with respect to any norm that makes $X \times Y$ a product Banach space.

Let $(x_0, h_0) \in X \times C^1(X, Y)$ we then have that

$$D_1(\mathrm{ev}_{X,Y})(x_0, h_0) = Dh_0(x_0) \in \mathcal{L}(X, Y)$$

as $Dh_0(x_0)$ is bounded by definition of h_0 being C^1, and this is the partial Fréchet derivative as we see that

$$\begin{aligned} &\lim_{x \to x_0} \frac{||\mathrm{ev}_{X,Y}(x, h_0) - \mathrm{ev}_{X,Y}(x_0, h_0) - Dh_0(x_0)(x - x_0)||_Y}{||x - x_0||_X} \\ &= \lim_{x \to x_0} \frac{||h_0(x) - h_0(x_0) - Dh_0(x_0)(x - x_0)||_Y}{||x - x_0||_X} \\ &= 0, \end{aligned}$$

which follows by definition of $Dh_0(x_0)$.

Next we have that

$$D_2(\mathrm{ev}_{X,Y})(x_0, h_0)(h) = \mathrm{ev}_{X,Y}(x_0, h) = h(x_0),$$

as the evaluation map is linear in the second component, and so the partial derivative is given by that linear transformation. This map is bounded as we see that

$$||D_2(\mathrm{ev}_{X,Y})(x_0, h_0)(h)|| = ||h(x_0)|| \leq ||h||_{C^1(X,Y)}$$

so the bounding constant is 1.

Now we will show that each partial derivative is continuous. First we are showing the following map

$$D_1(\mathrm{ev}_{X,Y}) : X \times C^1(X, Y) \to \mathcal{L}(X, Y)$$

is continuous. Let $(x_0, h_0) \in X \times C^1(X, Y)$ we then have given $\varepsilon > 0$ that by the continuity of Dh_0 there exists $\delta_1 > 0$ such that $||x_1 - x_0||_X < \delta_1$ implies

$$||Dh_0(x_1) - Dh_0(x_0)||_{\mathcal{L}(X,Y)} < \frac{\varepsilon}{2}$$

and then letting $\delta = \min(\delta_1, \frac{\varepsilon}{2})$ it follows that $||(x_1, h_1) - (x_0, h_0)||_{X \times C^1(X,Y)} < \delta$ implies

$$\begin{aligned}
||D_1(\mathrm{ev}_{X,Y})(x_1, h_1) - D_1(\mathrm{ev}_{X,Y})(x_0, h_0)||_{\mathcal{L}(X,Y)}& \\
&= ||Dh_1(x_1) - Dh_0(x_0)||_{\mathcal{L}(X,Y)} \\
&\leq ||Dh_1(x_1) - Dh_0(x_1)||_{\mathcal{L}(X,Y)} + ||Dh_0(x_1) - Dh_0(x_0)||_{\mathcal{L}(X,Y)} \\
&\leq ||h_1 - h_0||_{C^1(X,Y)} + ||Dh_0(x_1) - Dh_0(x_0)||_{\mathcal{L}(X,Y)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Then to show continuity for

$$D_2(\mathrm{ev}_{X,Y}) : X \times C^1(X,Y) \to \mathcal{L}(C^1(X,Y), Y)$$

we let $\varepsilon > 0$ and $(x_0, h_0) \in X \times C^1(X,Y)$. Then let $\delta_1 = \varepsilon$ and using the local convexity of X, we know there exists $\delta_2 > 0$ such that for all $x \in \widetilde{X}$ where $||x - x_0||_X < \delta_2$ it follows that

$$x_0 + (x - x_0)t \in X \text{ for all } t \in [0, 1].$$

Now we can let $\delta = \min(\delta_1, \delta_2)$ and then we have that

$$||(x_1, h_1) - (x_0, h_0)||_{X \times C^1(X,Y)} < \delta$$

implies, using the local convexity of X to apply the Fundamental Theorem of Calculus and integrate along the linear path in X given by $x_0 + (x - x_0)t$ from $t = 0$ to $t = 1$, that we have

$$\begin{aligned}
&||D_2(\mathrm{ev}_{X,Y})(x_1, h_1) - D_2(\mathrm{ev}_{X,Y})(x_0, h_0)||_{\mathcal{L}(C^1(X,Y),Y)} \\
&= \sup_{||h||_{C^1(X,Y)} \leq 1} ||D_2(\mathrm{ev}_{X,Y})(x_1, h_1)(h) - D_2(\mathrm{ev}_{X,Y})(x_0, h_0)(h)||_Y \\
&= \sup_{||h||_{C^1(X,Y)} \leq 1} ||h(x_1) - h(x_0)||_Y \\
&= \sup_{||h||_{C^1(X,Y)} \leq 1} \left|\left| \int_{t=0}^{1} \frac{d}{dt}(h((x_1 - x_0)t + x_0)) dt \right|\right|_Y \\
&= \sup_{||h||_{C^1(X,Y)} \leq 1} \left|\left| \int_{t=0}^{1} Dh((x_1 - x_0)t + x_0)(x_1 - x_0) dt \right|\right|_Y \\
&\leq \sup_{||h||_{C^1(X,Y)} \leq 1} \int_{t=0}^{1} ||Dh((x_1 - x_0)t + x_0)||_{\mathcal{L}(X,Y)} \cdot ||(x_1 - x_0)||_X dt \\
&\leq \sup_{||h||_{C^1(X,Y)} \leq 1} \int_{t=0}^{1} ||h||_{C^1(X,Y)} \cdot ||(x_1 - x_0)||_X dt \\
&\leq ||(x_1 - x_0)||_X \\
&< \delta \\
&= \varepsilon.
\end{aligned}$$

Thus we can conclude that $\mathrm{ev}_{X,Y}$ is C^1. ■

3.2 Smoothness for a Single Stripe

Recall that our goal is to show that the area feature function $f_2 : \mathcal{X}_{\mathrm{reg}} \to \mathbb{R}$ is C^1.

We have by definition of integration of characteristic functions that

$$f_2(u) := m_2(u^{-1}((-\infty, c])) = \int_{v\in[0,1]^2} K_{u^{-1}((-\infty,c])}(v)dm_2.$$

Next we will apply Fubini's Theorem from [3]:

Fubini's Theorem : Let $(X, A, \mu), (Y, B, \nu)$ be two measure spaces and ν be complete. Let f be integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$. Then for almost all $x \in X$, the x-section of f, $f(x, -)$, is integrable over Y with respect to ν and

$$\int_{X\times Y} fd(\mu \times \nu) = \int_{x\in X} \left(\int_{y\in Y} f(x, y)d\nu(y) \right) d\mu(x).$$

The conditions for Fubini's Theorem are satisfied in our case, as first the Lebesgue measure is complete on $\mathbb{R}$. Then the characteristic function $K_{u^{-1}((-\infty,c])}$ is integrable on $[0, 1]^2$ because $u^{-1}((-\infty, c])$ is defined over a closed set, so because closed sets are measurable, this function will be measurable, and because it is non-negative, the integral of its absolute value is just its integral, which is finite as

$$\int_{v\in[0,1]^2} K_{u^{-1}((-\infty,c])}(v)dm_2 \leq 1 < \infty.$$

So applying Fubini's Theorem gives us

$$f_2(u) = \int_{x=0}^{1} \int_{y=0}^{1} K_{u^{-1}((-\infty,c])}(x, y)dydx.$$

Thus this allows us to justify integrating the length of the slices of our pattern for each $x \in [0, 1]$.

Now with all of these complexities that arise in two dimensions we decided to try and prove that f_2 is C^1 for a particular function $u_0 \in \mathcal{X}_2$. We decided that we would let $c = 1$ and define

$$u_0(x, y) = 2y$$

where the following plot and pattern we get is:

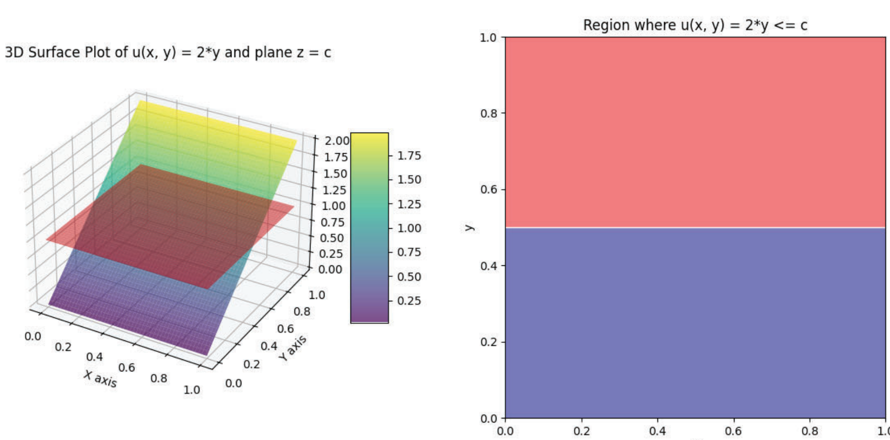


Figure 7 : The Rectangle Pattern

The construction for how we will show f_2 is C^1 at u_0 is as follows. We can treat the curve between the blue and red regions as a function of x that outputs the corresponding y value, and then use the Fubini Theorem and get the area of the blue region by integrating this function of x i.e. the curve. However, the feature function does not act on this curve, but instead the functions u locally around u_0. So what we need is a function that takes in a u locally around u_0 and spits out this continuous curve as a function of x.

Precisely we need an open set $U_0 \subseteq \mathcal{X}_2$ and a C^1 function

$$G : U_0 \to C^0([0,1],(0,1))$$

such that

$$G(u)(x) = y \text{ if and only if } u(x,y) = c.$$

This will allow us to conclude that for $u \in \mathcal{X}_2$ locally around u_0 we have

$$f_2(u) = \int_{x=0}^{1} G(u)(x)dx.$$

Unpacking this, we see that at u_0 we have that

$$G(u_0)(x) = \frac{1}{2} \text{ for all } x \in [0,1].$$

For $u \in \mathcal{X}_2$ around u_0 consider the example below:

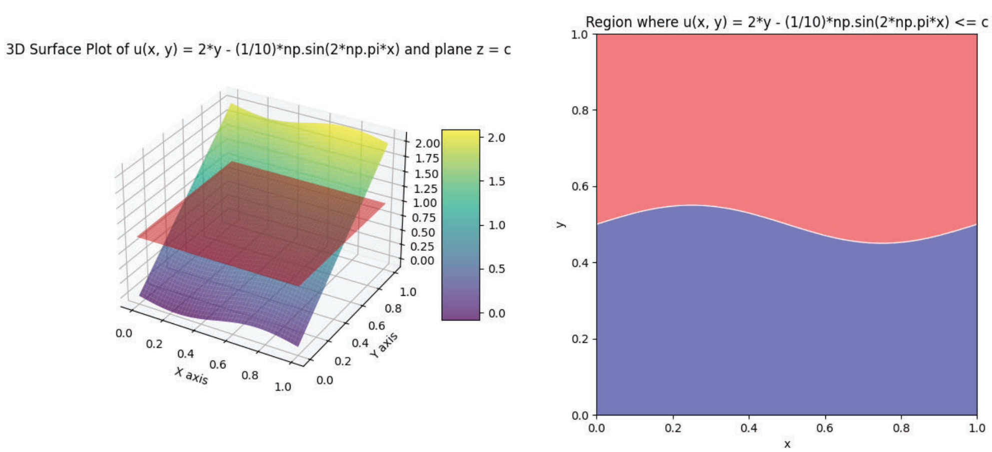


Figure 8 : Perturbed Wiggly Pattern from Rectangle

The wiggly white line between the red and blue regions is $G(u)(x)$ for this choice of u, and recall the area of the blue region is $f_2(u)$.

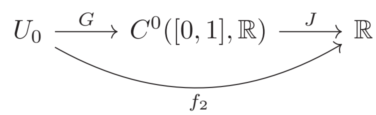
Notice that f_2 can be written as

$$f_2(u) = \int_{x=0}^{1} G(u)(x)dx,$$

so defining $J : C^0([0,1],\mathbb{R}) \to \mathbb{R}$ where

$$J(\phi) = \int_{x=0}^{1} \phi(x)dx,$$

it follows that $f_2 = J \circ G$ where we have the following diagram



Then J is linear, so it is automatically C^1 and thus if we show that G is C^1 we will have that f_2 is C^1 at u_0.

Lemma 3.2:
Let $\widetilde{X}, \widetilde{Y}, Z$ be Banach spaces. Let $X \subseteq \widetilde{X}$ be compact and $Y \subseteq \widetilde{Y}$ be open. Let $F : X \times Y \to Z$ be continuous. Then for all $y_0 \in Y$ we have that

$$\lim_{y \to y_0} \sup_{x \in X} ||F(x,y) - F(x,y_0)||_Z = 0,$$

where the limit is taken as $y \to y_0$ in $\widetilde{Y}$.

Proof: Note that similarly to Lemma 3.1, we will use the sup norm on $X \times Y$ and by finite dimensional norm equivalence, where we are treating $X \times Y$ as two dimensional, this will imply F is continuous independent of the choice of norm on $X \times Y$.

Also, note that we require Y to be open so that taking a limit of F as $y \to y_0$ is defined on $\widetilde{Y}$ and is equivalent to taking the limit as $y \to y_0$ in Y.

First let $\varepsilon > 0$ and $x \in X$. Next F is continuous on $X \times Y$, so because $(x, y_0) \in X \times Y$, there exists a $\delta_x > 0$ dependent on x such that for $(u, v) \in X \times Y$ where

$$||(u,v) - (x,y_0)||_{X \times Y} = \max(||u - x||_X, ||v - y_0||_Y) < \delta_x$$

it follows

$$||F(u,v) - F(x,y_0)||_Z < \frac{\varepsilon}{2}.$$

Now each open ball $B((x,y_0),\delta_x)$ allows us to form an open cover of $X \times \{y_0\}$ i.e.

$$X \times \{y_0\} \subseteq \bigcup_{x \in X} B((x,y_0),\delta_x),$$

and $X \times \{y_0\}$ is compact because X is compact and $\{y_0\}$ is compact as it is a singleton set, so this implies that $X \times \{y_0\}$ is compact, and thus there exists a finite subcover. Let $\{x_n\}_{n=1}^N \subseteq X$ be the collection of x-values in this finite subcover such that

$$X \times \{y_0\} \subseteq \bigcup_{n=1}^{N} B((x_n,y_0),\delta_{x_n}).$$

Then for any $x \in X$ we can find an x_n such that

$$(x, y_0) \in B((x_n,y_0),\delta_{x_n})$$

for some $n = 1, ..., N$.

We now let $\delta = \min(\{\delta_{x_n}\}_{n=1}^N)$. Then for any $y \in Y$ such that $||y - y_0||_Y < \delta$ we have

$$\begin{aligned} ||(x,y) - (x_n,y_0)||_{X \times Y} &= \max(||x - x_n||_X, ||y - y_0||_Y) \\ &< \max(\delta_{x_n}, \delta) \\ &= \delta_{x_n}. \end{aligned}$$

and so by the continuity of F we see that for any $(x, y) \in X \times Y$ where $||y - y_0||_Y < \delta$ it follows that there exists x_n such that

$$||F(x,y) - F(x_n, y_0)||_Z < \frac{\varepsilon}{2}.$$

Then by the triangle inequality we see that

$$\begin{aligned} ||F(x,y) - F(x,y_0)||_Z &\leq ||F(x,y) - F(x_n,y_0)||_Z + ||F(x,y_0) - F(x_n,y_0)||_Z \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

So then because X is compact and F is continuous, it follows that the difference $||F(x,y) - F(x,y_0)||_Z$ obtains its supremum on X, so we can conclude that for all $y \in Y$ where $||y - y_0||_Y < \delta$

$$\sup_{x \in X} ||F(x,y) - F(x,y_0)||_Z < \varepsilon$$

and thus because δ only depended on ε and y_0, and Y is open, we can conclude that

$$\lim_{y \to y_0} \sup_{x \in X} ||F(x,y) - F(x,y_0)||_Z = 0.$$

■

Lemma 3.3:
Let $\widetilde{X}, \widetilde{Y}$, and Z be Banach spaces, $X \subseteq \widetilde{X}$ be compact, and $Y \subseteq \widetilde{Y}$ be open. Let $F : X \times Y \to Z$ be continuous, and the partial derivative of F in the second component

$$D_2F : X \times Y \to \mathcal{L}(Y, Z)$$

be continuous.

Then the map $\widetilde{F} : Y \to C^0(X, Z)$ defined as

$$\widetilde{F}(y)(x) = F(x, y)$$

is C^1.

Proof: Note that $\widetilde{F}(y) \in C^0(X, Z)$ because

$$\widetilde{F}(y) = F(-, y)$$

and we know that F is continuous, and any continuous function on the product of Banach spaces will be continuous in each component when the other is fixed.

Let $y_0 \in Y$. We claim that

$$D\widetilde{F}(y_0)(y)(x) = D_2F(x, y_0)(y).$$

To show this we need to show that

$$\lim_{y \to y_0} \frac{||\widetilde{F}(y) - \widetilde{F}(y_0) - D_2F(-, y_0)(y - y_0)||_{C^0(X,Z)}}{||y - y_0||_Y} = 0.$$

Now we will unpack the $C^0(X, Z)$ norm and use the Fundamental Theorem of Calculus to integrate, and then apply Lemma 3.2 allowing us to swap the limit and supremum.

$$
\begin{aligned}
&\frac{||\widetilde{F}(y) - \widetilde{F}(y_0) - D_2F(-, y_0)(y - y_0)||_{C^0(X,Z)}}{||y - y_0||_Y} \\
&= \sup_{x \in X} \frac{||\widetilde{F}(y)(x) - \widetilde{F}(y_0)(x) - D_2F(x, y_0)(y - y_0)||_Z}{||y - y_0||_Y} \\
&= \sup_{x \in X} \frac{||F(x, y) - F(x, y_0) - D_2F(x, y_0)(y - y_0)||_Z}{||y - y_0||_Y} \\
&= \sup_{x \in X} \frac{|| \int_{t=0}^{1} \frac{d}{dt} F(x, t(y - y_0) + y_0)dt - D_2F(x, y_0)(y - y_0)||_Z}{||y - y_0||_Y} \\
&= \sup_{x \in X} \frac{|| \int_{t=0}^{1} D_2F(x, t(y - y_0) + y_0)(y - y_0)dt - D_2F(x, y_0)(y - y_0)||_Z}{||y - y_0||_Y} \\
&\leq \sup_{x \in X} \frac{(\int_{t=0}^{1} ||D_2F(x, t(y - y_0) + y_0) - D_2F(x, y_0)||_{\mathcal{L}(Y,Z)}dt)||y - y_0||_Y}{||y - y_0||_Y} \\
&= \sup_{x \in X} \int_{t=0}^{1} ||D_2F(x, t(y - y_0) + y_0) - D_2F(x, y_0)||_{\mathcal{L}(Y,Z)}dt.
\end{aligned}
$$

Then because $D_2F : X \times Y \to \mathcal{L}(Y, Z)$ is continuous into its respective space, X is compact, and Y is open, we can apply Lemma 3.2 on D_2F to conclude that given $\varepsilon > 0$ and $y_0 \in Y$ there exists a $\delta_1 > 0$ such that for any $x \in X$ and $y \in Y$ where $||y - y_0||_Y < \delta_1$ it follows that

$$
||D_2F(x, y) - D_2F(x, y_0)||_{\mathcal{L}(Y,Z)} < \frac{\varepsilon}{2}.
$$

Then because Y is open there exists a $\delta_2 > 0$ such that for any $y \in \widetilde{Y}$ where $||y - y_0||_Y < \delta_2$ it follows that $y \in Y$. So we can let $\delta = \min(\delta_1, \delta_2)$ and see that for all $t \in [0, 1]$ we have

$$
\begin{aligned}
||t(y - y_0) + y_0 - y_0||_Y &= ||t(y - y_0)||_Y \\
&= t||y - y_0||_Y \\
&\leq ||y - y_0||_Y \\
&< \delta,
\end{aligned}
$$

and so $t(y - y_0) + y_0 \in Y$ and $||t(y - y_0) + y_0 - y_0||_Y < \delta$ which implies that

$$
||D_2F(x, t(y - y_0) + y_0) - D_2F(x, y_0)||_{\mathcal{L}(Y,Z)} < \frac{\varepsilon}{2} \text{ for all } t \in [0, 1].
$$

Therefore we can conclude that

$$
\begin{aligned}
&\sup_{x \in X} \int_{t=0}^{1} ||D_2F(x, t(y - y_0) + y_0) - D_2F(x, y_0)||_{\mathcal{L}(Y,Z)}dt \\
&\qquad \leq \sup_{x \in X} \int_{t=0}^{1} \frac{\varepsilon}{2}dt \\
&\qquad = \frac{\varepsilon}{2} \\
&\qquad < \varepsilon.
\end{aligned}
$$

Now all we have to show is that $D\widetilde{F} : Y \to \mathcal{L}(Y, C^0(X, Z))$ is continuous, and we will be done. Let

$y_0 \in Y$ and we see that for any $y_1 \in Y$ that

$$\begin{aligned}
&||D\widetilde{F}(y_1) - D\widetilde{F}(y_0)||_{\mathcal{L}(Y,C^0(X,Z))} \\
&= \sup_{||y|| \le 1} ||D_2F(-,y_1)(y) - D_2F(-,y_0)(y)||_{C^0(X,Z)} \\
&= \sup_{||y|| \le 1} \sup_{x \in X} ||D_2F(x,y_1)(y) - D_2F(x,y_0)(y)||_Z \\
&\le \sup_{x \in X} ||D_2F(x,y_1) - D_2F(x,y_0)||_{\mathcal{L}(Y,Z)}.
\end{aligned}$$

Then because D_2F is continuous, we can again apply Lemma 3.2 again, and see that

$$\begin{aligned}
\lim_{y_1 \to y_0} ||D\widetilde{F}(y_1) - D\widetilde{F}(y_0)||_{\mathcal{L}(Y,C^0(X,Z))} &\le \lim_{y_1 \to y_0} \sup_{x \in X} ||D_2F(x,y_1) - D_2F(x,y_0)||_{\mathcal{L}(Y,Z)} \\
&= 0,
\end{aligned}$$

and so we can conclude that $D\widetilde{F}$ is continuous, and thus $\widetilde{F}$ is C^1.

■

Recall that we chose Figure 7 with function $u_0(x, y) = 2y$ with $c = 1$ as what we figured out was the simplest example.

We proved it for this function and then realized that proof quickly generalized to any stripe such that the boundary of the pattern was uniquely given by a continuous curve, and had good boundary behavior, for example we would have smoothness of f_2 at the function used to create the plot and pattern for Figure 8.

The boundary behavior we want in particular as well as the proof for this are outlined below.

Theorem 3.1:
Let $u_0 \in \mathcal{X}_2$ be such that there exists an $h_0 \in C^0([0, 1], (0, 1))$ where

$$\{(x, h_0(x)) \subseteq [0, 1] \times (0, 1)\} = u_0^{-1}(\{c\}),$$

and we have that

$$\frac{\partial u_0}{\partial y}(x, h_0(x)) \neq 0 \text{ for all } x \in [0, 1].$$

Then there exists an open set $U_0 \subseteq \mathcal{X}_2$ where $u_0 \in U_0$ and a unique C^1 map $G : U_0 \to C^0([0, 1], (0, 1))$ such that

$$y = G(u)(x) \text{ if and only if } u(x, y) = c$$

for all $(x, y) \in [0, 1]^2$ and $u \in U_0$.

Proof: Let $I = [0, 1]$ and $H = C^1(I, (0, 1))$ and $F : I \times H \times \mathcal{X}_2 \to \mathbb{R}$ be given by

$$F(x, h, u) = u(x, h(x)) - c.$$

We then define $U_1 = B(u_0, 1) \subseteq \mathcal{X}_2$. Now we claim that F is continuous on $I \times H \times U_1$, as we will show F can be written as a composition of continuous maps.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = x - c,$$

this map is affine and thus is continuous.

Let $\pi : I \times H \times \mathcal{X}_2 \to I \times (I \times H) \times \mathcal{X}_2$ be defined as

$$\pi(x, h, u) = (x, (x, h), u).$$

We see that π is linear and thus is continuous.

Then consider the following diagram

$$\begin{array}{ccc}
I \times H \times \mathcal{X}_2 & & (x, h, u) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\
I \times (I \times H) \times \mathcal{X}_2 & & (x, (x, h), u) \\
\Big\downarrow{\scriptstyle \mathrm{id}_I \times \mathrm{ev}_{I,(0,1)} \times \mathrm{id}_{U_1}} & & \Big\downarrow \\
I \times I \times \mathcal{X}_2 & & (x, h(x), u) \\
\Big\downarrow{\scriptstyle \mathrm{ev}_{I^2,\mathbb{R}}} & & \Big\downarrow \\
\mathbb{R} & & u(x, h(x)) \\
\Big\downarrow{\scriptstyle \phi} & & \Big\downarrow \\
\mathbb{R} & & u(x, h(x)) - c
\end{array}$$

This allows us to conclude that

$$F = \phi \circ \mathrm{ev}_{I^2,\mathbb{R}} \circ (\mathrm{id}_I \times \mathrm{ev}_{I,I} \times \mathrm{id}_{U_1}) \circ \pi,$$

and so F is continuous because each evaluation map is continuous, the cartesian product of continuous maps is continuous and the composition of continuous maps is continuous.

Then we see that treating $H \times \mathcal{X}_2$ as an open subset of the product Banach space, and using the chain rule, we can take the partial derivative in this component, and using the chain rule we see that

$$D_{H \times \mathcal{X}_2} F(x_0, h_0, u_0)(h, u) = \frac{\partial u_0}{\partial y}(x_0, h_0(x_0))h(x_0) + u(x_0, h_0(x_0)),$$

and we see that this is continuous as mapping into each evaluation map is continuous.

Now we can define the function $\widetilde{F} : H \times \mathcal{X}_2 \to C^0([0, 1], \mathbb{R})$ as

$$\widetilde{F}(h, u)(x) = F(x, h, u),$$

and then because F is continuous and $D_2 F$ is continuous, we can apply Lemma 3.3, and this implies that $\widetilde{F}$ is C^1 on $H \times \mathcal{X}_2$.

The idea now is that we will use the Implicit Function Theorem to solve for h as a function of u locally around u_0. We have that by the chain rule for any $h_0 \in H$ and $u_0 \in \mathcal{X}_2$ we have

$$D_1 \widetilde{F}(h_0, u_0)(h)(x) = \frac{\partial u_0}{\partial y}(x, h_0(x))h(x).$$

Then we will show the inverse of $D_1 \widetilde{F}(h_0, u_0)$ is bounded. First because

$$\frac{\partial u_0}{\partial y}(x, h_0(x)) \neq 0 \text{ for all } x \in [0, 1],$$

by assumption on h_0, and u_0 is C^2, we can conclude that $\left|\frac{\partial u_0}{\partial y}\right|$ is continuous. So because $u_0^{-1}(\{c\})$ is compact, $\left|\frac{\partial u_0}{\partial y}\right|$ must obtain its minimum on $u_0^{-1}(\{c\})$, and thus we can let $m > 0$ be this minimum value, and then we see that for any $\psi \in C^0([0,1],\mathbb{R})$ it follows

$$\begin{aligned} ||D_1\widetilde{F}(h_0,u_0)^{-1}(\psi)||_{C^0([0,1],\mathbb{R})} &= \sup_{x\in[0,1]} \left| \frac{1}{\frac{\partial u_0}{\partial y}(x,h_0(x))}\psi(x) \right| \\ &\leq \sup_{x\in[0,1]} \frac{1}{m}|\psi(x)| \\ &= \frac{1}{m}||\psi||_{C^0([0,1],\mathbb{R})}. \end{aligned}$$

Now H and $\mathcal{X}_2$ are open subsets of Banach spaces $C^0([0,1],\mathbb{R})$ and $C^2([0,1]^2,\mathbb{R})$ respectively, and the codomain of $\widetilde{F}$ given by $C^0([0,1],\mathbb{R})$ is a Banach space, $\widetilde{F}$ is C^1, and for any $x \in [0,1]$ we have

$$\begin{aligned} \widetilde{F}(h_0,u_0)(x) &= F(x,h_0,u_0) \\ &= u_0(x,h_0(x)) - c \\ &= c - c \\ &= 0. \end{aligned}$$

So we see that because this is true for all $x \in [0,1]$, it follows

$$\widetilde{F}(h_0,u_0) = 0 \in C^0([0,1],\mathbb{R}).$$

Thus because $D_1\widetilde{F}(h_0,u_0)$ is invertible with bounded inverse, we can apply the Implicit Function Theorem and conclude that there exist open sets $U_0 \subseteq \mathcal{X}_2$ and $H_0 \subseteq H$ such that $u_0 \in U_0$ and $h_0 \in H_0$ and a unique C^1 function $G : U_0 \to H_0$ such that for all $u \in U_0$ and $h \in H_0$ we have

$$G(u) = h \text{ if and only if } \widetilde{F}(h,u) = 0.$$

This then implies that

$$G(u)(x) = h(x) \text{ if and only if } \widetilde{F}(h,u)(x) = 0,$$

so

$$G(u)(x) = h(x) \text{ if and only if } u(x,h(x)) - c = 0,$$

thus because $h(x)$ is a function this implies that

$$G(u)(x) = y \text{ if and only if } u(x,y) = c.$$

■

3.3 Smoothness for Finite Striped Patterns

Recall that our goal is to show that the area feature function $f_2 : \mathcal{X}_{\text{reg}} \to \mathbb{R}$ is C^1.

By using a very similar process as in Theorem 3.1, we can show that f_2 is C^1 for functions $u_0 \in \mathcal{X}_2$ with patterns that have a finite number of stripes. When we say a finite number of stripes, we mean for example a function like

$$u_0(x,y) = \cos\left(10\pi\left(\left(y + \frac{1}{10}\right)^2\right)\right) - \frac{1}{7}\sin(5\pi x)$$

where we get the following plot and pattern:

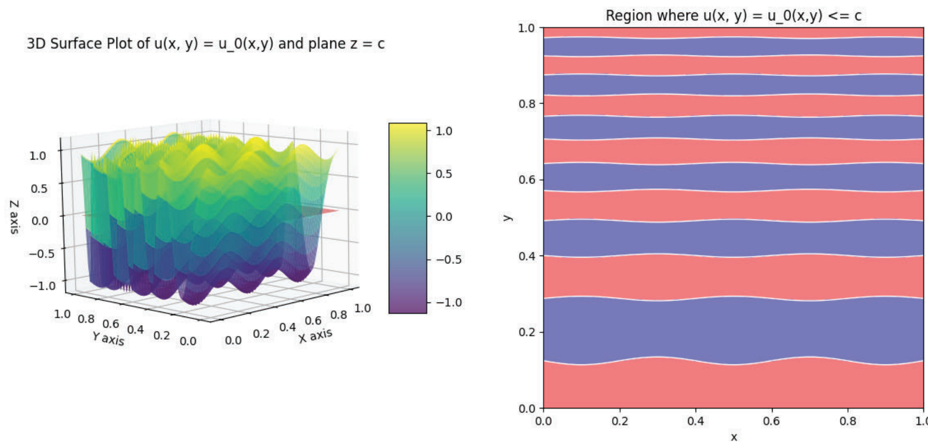


Figure 9 : Finite Stripes Pattern

So by stripes, here we mean each blue region, such that they are given by two functions from $[0,1]$ into $(0,1)$ which trace out the boundary of the top and bottom respectively of each stripe. It turns out the only condition we need is that along each of these boundary functions for the stripes $\frac{\partial u_0}{\partial y} \neq 0$.

Theorem 3.2:
Let $u_0 \in \mathcal{X}_2$ be such that u_0 gives us a finite striped pattern. More precisely, let u_0 satisfy that

$$\frac{\partial u_0}{\partial y}^{-1}(\{0\}) \cap u_0^{-1}(\{c\}) = \varnothing,$$

and let the set $u_0^{-1}(\{c\})$ be uniquely determined by a finite number of curves $\{h_k\}_{k=1}^n \subseteq C^0([0,1],(0,1))$.

Then the feature function $f_2 : \mathcal{X}_2 \to \mathbb{R}$ is C^1 at u_0.

Proof: Recall from Theorem 3.1 the function $\widetilde{F} : H \times \mathcal{X}_2 \to C^0([0,1],\mathbb{R})$ defined as

$$\widetilde{F}(h,u)(x) = F(x,h,u).$$

We showed that $\widetilde{F}$ is C^1 on $H \times \mathcal{X}_2$. The idea now is that we will use the implicit function theorem on each $\{h_k\}_{k=1}^n \subseteq H$ where these are the finite number of curves that trace out the boundaries of all the stripes.

Recall the formula we calculated using the chain rule that for any $h_0 \in H$ and $u_0 \in \mathcal{X}_2$ we have

$$D_1\widetilde{F}(h_0,u_0)(h)(x) = \frac{\partial u_0}{\partial y}(x,h_0(x))h(x).$$

We then have that H and $\mathcal{X}_2$ are both open subsets of a Banach space, $\widetilde{F}$ is C^1 on $H \times \mathcal{X}_2$, and for each $k \in \{1,2,...,n\}$ we have by assumption on each $h_k \in H$ that

$$\widetilde{F}(h_k,u_0) = 0 \in C^0([0,1],\mathbb{R}).$$

Now we can apply Theorem 3.1 and it follows that for each $k \in \{1,...,n\}$ there exists an open neighborhood $H_k \subseteq H$ and $U_k \subseteq \mathcal{X}_2$ with $h_k \in H_k$, and $u_0 \in U_k$, and a unique C^1 function $G_k : U_k \to H_k$ such that for all $u \in U_k$ and $h \in H_k$ we have that

$$G(u) = h \text{ if and only if } \widetilde{F}(h,u) = 0.$$

Now we can create a relatively open subset of $[0,1]^2$ using that each H_k is an open subset of $C^0([0,1],(0,1))$ by taking each h_k and finding an open ball $B(h_,\delta_k) \subseteq H_k$ for some $\delta_k > 0$. Then our relatively open subset around the graphs of each h_k is given by $V_k \subseteq [0,1]^2$ where

$$V_k = \{(x, h(x)) : x \in [0,1], h \in B(h_k, \delta_k)\}.$$

and then let

$$V = \bigcup_{k=1}^{n} V_k.$$

Next $V^c = [0,1]^2 \setminus V$ is compact, and u_0 is continuous, and it follows that u_0 is only equal to c along each h_k by assumption. Thus we can conclude that

$$u_0(x,y) - c \neq 0 \text{ for all } (x,y) \in V^c.$$

Then u_0 is C^2 and $|u_0(x,y)-c|$ is continuous, so $|u_0(x,y)-c|$ obtains its minimum on V^c, and that minimum is positive.

So we can let

$$\delta := \frac{1}{2} \min_{(x,y)\in V^c} |u_0(x,y) - c|,$$

and let

$$U := \left(\bigcap_{k=1}^{n} U_k \right) \cap B(u_0, \delta).$$

Now it follows that for any $u \in U$, by the uniqueness of each G_k, there exists exactly one curve where u_0 intersects with c inside each V_k, and therefore on all of V, we have exactly n curves.

Then on V^c, because $u \in B(u_0,\delta)$, we can conclude that $u(x,y) \neq c$ as for any $(\widetilde{x},\widetilde{y}) \in V^c$ we have that

$$|u_0(\widetilde{x},\widetilde{y}) - c| \geq \min_{(x,y)\in V^c} |u_0(x,y) - c| = 2\delta,$$

but notice that

$$|u_0(\widetilde{x},\widetilde{y}) - u(\widetilde{x},\widetilde{y})| \leq \max_{(x,y)\in V^c} |u_0(x,y) - u(x,y)| < \delta.$$

Therefore it follows that

$$u(\widetilde{x},\widetilde{y}) \neq c$$

for all $(\widetilde{x},\widetilde{y}) \in V^c$.

Thus this tells us that for each $u \in U$ the set $u^{-1}(\{c\})$ is given by exactly n curves each given by G_k. Then assuming that u_0 is zero on $[0,1]\times\{0\}$ and zero on $[0,1]\times\{1\}$ i.e. on the top and bottom of the unit square, and that c is positive, which allows us to negate case work, we can write f_2 for all $u \in U$ by

$$f_2(u) = \int_{x=0}^{1} \left(1 + \sum_{k=1}^{n} (-1)^{k-1} G_k(u)(x) \right) dx,$$

and so then we see that

$$f_2(u) = 1 + \sum_{k=1}^{n} (-1)^{k-1} \int_{x=0}^{1} G_k(u)(x) dx,$$

and then this implies that f_2 is the finite sum and difference of C^1 functions each given by $G_k \circ J$ where recall that

$$J : C^0([0,1],\mathbb{R}) \to \mathbb{R}$$

is defined by

$$J(\psi) = \int_{x=0}^{1} \psi(x).$$

Thus we can conclude that f_2 is C^1 at u_0.

■

3.4 Smoothness for Finite Spotted Patterns

Recall that our goal is to show that the area feature function $f_2 : \mathcal{X}_{\text{reg}} \to \mathbb{R}$ is C^1.

Now we will first give a rough sketch of how we will prove that f_2 is C^1 for a spotted pattern. As an example, we can let $c = \frac{1}{16}$ and define $u_0(x, y) = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2$. This gives us a pattern with a single spot, with the following plot and pattern:

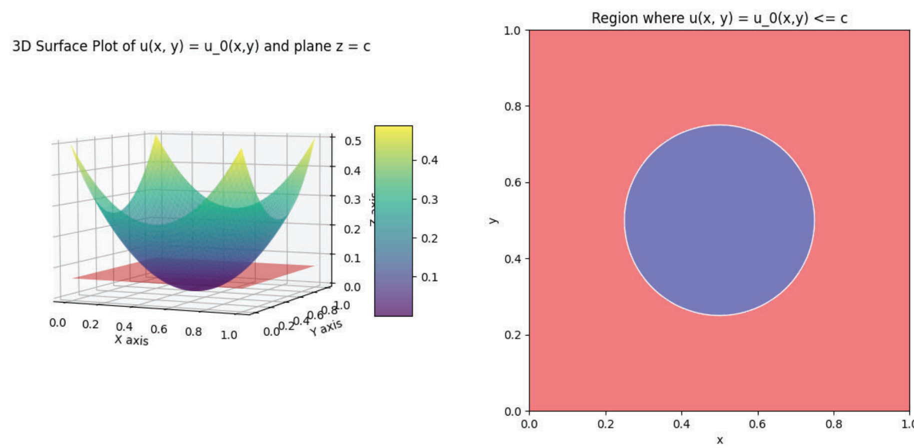


Figure 10 : Single Circular Spot Pattern

Then the idea is that we can index each point in $u_0^{-1}(\{c\})$ by (x_α, y_α), and apply the Implicit Function Theorem on each $(x_\alpha, y_\alpha) \in u_0^{-1}(\{c\})$ to create a rectangle such that in each rectangle we have essentially these four cases:

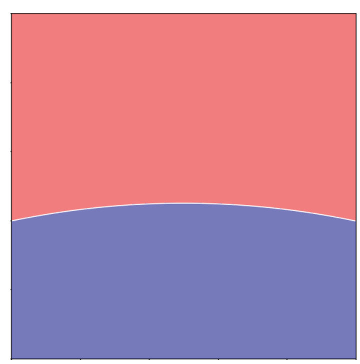


Figure 11 : Case with Blue on the Bottom

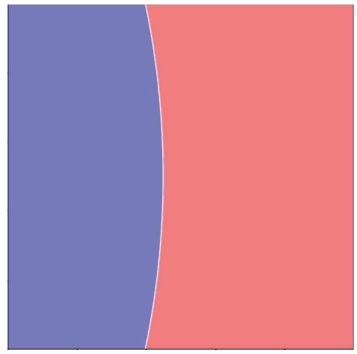


Figure 12 : Case with Blue on the Left

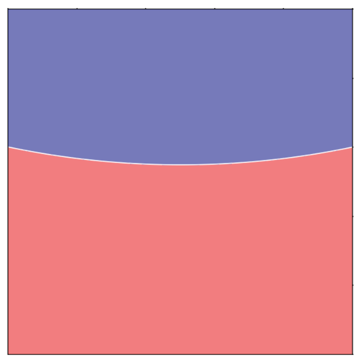


Figure 13 : Case with Blue on the Top

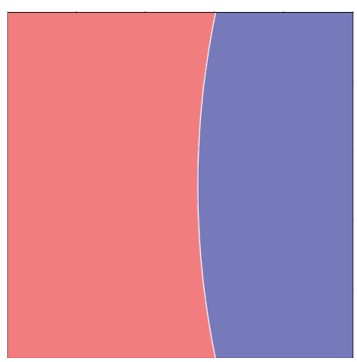


Figure 14 : Case with Blue on the Right

as consider the following cover of the circle:

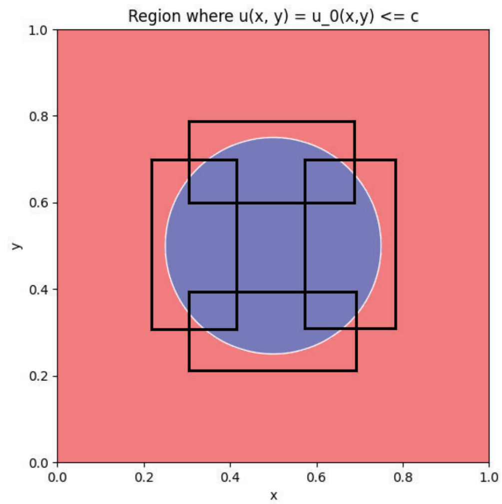


Figure 15 : Cover of Spot with Rectangles

where we have either y as a function of x or x as a function of y, and the white circle is such that it enters and exits each rectangle on opposite sides.

Then this allows us to create an open cover of $u_0^{-1}(\{c\})$ with open rectangles, and then to negate case work, we focus just on Figure 11, the case where blue is on the bottom, and we can define $y = G_\alpha(x, u)$. This allows us to essentially just solve the same problem we have been solving where we apply the Implicit Function Theorem and previous Lemmas to solve for the curve in the box defining $u^{-1}(\{c\})$ as a function of u locally around u_0. Then integrating this function gives us the area intersecting the blue disk with the box.

However, notice that we cannot just add the integrals of each box together as these rectangular covers will almost certainly overlap. To fix this we use a partition of unity and then this allows us to integrate each partition of unity in each box, and then add these integrals and we get the area intersecting all the boxes. Then to get the entire area of the blue disk, we add the leftover area inside the disk at the end, which is constant, and then conclude that f_2 is the sum of C^1 functions, and so f_2 will be C^1.

This technique immediately generalizes to any $u_0 \in \mathcal{X}_2$ such that $u_0^{-1}(\{c\}) \subseteq (0,1)^2$, as for multiple spots, we can treat each connected component separately as the disk by covering it with boxes, and using a partition of unity to glue the integrals together. Then to get the entire area, we can just add the area of each connected component, and again the finite sum of C^1 functions is C^1 and thus we will be done.

Theorem 3.3:
The feature function f_2 is C^1 at $u_0 \in \mathcal{X}_2$ where the level set $u_0^{-1}(\{c\}) \subseteq (0,1)^2$.

Proof: Let $(x_\alpha, y_\alpha) \in u_0^{-1}(\{c\})$. Then because $u_0 \in \mathcal{X}_2$ it follows that either

$$\frac{\partial u_0}{\partial x}(x_\alpha, y_\alpha) \neq 0 \text{ or } \frac{\partial u_0}{\partial y}(x_\alpha, y_\alpha) \neq 0.$$

If $\frac{\partial u_0}{\partial y}(x_\alpha, y_\alpha) \neq 0$ then we consider the following function $F : (0,1)^2 \times \mathcal{X}_2 \to \mathbb{R}$ given by

$$F(x, y, u) = \text{ev}_{[0,1]^2,\mathbb{R}}(x, y, u) - c.$$

We see that F is the composition of C^1 maps, and thus is C^1, and its partial derivative in the y component at (x_α, y_α, u) is given by

$$D_2F(x_\alpha, y_\alpha, u_0) = \frac{\partial u_0}{\partial y}(x_\alpha, y_\alpha),$$

and thus because this is a non-zero real number, we can conclude that the inverse is given by

$$D_2F(x_\alpha, y_\alpha, u_0)^{-1} = \frac{1}{\frac{\partial u_0}{\partial y}(x_\alpha, y_\alpha)}$$

and the inverse is bounded by $\left|\frac{1}{\frac{\partial u_0}{\partial y}(x_\alpha, y_\alpha)}\right|$.

Thus we can apply the Implicit Function Theorem and conclude that there exist open sets $O_\alpha \times U_\alpha \subseteq (0,1) \times \mathcal{X}_2$ and $V_\alpha \subseteq (0,1)$ such that $(x_\alpha, u_0) \in O_\alpha \times U_\alpha$ and $y_\alpha \in V_\alpha$, and there exists a unique C^1 function $G_\alpha : O_\alpha \times U_\alpha \to V_\alpha$ such that for all $(x, y, u) \in O_\alpha \times V_\alpha \times U_\alpha$ it follows

$$y = G_\alpha(x, u) \text{ if and only if } F(x, y, u) = 0.$$

This then implies that

$$y = G_\alpha(x, u) \text{ if and only if } u(x, y) = c.$$

Now because $x_\alpha \in O_\alpha$ and $O_\alpha \subseteq (0,1)$ is open in the standard Euclidean topology, we can conclude that there exists open interval $I_\alpha \subseteq O_\alpha$ such that $x_\alpha \in I_\alpha$ and

$$\overline{I_\alpha} \subseteq O_\alpha.$$

Next because $\overline{I_\alpha}$ is a closed interval, it is compact and connected, so because the function $G_\alpha(-, u_0)$ is continuous in x, it follows that the image $G_\alpha(-, u_0)(\overline{I}_\alpha) \subseteq V_\alpha$ is compact and connected, which because it is a subset of $\mathbb{R}$ implies it is a closed interval. Then $G_\alpha(-, u_0)(\overline{I}_\alpha)$ is contained in V_α, so even though V_α might not be an interval, the connected component of y_α in V_α is, so we can construct an open interval J_α such that

$$G_\alpha(-, u_0)(\overline{I}_\alpha) \subsetneq J_\alpha \subsetneq V_\alpha$$

as consider the following picture below:

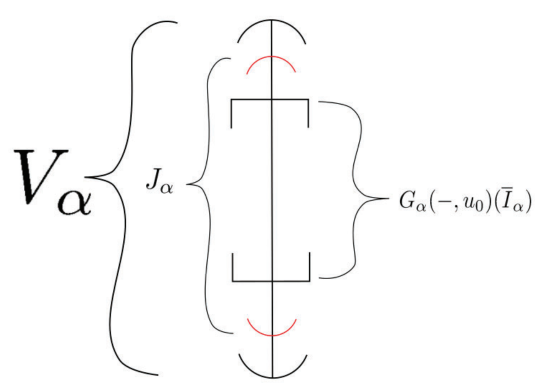


Figure 16 : Construction of Intervals for Rectangular Cover

Now if $\frac{\partial u_0}{\partial y}(x_\alpha, y_\alpha) = 0$, we can still solve for x as a C^1 function $x = H_\alpha(y, u)$ because $u_0 \in \mathcal{X}_2$ implies that $\frac{\partial u_0}{\partial x}(x_\alpha, y_\alpha) \neq 0$. So we can repeat the exact same process, and arrive at intervals I_α and J_α similarly such that $x = H_\alpha(y, u)$ for $x \in I_\alpha$ and $(y, u) \in J_\alpha \times U_\alpha$ where

$$H_\alpha(-, u_0)(\overline{J}_\alpha) \subsetneq I_\alpha \subsetneq O_\alpha.$$

This implies that the collection of rectangles $\{I_\alpha \times J_\alpha\}_{\alpha \in \mathcal{A}}$ creates an open cover of $u_0^{-1}(\{c\})$, and so because $u_0^{-1}(\{c\})$ is compact, this implies that there exists a finite subcover

$$\{I_k \times J_k\}_{k=1}^n.$$

Then indexing with k instead of α, we have that the open set

$$M = \bigcup_{k=1}^n I_k \times J_k \subseteq (0,1)^2$$

is an open subset of $\mathbb{R}^2$ and so it is a C^∞ manifold.

Now M is open so $M^c = [0,1]^2 \setminus M$ is compact. Then M forms a cover for $u_0^{-1}(\{c\})$, so the function $|u_0(x,y) - c|$ is non-zero on M^c. Then it is continuous in x and y as it is a composition of continuous functions, so it must achieve its minimum, and that minimum will be positive. So we can let

$$\delta := \frac{1}{2} \min_{(x,y) \in M^c} |u_0(x,y) - c|.$$

Then let

$$U_0 = \left(\bigcap_{k=1}^n U_k \right) \cap B(u_0, \delta)$$

where each U_k is the open neighborhood of u_0 in $\mathcal{X}_2$ from the Implicit Function Theorem.

Now it follows for any $u \in U_0$, the level set $u^{-1}(\{c\}) \subseteq M$ because for any $(x,y) \in M^c$ we have

$$|u_0(x,y) - c| \geq \min_{(x,y) \in M^c} |u_0(x,y) - c| = 2\delta,$$

and

$$\begin{aligned} |u_0(x,y) - u(x,y)| &\leq \max_{(x,y) \in M^c} |u_0(x,y) - u(x,y)| \\ &< ||u - u_0||_{\mathcal{X}_2} \\ &= \delta. \end{aligned}$$

so we can conclude that

$$|u_0(x,y) - c| \neq |u_0(x,y) - u(x,y)|,$$

and thus

$$u(x,y) \neq c$$

for all $(x,y) \in M^c$.

Then this implies that $u^{-1}(\{c\}) \subseteq M$ and so the area of the pattern outside of M does not change for any $u \in U_0$. Thus we can conclude that

$$\int_{u^{-1}((-\infty,c]) \setminus M} 1 dxdy = \int_{u_0^{-1}((-\infty,c]) \setminus M} 1 dxdy,$$

and so it is constant, i.e. it does not depend on u, only on u_0.

This allows us to see that

$$\begin{aligned} f_2(u) &= \int_{u^{-1}((-\infty,c])} 1 dxdy \\ &= \int_{u^{-1}((-\infty,c])\setminus M} 1 dxdy + \int_{u^{-1}((-\infty,c])\cap M} 1 dxdy \\ &= \int_{u_0^{-1}((-\infty,c])\setminus M} 1 dxdy + \int_{u^{-1}((-\infty,c])\cap M} 1 dxdy. \end{aligned}$$

Thus we only need to show that

$$\int_{u^{-1}((-\infty,c])\cap M} 1 dxdy$$

is a C^1 function of u and we will be done.

Now we have that $\{I_k \times J_k\}_{k=1}^n$ forms an open cover of M, so by Theorem 2.23 in [1], we can conclude there exists a C^∞ partition of unity subordinate to $\{I_k \times J_k\}_{k=1}^n$. This tells us that there exists maps

$$\{\psi_k\}_{k=1}^n \subseteq C^\infty(M, [0,1])$$

such that they are compactly supported i.e. the support defined as

$$\text{supp}(\psi_k) = \overline{\{(x,y) \in M : \psi_k(x,y) \neq 0\}}$$

is compact and satisfies

$$\text{supp}(\psi_k) \subseteq I_k \times J_k,$$

and

$$\sum_{k=1}^n \psi_k(x,y) = 1 \text{ for all } (x,y) \in M.$$

So using our partition of unity we have

$$\begin{aligned} \int_{u^{-1}((-\infty,c])\cap M} 1 dxdy &= \int_{u^{-1}((-\infty,c])\cap M} \sum_{k=1}^n \psi_k(x,y) dxdy \\ &= \sum_{k=1}^n \int_{u^{-1}((-\infty,c])\cap M} \psi_k(x,y) dxdy \\ &= \sum_{k=1}^n \int_{u^{-1}((-\infty,c])\cap (I_k \times J_k)} \psi_k(x,y) dxdy. \end{aligned}$$

Thus all we need to do is show that each of these integrals

$$\int_{u^{-1}((-\infty,c])\cap (I_k \times J_k)} \psi_k(x,y) dxdy$$

is C^1 as a function of u and then this integral will be a finite sum of C^1 functions in u and thus will be C^1.

To show this without loss of generality we will focus on if $y = G_k(x,u)$, and where the blue region is below the function, keep in mind Figure 11, the case with blue on the bottom.

The blue region is the set $u^{-1}((-\infty,c]) \cap (\overline{I_k} \times J_k)$. Note that whether or not we use I_k or $\overline{I_k}$ for bounds of integration is irrelevant because the difference has measure zero in $\mathbb{R}^2$. Now by our arguments using the Implicit Function Theorem the blue region is uniquely defined by

$$u^{-1}((-\infty,c]) \cap (\overline{I_k} \times J_k) = \{(x,y) \in (0,1)^2 : x \in \overline{I_k}, y \in (\inf(J_k), G_k(u,x))\}.$$

Thus we can plug in these bounds of integration and see that

$$\int_{u^{-1}((-\infty,c])\cap(I_k\times J_k)} \psi_k(x,y)dxdy = \int_{x\in\overline{I_k}} \int_{y=\inf(J_k)}^{G_k(x,u)} \psi_k(x,y)dydx$$

Now we can define the map $P_k : \overline{I_k} \times \overline{J_k} \to \mathbb{R}$ to be

$$P_k(x,z) = \int_{y=\inf(J_k)}^{z} \psi_k(x,y)dy.$$

Note this map exists as ψ_k has a unique extension to a C^∞ function defined on $\overline{I_k} \times \overline{J_k}$ because it is compactly supported, so we can just define it to be zero on the boundary of the rectangle $\overline{I_k} \times \overline{J_k}$.

Now we will show P_k is C^1 by showing each partial derivative exists and is continuous.

The first partial derivative in x is continuous because if z is fixed, we are looking at the composition

$$x \to \psi_k(x,-) \to \int_{y=\inf(J_k)}^{z} \psi_k(x,y)dy,$$

which then because $\overline{J_k}$ is compact, we can apply Lemma 3.3 to conclude that the map

$$x \to \psi_k(x,-)$$

is C^1 into $C^0(\overline{J_k},\mathbb{R})$. Next the integral on $(\inf(J_k), z)$ is linear and thus is C^1 on $C^0(\overline{J_k},\mathbb{R})$. Together we now can use that the composition of C^1 functions is C^1, and so the partial derivative of P_k in the first component exists and is continuous.

Then the other partial derivative is continuous because we can apply the Fundamental Theorem of Calculus as $\psi_k(x,y)$ is continuous in y and so the partial derivative of $P_k(x,z)$ in the second component is given by $\psi_k(x,z)$. Thus we can conclude that P_k is C^1.

Now we can define the map $\phi_k : \overline{I_k} \times U_k$ given by

$$\phi_k(x,u) = P_k(x, G_k(x,u)),$$

and this map is C^1 because it is a composition of P and a map that is a projection in the first component, and in the second component is G_k, which is C^1 by the Implicit Function Theorem.

Then because $\overline{I_k}$ is compact, U_k is open, by Lemma 3.3 it follows that the map $\widetilde{\phi}_k : U_k \to C^0(\widetilde{I}_k, J_k)$ given by $\widetilde{\phi}_k(u)(x) = \phi_k(x,u)$ is C^1.

Now the map given by the integral $\int_{x\in\overline{I_k}}(-)dx : C^0(\overline{I_k},\mathbb{R}) \to \mathbb{R}$ is C^1 because the map is linear, so then we see that

$$\int_{x\in\overline{I_k}} \int_{y=\inf(J_k)}^{G_k(x,u)} \psi_k(x,y)dydx = \int_{x\in\overline{I_k}} \widetilde{\phi}(u)(x)dx,$$

and so the composition

$$\int_{x\in\overline{I_k}} (-)dx \circ \widetilde{\phi},$$

is also C^1 as it is a composition of C^1 maps.

Now we can then apply this exact same logic both if the red and blue regions are swapped or if $x = H_k(y,u)$, so we can conclude that for $u \in U_0$ we have that $f_2(u)$ is the finite sum of C^1 functions, and thus that f_2 is C^1 at u_0. ■

3.5 Smoothness for Periodic Patterns

Finally we now have enough ammunition to be able to achieve our goal of proving that the area feature function $f_2 : \mathcal{X}_{\text{reg}} \to \mathbb{R}$ is C^1.

In [2] they only considered functions defined on the two torus $D = (\mathbb{R} \setminus 2\pi\mathbb{Z})^2$, and they defined their space of functions to be

$$\mathcal{X}_{\text{reg}} = \{u \in C^2(D, \mathbb{R}) : u^{-1}(\{c\}) \neq \varnothing, \text{ and } \nabla u(x) \neq 0 \text{ for all } x \in u^{-1}(\{c\})\}.$$

Note that the gradient condition they use is the same as the definition of c being a regular value of u. So with what we have shown, we can now prove smoothness of the area of a pattern to be smooth for functions in $\mathcal{X}_{\text{reg}}$.

It turns out that we only have to prove smoothness on the space of functions on the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ created by identifying the opposite ends of the square $[0, 1]^2$. We define this space as

$$\mathcal{X}_T = \{u \in C^2(T, \mathbb{R}) : u^{-1}(\{c\}) \neq \varnothing, \text{ and } c \text{ is a regular value of } u\}.$$

This is because we can just scale any function from $\mathcal{X}_{\text{reg}}$ into this space by just scaling D to be T as both are similar two tori. This scaling action would allow us to convert back and forth between functions in $\mathcal{X}_{\text{reg}}$ and $\mathcal{X}_T$, and so then the area of the pattern for a function in $\mathcal{X}_{\text{reg}}$ would be $4\pi^2$ multiplied by the area for the corresponding function in $\mathcal{X}_T$ and so then f_2 will be smooth for elements in $\mathcal{X}_{\text{reg}}$ if and only if f_2 is smooth for their rescaled version in $\mathcal{X}_T$. Thus we only need to show that f_2 is C^1 on $\mathcal{X}_T$ and we will be done.

Lemma 3.4 (Number of Connected Components is Finite)**:**
Every compact manifold has a finite number of connected components

Proof: Manifolds are locally path connected by Proposition 1.11a [1] , and the path components of a locally path connected space are open and are the same as the connected components in that space by Proposition A.43ab [1]. Thus for any compact manifold we can create an open cover using the connected components and then find a finite subcover, and thus every compact manifold has a finite number of connected components.

■

Theorem 3.4:
The feature function f_2 is C^1 for all functions $u_T \in \mathcal{X}_T$.

Proof: Now notice that we first need to show that we can treat $\mathcal{X}_T$ as a subset of $\mathcal{X}_2$ in order for it to make sense for f_2 to act on $\mathcal{X}_T$. We can do this naturally, by seeing that we have a natural quotient map $q : \mathbb{R}^2 \to T$ where we treat T as $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Then for any $u_T \in \mathcal{X}_T$ we can map it to a function $u \in C^2(\mathbb{R}^2, \mathbb{R})$ by letting

$$u(x, y) = u_T(q(x, y)).$$

We now want to show that $u \in \mathcal{X}_2$.

First $u \in C^2(\mathbb{R}^2, \mathbb{R})$ because the quotient map is C^∞ by construction of the smooth structure on T.

Then we can restrict u to just $[0, 1]^2$ and because $[0, 1]^2 \subseteq \mathbb{R}^2$ it follows that $u|_{[0,1]^2} \in C^2([0, 1]^2, \mathbb{R})$.

Now note that u is in $\mathcal{X}_2$ if and only if

$$Du(x, y) \neq 0 \text{ for all } (x, y) \in u^{-1}(\{c\}).$$

This is because the gradient and the Fréchet derivative of u are the transpose of each other as linear maps, so because the transpose of a map is zero if and only if it is zero, we can conclude that this definition is equivalent.

Now the quotient map q is a C^∞ submersion because it is a smooth covering map, which follows by construction of the smooth structure on T. Thus for any $(x_0, y_0) \in u^{-1}(\{c\})$ we have that

$$u(x_0, y_0) = c,$$

and so by construction of u we know that

$$u(x_0, y_0) = u_T(q(x_0, y_0)),$$

and thus

$$q(x_0, y_0) \in u_T^{-1}(\{c\}).$$

Then the chain rule gives us that

$$Du(x_0, y_0) = Du_T(q(x_0, y_0)) \circ Dq(x_0, y_0),$$

and so because $q(x_0, y_0) \in u_T^{-1}(\{c\})$ it follows $Du_T(q(x_0, y_0))$ is surjective by definition of u_T being an element in $\mathcal{X}_T$, and then because q is a smooth submersion, this implies that $Dq(x_0, y_0)$ is surjective, and so then the composition of surjective maps is surjective, and thus we can conclude that

$$Du(x_0, y_0) \neq 0,$$

and so $u \in \mathcal{X}_2$.

Thus we can naturally define

$$f_2(u_T) = f_2(u).$$

Then notice that the set $u_T^{-1}(\{c\})$ is compact because it is a closed subset of the compact space T, and is a submanifold of T, and so using Lemma 3.4 because it is a compact manifold it must have a finite number of connected components. Then by Exercise 15-13 in Lee's Smooth Manifolds [1] it follows that every connected compact manifold is diffeomorphic to S^1. Thus we can do the exact same thing we did for spots in Theorem 3.3 for each connected component, and because the number of connected components is finite, we can again cover $u_T^{-1}(\{c\})$ with a finite number of rectangles as given below:

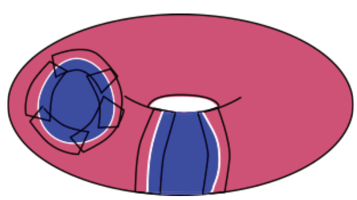


Figure 17 : Torus Pattern Covered with Rectangles

The only other kind of pattern than the two above that could form on the torus is a stripe that has the boundary circles going around the larger circle of the torus instead of the smaller like we have in the above picture. However, in any of these three cases, because each connected component is diffeomorphic to a circle contained in an open subset of T, we can integrate in each box, and sum them together using a partition of unity, as in the spots case Theorem 3.3, and add the remaining constant interior area of the pattern, and get that f_2 is a sum of C^1 functions, and so f_2 is C^1. ■

4 Conclusion

Throughout this journey we have learned that the area of a sublevel set is C^1 as a function on the space of functions on $\mathcal{X}_{\text{reg}}$. The remaining questions lie with filling in the remaining gaps about the map from the parameter space into more complex feature functions.

In reality, most of the outputs of the feature functions are actually measures, not real numbers, so the smoothness in question is actually from the parameter space, all the way into the the Wasserstein distance between outputs of the feature functions, which is a much harder statement to prove. Also, some patterns generated from the model will not actually be in $\mathcal{X}_{\text{reg}}$, which further puts into question if smoothness can be shown in these more general cases.

However, smoothness of the area feature function is still an interesting problem in its own right, and still does greatly suggest the effectiveness of these data driven techniques for future pattern formation analysis.

We have actually shown that the space of functions where f_2 is C^1 is larger than the space of periodic functions $\mathcal{X}_{\text{reg}}$ as even the so called simplest example of a single stripe given in Theorem 3.1 does not satisfy the periodic boundary conditions. Therefore this leaves us with the following question: What is the space of all functions in $\mathcal{X}_2$ where f_2 is C^1?

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