Predicting Market Share: Application of Alternative Lotka-Volterra Competition Model*

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Abstract. In this paper, we apply an alternative two-dimensional Lotka-Volterra system, traditionally used in ecology to model species competition, to the field of industrial organization. We investigate whether this model can effectively represent the competitive interactions between firms in technology sector markets. The study focuses on determining if the model can provide realistic predictions of future market shares, aligning with observed trends. Additionally, we explore the general implications of changes in dynamic variables within the competitive framework. This approach helps draw parallels between ecological and economic competition, suggesting that this Lotka-Volterra model offers valuable insights for market strategy development. This paper lays the groundwork for further research, including the incorporation of stochastic elements and the analysis of more complex competitive scenarios involving multiple firms.

Key words. Lotka-Volterra Model, Ecological Competition, Market Competition, Industrial Organization, Linear Algebra, Ordinary Differential Equations, and Dynamical Systems.

AMS subject classifications. 34C60 (Qualitative investigation and simulation of ordinary differential equation models), 91B74 (Economic models of real-world systems).

1. Introduction.

The application of mathematical models to understand complex systems has long been a staple in both natural and social sciences. This paper presents an approach to modeling market dynamics using a unique governing set of equations that have traditionally been used only in ecological contexts. This paper extends beyond the conventional use of these specific equations, offering a novel perspective on industrial organization and competition within the technology sector.

In exploring this model, we succinctly address the competitive dynamics between two major players in the smartphone market. The unique contribution of this paper lies in the modification of the traditional Lotka-Volterra system to incorporate explicitly defined carrying capacities, providing a more nuanced and realistic representation of market share and growth limitations. This approach enables us to not only simulate current market conditions but also to forecast future trends in market shares with a degree of precision not previously explored.

This research has the goal of adding to the knowledge of current economic modeling techniques, where traditional models can at times fall short in capturing the intricacies of technological market competition. By applying an unconventional version of the Lotka-Volterra model to an economic context, we aim to bring more modeling options to the table, options that further cultivate our predictive capabilities.

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The findings of this study are not only relevant for economists and mathematicians, but also for industry practitioners. They provide a framework for understanding market behavior and strategy development in a highly competitive and dynamic industry. This paper lays a foundation for further exploration into more complex competitive scenarios, potentially including stochastic processes and higher-dimensional systems, thereby broadening the applicability of ecological models to not only model competition dynamics, but to explore uncertainty as well.

2. Background. Mathematics is the fundamental language of precision and provides a crucial framework for communicating in absolute terms. As a discipline, mathematics provides numerous empirical methods that both the natural sciences and social sciences rely on. Modeling is one of these empirical methods, and is a critical one at that. When models are accurate in capturing their target phenomenon, they have a powerful predictive capability and can be utilized to give an idea of what could happen in the future.

The original Lotka-Volterra model was pioneered independently by the American mathematician Alfred James Lotka in 1925 and Italian mathematician Vito Volterra in 1926. Originally conceived to model the various dynamics of predator-prey interactions over time in an ecosystem, the Lotka-Volterra equations have found applications beyond the realm of ecology. This paper proposes to apply a modified version of this system of nonlinear differential equations to the realm of market dynamics. Specifically, it has the goal of predicting market share. Market share could have a few interpretations, but in the case of this paper it means something specific. It is defined as the proportion of total sales of a product accounted for by an individual brand of the product or all brands of the product offered by a firm in a particular market [8].

The Lotka-Volterra model and its application to economics is far from revolutionary. It has been widely employed to predict trends among competing species, or businesses in this case, within various market ecosystems. One piece of literature in particular highlights the numerous examples of the model's application to various real-world market dynamics [5].

In industrial organization, a common theme includes the use of the model to study the interactions between competing technologies. This includes utilizing the Lotka-Volterra model to analyze the competition between semiconductor manufacturing technologies [2]. Additionally, the model has also found applications in studying the relationships between competing technologies in information and communications engineering [6] and has been used to study the competitive behavior between two well-known major global CPU suppliers, Intel and AMD [12]. Yet another example entails utilizing the model to study interactions between integrated circuit industries [13]. In financial economics, it has been employed to analyze the dynamic relationship between competing Korean stock markets [7].

For some, the predictive output was realistic. For others, it was not. Nevertheless, these pieces of literature highlight the versatility of the Lotka-Volterra model and its various applications for capturing competition outside of ecology. However, these pieces of literature do not use the same model that will be tested here nor do they directly relate to predicting market share.

One paper in particular deploys the model to examine competitive relationships among four smartphone operating systems and forecasts the sales volume of each over a five-year period [14]. This is a particularly relevant piece of literature because it focuses on forecasting operating system smartphone sales volumes. This could also be framed as the forecasting of operating system market shares in a smartphone market, something quite similar to what is being done here. However, there is a crucial difference. They utilize the more traditional Lotka-Volterra system (2.1).

$$\begin{aligned} \frac{dx}{dt} &= a_1x + b_1x^2 - c_1xy \\ \frac{dy}{dt} &= a_2y + b_2y^2 - c_2xy \end{aligned} \tag{2.1}$$

Their model, while accounting for a form of growth limitation through the b_1 and b_2 parameters, lacks an explicit carrying capacity. As such, the unique contribution of this paper comes with the fact that the model applied in this paper utilizes an explicitly defined carrying capacity, providing a distinct boundary on growth. This explicit limit has the potential to yield different and more absolute long run results for market share predictions.

The interchangeability between ecological variables and economic variables is fundamentally due to the parallels they share. Although an older source, research from the University of British Columbia presents an intriguing example demonstrating this [3]. It involves fish population and the concept of capital in modern capital theory. Capital, one of the key means of production in classical and neoclassical economic theory, can yield products or services over time. The population of fish, when harvested, can also yield products or services over time.

Maintaining a stable fish population, much like preserving capital, ensures a sustained flow of resources. Mismanagement or over-harvesting, on the other hand, can jeopardize this flow. In a similar manner, over utilization and poor management of capital can lead to a business's downfall.

Now consider the sale of smartphones as it is the market of choice for which the model will be tested. The quantity sold annually by a firm, or the "consumption flow" of smartphones, mirrors the sustainable harvesting of fish. This is because the common agenda in both scenarios is to maximize consumption, subject to a constraint. For fish-harvesting, this constraint would be the point at which the fish population begins to decline, making the consumption flow no longer sustainable. For the smartphone market, the constraint would be the total sales of smartphone products in the smartphone market, also known as the market share carrying capacity. Economics studies the allocation of resources, resources which are scarce. Even more fundamentally, in a market economy, competition plays a significant role in how these scarce resources are allocated. In nature, competition amongst species plays a significant role in how resources crucial for survival are allocated. Given the similar bearing of scarcity and competition, a ecological system's variables can be substituted with economic related ones to reflect economic competition between firms instead.

A standard Lotka-Volterra model for competition (2.2) involving rabbit population x and deer population y at time t is given below [11]. Suppose both species are entrenched in competition for the same finite resource (vegetation in this case).

$$\frac{dx}{dt} = x(3 - x - 2y)$$
$$\frac{dy}{dt} = y(2 - x - y) \tag{2.2}$$

Traditional models in mathematical ecology can undergo several possible modifications [10]. Here, the standard Lotka-Volterra model can be modified to include an explicitly defined carrying capacity M and N for each species with coefficients β and δ continuing to reflect pure competition dynamics between the two species (2.3). Let α and γ be growth rates that govern the populations.

$$\frac{dx}{dt} = \alpha x(1 - \frac{x}{M}) - \beta xy$$
$$\frac{dy}{dt} = \gamma y(1 - \frac{y}{N}) - \delta xy \tag{2.3}$$

There is a high degree of versatility to this model. Table 1 provides an interpretation for the β and δ coefficients where different dynamics are captured by different combinations of the coefficients.

Table 1: Competitive relationship classifications as determined by competition coefficients.

β	δ	Type	Explanation
-	-	Pure competition	Both firms negatively impact each other's growth
+	-	Predator–prey	One firm benefits from the other's presence, while the other is negatively impacted
-	+	Predator–prey	
+	+	Mutualism	Both firms benefit from each other's presence, but the degree of benefit may vary, potentially leading to competitive dynamics if one firm benefits significantly more
-	0	Amensalism	One firm negatively impacts the growth of the other while remaining unaffected themselves.
0	-	Amensalism	
+	0	Commensalism	One firm benefits from the presence of the other while the other remains unaffected
0	+	Commensalism	
0	0	Neutralism	There is no interaction, system is decoupled

This table is inspired by the various dynamics possible from varying coefficients for two predator-prey retail formats [5] but has been altered to better describe the possibilities of this particular system (2.3).

The remainder of the paper is organized with a deeper overview of the modified Lotka-Volterra model in section 3, the parameters in section 4, an analysis in section 5, and finishes off with a summary in section 6.

3. The Model. Both Apple and Samsung have been dominant and at the forefront of the smartphone market whether it be due to the Apple iPhone or Samsung Galaxy. They are also both publicly traded companies. Apple is traded primarily on the National Association of Securities Dealers Automated Quotations, better known as the NASDAQ Stock Exchange, while Samsung is traded primarily on the Korean Stock Exchange or KSE. Because they are publicly traded, their financial data is transparent and provides useful proxies for curve fitting. The modified system is now as follows with the appropriately notated variables and parameters.

$$\begin{aligned} \frac{dA}{dt} &= g_1 A(1 - \frac{A}{k_1}) - c_1 AS \\ \frac{dS}{dt} &= g_2 S(1 - \frac{S}{k_2}) - c_2 AS \end{aligned} \tag{3.1}$$

Here, $A(t)$ and $S(t)$ represent Apple and Samsung's smartphone market share percentage in the United States at time t in years. The parameter g represents each company's growth in the market. The logistic growth term of the form $(1 - \frac{\Omega}{k})$ shows that growth can be limited. In this case it is limited by k, the carrying capacity or the maximum market share a firm Ω could achieve. Finally, c is the interaction parameter representing pure competition per Table 1.

This system contains a few assumptions. First, it is assumed that both Apple and Samsung follow logistic growth. Growth slows down as both Apple and Samsung approach their respective market share carrying capacities. In reality, this may not be the case. If Apple controls 50% of the market, it is entirely plausible that their market share growth continues to increase at the same rate as if they controlled 40%. Also, the carrying capacities k_1, k_2 are independent of one another. There is a reasonable case to be made that Apple and Samsung ought to share the same carrying capacity k as they are competing for control of the same market. However, independent carrying capacities capture nuances better. This is because the use of the same carrying capacity k would imply that both companies have the ability to hit the same market share carrying capacity. In reality, companies have various resource constraints which would mean their market share potentials are independent. Thirdly, the growth rates, carrying capacities, and competition coefficients are not dynamic and are all constant over time. In the real world, these coefficient certainly change with time and are subject to ebb and flow. Finally, market dynamics are simple. Elements like technological advancement, brand image, consumer preference, and others are not explicitly factored into the model. The key idea here is that the competition coefficients could capture these elements. albeit implicitly. Finally, this system only captures the dynamics of two firms competing in a market. The competition dynamics between three or more firms and the implication of higher dimensions is not implemented here.

4. Parameters. Utilizing SciPy, the model was fitted to the historic annual U.S. smartphone market share percentages for Apple and Samsung. These figures were calculated by summing the annual U.S. smartphone products sold by Apple, Samsung, LG, Motorola, RIM, HTC, Google, Nokia, ZTE, Sony and all others and obtaining the respective market share ratios for each company. Refer to Table 6 in the appendix for relevant data and sources.

Numerically, this was done utilizing SciPy's optimize.curvefit function. The mathematics behind this involves applying non-linear least squares to fit a model's function f to the data [4] with a Levenberg-Marquadt algorithm. The Levenberg-Marquadt method is made up of two sub-methods. One is the Inverse-Hessian method and the other the gradient or steepest descent method. In this case, the Inverse-Hessian method is the Gauss-Newton method (this can be thought of as an approximation to the Inverse-Hessian method when dealing with least square problems). A dynamic dampening or fudge parameter λ affects the strategy of how the Levenberg-Marquadt algorithm continues to minimize the error between the data and the function. When the SciPy curve fit function starts its initial iteration but does not result in a satisfactory minimum, the dampening parameter λ is increased causing the next iteration to behave more like the gradient descent method. If the iterations begin approaching a satisfactory minimum, then the λ parameter is reduced, making the function behave more like the Gauss-Newton method [9].

Each differential equation from the system is fitted independently from one another and utilizes an initial guess for each parameter. All sequential guesses by the function are non-negatively bounded. For all growth rate g guesses, there is an explicit bound between 0 and 1. The rationality behind this growth rate bound is as follows. First, the growth coefficient being positive is necessary for competition. Nevertheless, it cannot be greater than 1 or 100% because no company possesses a year-to-year smartphone market share growth rate with an order of magnitude in the triple digits. Additionally, the average compound annual growth rate (CAGR) for companies in the smartphone market as a whole is negative. In other words, most companies are losing. However, as stated for the intent of the model, an initial guess with negative growth (decay) would not make sense. Therefore, a conservative positive growth rate g of 1%, is initialized for both. The carrying capacities k is also explicitly bounded between 0 and 1 because a firm cannot control more than 100% of the market. The initial guess for the carrying capacities k start at 1 as this simulates both companies having the potential to capture the whole market. Finally, there is the competition coefficient. It is a fairly abstract parameter which implicitly entails a variety of factors. Because of its abstract nature, no bound is explicitly defined, the only requirement is that it is non-negative. Given this, the initial guesses for the competitions coefficients c of 0.01 are notional. Refer to Table 2 for the initial guesses.

Parameter	**Initial Guess**
g_1	0.01
k_1	1
c_1	0.01
g_2	0.01
k_2	1
c_2	0.01

Table 2: Initial parameter guesses for Apple and Samsung.

The iteration count is 10,000 and the outputs are as followed and are rounded to the hundredths for simplicity purposes. Refer to Table 3 below.

Apple	**Samsung**
$g_1 = 0.24$	$g_2 = 0.90$
$k_1 = 0.58$	$k_2 = 0.30$
$c_1 = 0.03$	$c_2 = 0.21$

Table 3: Fitted parameters for Apple and Samsung.

By curve fitting, it can be determined how well the differential equations can match a real world phenomenon. The empirical data showcases the true smartphone market share dynamics of the real world be it linear, oscillating, or exponential. If the model cannot fit to it, this tends to indicate problems and presents a probable cause that the model is not appropriate to capture said real world phenomenon.

Here, the real world data shows that Apple and Samsung's market share percentages both increase to a certain market share percentage range and then stabilize. The model seems to be able to capture this dynamic, a good indicator the model is working as intended. See Figure 1 for the visualization. Plugging these fitted parameters in, the system is now as followed (4.1).

$$\begin{aligned} \frac{dA}{dt} &= (0.24)A(1-\frac{A}{0.58}) - (0.03)AS \\ \frac{dS}{dt} &= (0.90)S(1-\frac{S}{0.30}) - (0.21)AS \end{aligned} \tag{4.1}$$

The realism of the parameters vary. As defined with the bounds, the carrying capacities k_1 and k_2 themselves should never exceed 1 because the maximum market share any company could have is 100%. Note also that the sum of k_1 and k_2 is 0.88 which is less than 1. Given that Apple and Samsung are competing for control of the same market, these parameters are quite reasonable. Regarding the competition coefficients, note that the Samsung's competition coefficient c_2 is greater than Apple's competition coefficient c_1. This does make sense as Apple controls a bigger share of the market. In other words, pure competition hurts the smaller firm Samsung more than it does the bigger firm Apple. The only parameters that are a stretch are the fitted growth rates. The rates of 24% for Apple and 90% for Samsung are simply too high to be realistic. Overall though, considering the simplicity of the model, the majority of the parameters are reasonable.

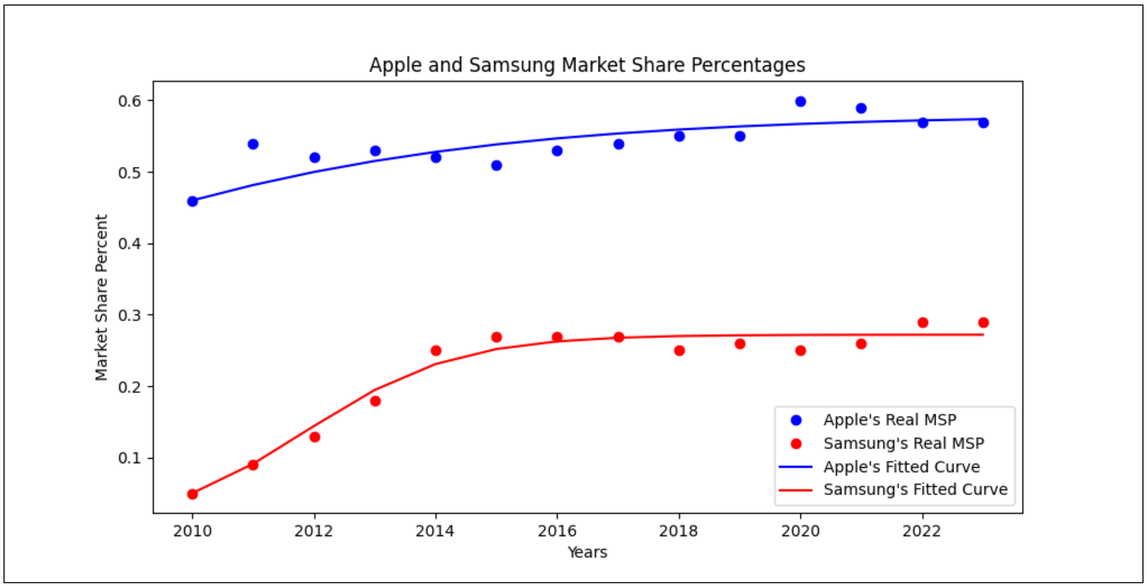


Figure 1: The Lotka-Volterra model fitted to smartphone market share data.

Employing the same fitted parameters, the curve itself can be extended an additional 100 years to observe what market share percentage Apple and Samsung would be expected to be in the long run. At the end point in time, Apple has a market share value of 58.02% and Samsung has a market share value of 27.19%. See Figure 2 below.

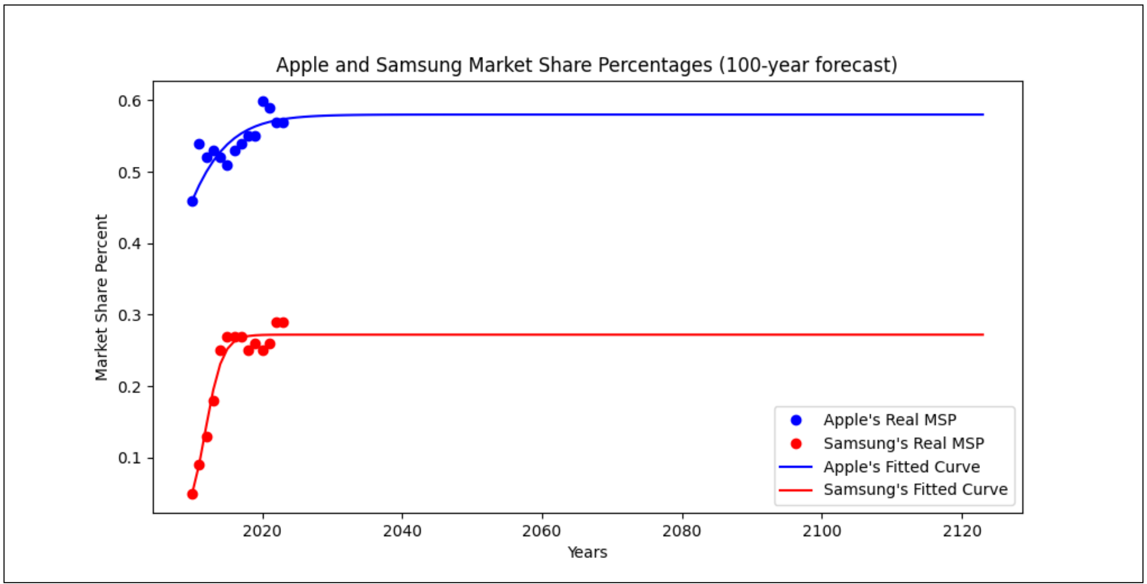


Figure 2: The Lotka-Volterra model fitted to smartphone market share data (extended 100 years).

Any model can have a fitted curve. However, a curve of best fit is not guaranteed to be a good fit (Figure 1). In order to determine whether or not the curve is a good fit, it is necessary to see the deviation between the predicted values and actual values. Market share is a continuous variable and as such, the root mean square error, or RMSE, and the mean absolute error, or MAE tests are employed to measure the error between the fitted parameter curves and the actual market data. The two tests are different in their error weightings, but both tests conclude that the fit is good and that the model's differential equations do a good job of capturing the real world market share phenomenon. Refer to Table 4 for RMSE and MAE values.

	Root Mean Square Error	**Mean Absolute Error**
Apple	0.02	0.02
Samsung	0.01	0.01

Table 4: RMSE and MAE for Apple and Samsung fitting.

5. Analysis. After reviewing the system of equations (3.1), it becomes apparent that there are four analytical equilibrium points that satisfies both $\frac{dA}{dt} = 0$ and $\frac{dS}{dt} = 0$. One equilibrium point is when (A^*, S^*) is $(0, 0)$. This equilibrium point would mean that both Apple and Samsung control none of the market. In real terms this could represent Apple and Samsung having exited the smartphone market willingly, them having gone out of business, or them having never entered the market in the first place.

The next two equilibrium points will be $(A^*, S^*) = (k_1, 0)$ and $(A^*, S^*) = (0, k_2)$. These two are more interesting as they represent the forming of a monopolistic market. It is possible that one firm falls leaving the other firm to dominate, or it could be the case that one firm dominates leaving the other firm unable to enter the market.

Finally, there is the non-zero fourth equilibrium point. This equilibrium point is the most insightful as it means both firms are coexisting and actively competing with each other and is located below (5.1) where $g_1g_2 \neq c_1c_2k_1k_2$.

$$(A^*, S^*) = \left(-\frac{-(g_1g_2k_1) + c_1g_2k_1k_2}{g_1g_2 - c_1c_2k_1k_2}, \frac{-(g_1g_2k_2) + c_2g_1k_1k_2}{-(g_1g_2) + c_1c_2k_1k_2}\right) \tag{5.1}$$

As an alternative to solving for them analytically, these equilibrium points could be approximated graphically by setting the horizontal and vertical nullclines equal to zero, i.e. $\frac{dA}{dt} = f(A, S) = 0$, plotting each function, and notating where they intersect.

From graphing the horizontal and vertical nullclines and seeing where they intersect, and by plugging in the fitted parameters into the analytical equilibrium points, the four equilibrium points (A^*, S^*) are the black point at $(0, 0)$, the blue point at $(0.58, 0)$, the red point at $(0, 0.30)$, and the green point at $(0.5611, 0.2607)$. As all the equilibrium points are now known, the system's behavior as a whole can better be understood graphically with the use of a phase portrait.

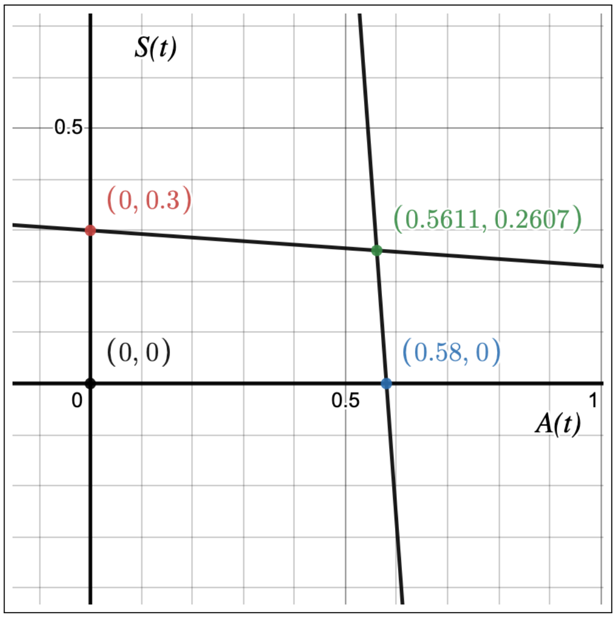


Figure 3: System's horizontal and vertical nullclines with equilibrium points.

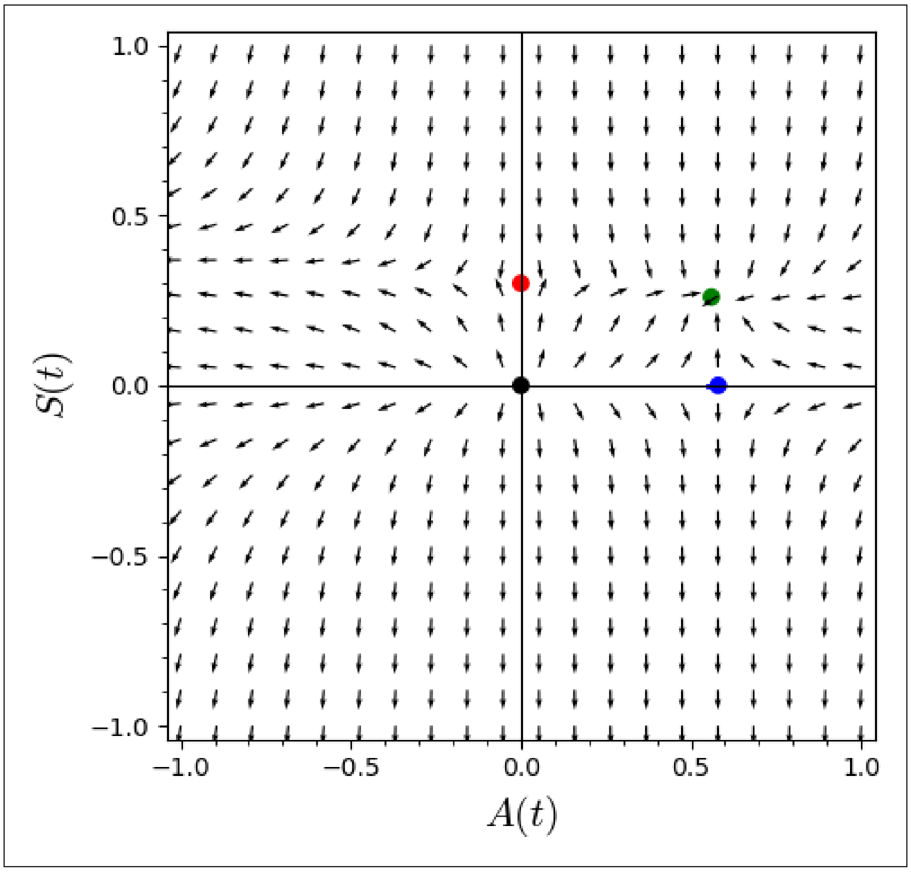


Figure 4: System's phase portrait with equilibrium points.

With all four equilibrium points plotted in the phase plane, the vector field produces some interesting insights. Any initial condition that starts anywhere besides quadrant 1 will result in either Apple, Samsung, or both's market share being negative for all forward time. Given that there is no such thing as a negative market share, these initial conditions are not economically sensible and should be ignored. Furthermore, because it is not economically sensible to control more than 100% of the market, the window of interest is from 0 to 1 for both $A(t)$ and $S(t)$.

Qualitatively, there are two saddles. One is the blue point $(0.58, 0)$ and the other is the red point $(0, 0.30)$. Saddles have stable and unstable manifolds. Stable manifolds are a set of points that form a trajectory such that any initial condition on this trajectory will approach the equilibrium point asymptotically in forward time. Unstable manifolds are a set of points that form a trajectory where any initial condition on this trajectory approaches the equilibrium point asymptotically in backward time.

In the case of this system, the equation for the stable manifold for $(0.58, 0)$ is $S = 0$. The interpretation is that if any initial condition starts with $S = 0$, Apple dominates the market for all forward time. Mathematically, for initial conditions with $0 < A \leq 1$ and $S = 0$, then the $\lim_{t\to\infty} A(t) = 0.58$. The equation for the stable manifold for $(0, 0.30)$ is $A = 0$. If any initial condition starts with $A = 0$, Samsung dominates the market for all forward time. In similar fashion, for initial conditions with $A = 0$ and $0 < S \leq 1$, then the $\lim_{t\to\infty} S(t) = 0.30$.

The interpretation of the saddles is that if you are a firm that starts with no market power, you stay with none forever. As the second firm, you simply cannot enter the market out of nowhere if one firm already dominates it. This is indicative of a high barrier to entry and a monopolistic market.

In quadrant 1, qualitative analysis shows that only the green point $(0.5611, 0.2607)$ is stable. It behaves as a nodal sink while the other points exhibit unstable behavior of some form. Any initial condition in quadrant 1 that is not on the $A(t)$ or $S(t)$ axis will approach it in forward time. However, going off of qualitative analysis alone is insufficient. It is vital to confirm the behavior of these points with another method.

$$\begin{aligned} \frac{dA}{dt} &= g_1 A(1 - \frac{A}{k_1}) - c_1 AS = f(A,S) \\ \frac{dS}{dt} &= g_2 S(1 - \frac{S}{k_2}) - c_2 AS = g(A,S) \end{aligned} \tag{5.2}$$

In order to confirm the behavior of the system with constant parameters, linearization will be utilized. A nonlinear system such as this can be approximated near each equilibrium point with a linear system by using a Taylor series expansion for f and g. Let u and v denote small disturbances from a fixed/equilibrium point (A^*, S^*).

$$\begin{aligned} u &= A - A^* \\ v &= S - S^* \end{aligned} \tag{5.3}$$

Differentiate u and v. Substitute A and S in terms of A^*, u, S^*, and v from (5.3) into $f(A,S)$ and $g(A,S)$.

$$\begin{aligned}\frac{du}{dt} &= \frac{dA}{dt}\\ \frac{dv}{dt} &= \frac{dS}{dt}\end{aligned} \tag{5.4}$$

$$\begin{aligned}\frac{du}{dt} &= f(A^*+u, S^*+v)\\ \frac{dv}{dt} &= g(A^*+u, S^*+v)\end{aligned} \tag{5.5}$$

Apply Taylor series expansion. Assuming higher order terms O are sufficiently small, and given that the functions at each fixed point $f(A^*, S^*)$ and $g(A^*, S^*)$ are 0, simplify.

$$\begin{aligned}\frac{du}{dt} &= f(A^*,S^*) + u\frac{\partial f}{\partial A} + v\frac{\partial f}{\partial S} + O(u^2,v^2,uv)\\ \frac{dv}{dt} &= g(A^*,S^*) + u\frac{\partial g}{\partial A} + v\frac{\partial g}{\partial S} + O(u^2,v^2,uv)\end{aligned} \tag{5.6}$$

$$\begin{aligned}\frac{du}{dt} &= u\frac{\partial f}{\partial A} + v\frac{\partial f}{\partial S}\\ \frac{dv}{dt} &= u\frac{\partial g}{\partial A} + v\frac{\partial g}{\partial S}\end{aligned} \tag{5.7}$$

Convert system to matrix notation. The 2x2 matrix of this linearized system evaluated at (A^*, S^*) is the Jacobian matrix of the system. Solving for the partial derivatives, the resulting form for the Jacobian matrix for the system is as follows.

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix} = \begin{bmatrix}\frac{\partial f}{\partial A} & \frac{\partial f}{\partial S}\\ \frac{\partial g}{\partial A} & \frac{\partial g}{\partial S}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} \tag{5.8}$$

$$J = \begin{bmatrix}\frac{\partial f}{\partial A} & \frac{\partial f}{\partial S}\\ \frac{\partial g}{\partial A} & \frac{\partial g}{\partial S}\end{bmatrix}\Bigg|_{(A^*,S^*)} \tag{5.9}$$

$$J = \begin{bmatrix}g_1 - \frac{2g_1A}{k_1} - c_1S & -c_1A\\ -c_2S & g_2 - \frac{2g_2S}{k_2} - c_2A\end{bmatrix}\Bigg|_{(A^*,S^*)} \tag{5.10}$$

Plugging in the known parameters of the system $g_1 = 0.24, k_1 = 0.58$, etc. while evaluating the Jacobian matrix at the four equilibrium points $(0,0)$, $(0.58,0)$, etc. yields the following matrices for each equilibrium point.

$$J(0,0) = \begin{bmatrix} 0.24 & 0 \\ 0 & 0.9 \end{bmatrix}$$

$$J(0.58,0) = \begin{bmatrix} -0.24 & -0.0174 \\ 0 & 0.7782 \end{bmatrix} \tag{5.11}$$

$$J(0,0.30) = \begin{bmatrix} 0.23 & 0 \\ -0.063 & -0.9 \end{bmatrix} J(0.5611, 0.2607) = \begin{bmatrix} -0.23217962 & -0.016833 \\ -0.054747 & -0.782031 \end{bmatrix}$$

The next step is to solve for the trace T and determinant D for each matrix. In the case of this system's Jacobian, the general form for the trace and determinant at each point would be as followed and are evaluated accordingly.

$$\mathrm{D}(J) = \left(g_1 - \frac{2g_1A^*}{k_1} - c_1S^* \right) \left(g_2 - \frac{2g_2S^*}{k_2} - c_2A^* \right) - (-c_1A^*)(-c_2S^*) \Bigg|_{(A^*,S^*)} \tag{5.12}$$

$$\mathrm{T}(J) = \left(g_1 - \frac{2g_1A}{k_1} - c_1S \right) + \left(g_2 - \frac{2g_2S}{k_2} - c_2A \right) \Bigg|_{(A^*,S^*)} \tag{5.13}$$

The placement of an ordered pair (T, D) relative to the parabola $T^2 - 4D = 0$ is utilized to classify the equilibrium points in the trace-determinant plane. The ordered pairs for each evaluated Jacobian matrix are given in Table 5 below and are rounded to the nearest thousandths.

Jacobian	Trace T	Determinant D
$J(0,0)$	1.14	0.216
$J(0.58,0)$	0.538	-0.187
$J(0,0.30)$	-0.669	-0.208
$J(0.5611,0.2607)$	-1.014	0.181

Table 5: Trace-determinant coordinates for each equilibrium point.

Graphing these in the trace-determinant plane (Figure 5), it can be confirmed with certainty that equilibrium point $(0,0)$ is a nodal source. It can also be confirmed that equilibrium points $(0.58,0)$ and $(0,0.3)$ are saddles. As qualitatively assessed, these three equilibrium points are considered unstable. Most importantly, the equilibrium point of interest, $(0.5611, 0.2607)$, is indeed stable as it is confirmed to be a nodal sink. The trace-determinant plane classification legend is a common figure in dynamics and can be used to verify the results of this system.

An important caveat to the linearization process is that the use of the Jacobian matrix is an approximation, one that neglects quadratic terms and all other higher order terms under the assumption that these sufficiently small nonlinear terms will not affect the classification outcome (5.6). This means that in some cases, linearization of a system can fail. The *Hartman-Grobman theorem* [11] states that precise linearization classifications holds true only for robust cases such as saddles, nodes, and spirals. These equilibrium points are also referred to as being hyperbolic because their stability types are unaffected by small nonlinear terms. If the linearized system predicts any borderline case such as a center, degenerate node, or line of fixed points, the neglected higher order terms can make a difference. These are nonhyperbolic. Fortunately, in this case, all Jacobian classifications are either a saddle or node, constituting the *Hartman-Grobman theorem*, and making their classification absolute. Proof of this result has been established for decades [1].

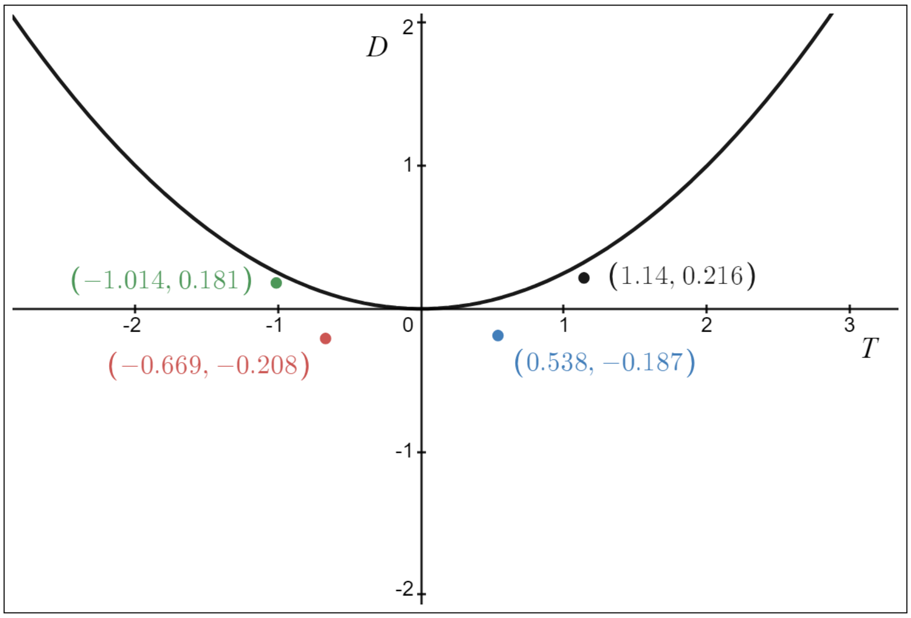
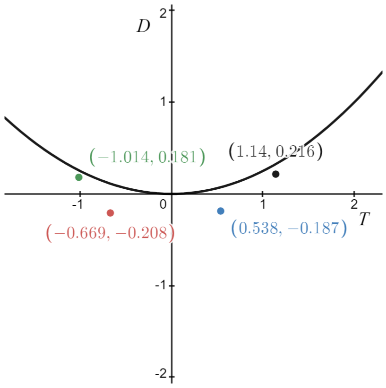


Figure 5: Trace-determinant plane for the system of fitted parameters (4.1).

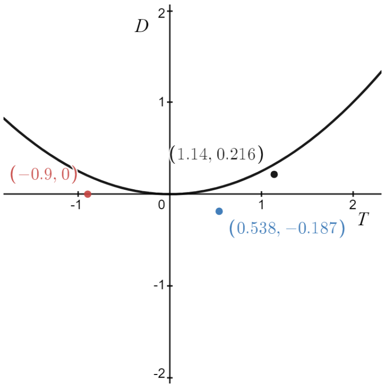
Alternatively, the eigenvalues λ_1 and λ_2 for the linearized system at each equilibrium point could be utilized to classify them as well. Point given, $J(0.58, 0)$ has eigenvalues $\lambda_1 = 0.9$ and $\lambda_2 = -0.24$. If one eigenvalue is positive and the other negative, as is the case here, the system has a saddle. This is a finding that is in tandem to the trace-determinant plane.

Section 3 mentions that one constraint of the model as a result of its simplicity is that parameter coefficients are constant throughout time. If parameters are no longer constant, there is a captivating implication. Bifurcations highlight critical changes in behavior and topological structure for the system depending on the value of some parameter. Say, Apple's competition coefficient c_1 is no longer constant. Then, a change in c_1 can alter the system behavior fundamentally. It is in Samsung's interest to drive c_1 up as much as possible because the higher the competition coefficient is for Apple, the lower Apple's positive market share rate of change becomes. This could be done in many ways. As an example, say the c_1 coefficient takes into account individual smartphone pricing. If Samsung employs an aggressive smartphone pricing strategy, the c_1 coefficient would be bound to increase.

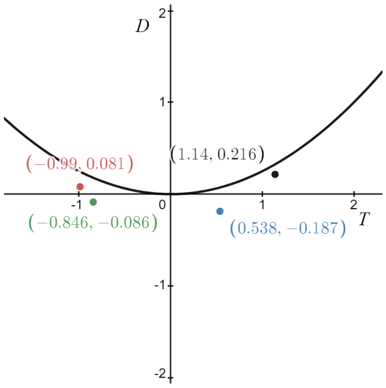
As c_1 increases, it will eventually hit a critical value of c_1 such that the stability and existence of certain equilibrium points alter. Hence, a critical change in the behavior of the system is realized. Analytically, this can be assessed with the Jacobian matrix's trace (5.13) and determinant (5.12). Graphically, it can be assessed with the use of phase portraits. We opt to show the qualitative change of the system and acknowledge that a more rigours algebraic computation could be done to show the bifurcation as well. Recall that the nontrivial fixed point (5.1) is the only fixed point dependent on c_1. In fact, it is the only fixed point where becoming undefined is a possibility. Analytically, it is clear that the critical value $c_1 = \frac{g_1}{k_2}$ results in a 0 value denominator for S^*, and hence a undefined fixed point. This critical value occurs at $c_1 = 0.80$ given the current parameters. See Figure 6 and Figure 7 for varying c_1 parameter results.



(a) $c_1 < 0.8$ ($c_1 = 0.03$)

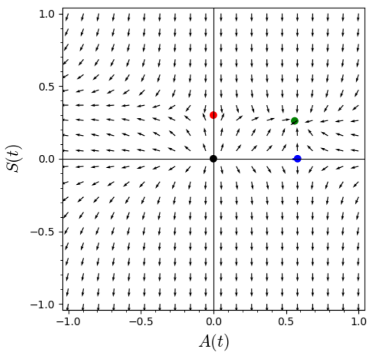


(b) $c_1 = 0.8$

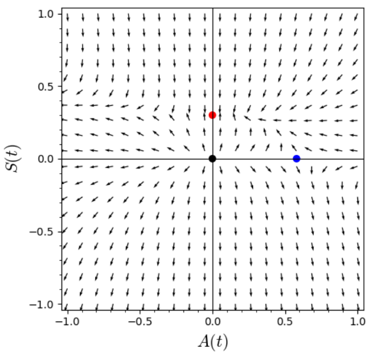


(c) $c_1 > 0.8$ ($c_1 = 1.1$)

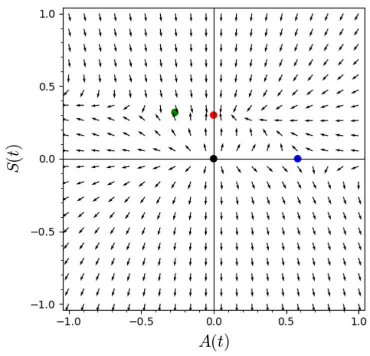
Figure 6: Trace-determinant plane diagram for varying c_1.



(a) $c_1 < 0.8$ ($c_1 = 0.03$)



(b) $c_1 = 0.8$



(c) $c_1 > 0.8$ ($c_1 = 1.1$)

Figure 7: Phase portrait diagram for varying c_1.

In observing the phase portraits, as c_1 increases, the nontrivial fixed point (green) shifts leftward eventually colliding with the fixed point $(0, k_2)$ (red). This occurs when $c_1 = 0.8$.

When this happens, the nontrivial fixed point becomes undefined as shown analytically above. At the same time, the fixed point $(0, k_2)$ (red) changes classification from a saddle to a line of stable fixed points as shown in the trace-determinant plane classification. This result is not immediately obvious from the phase portrait. Finally, as c_1 increases from 0.8, the nontrivial fixed point re-emerges. However, the trace-determinant plane shows a swap in stability for the nontrivial fixed point (green) as it becomes a saddle. Additionally, there is another swap in stability as $(0, k_2)$ (red) becomes a sink. This change is well reflected in the phase portrait. Interesting enough, the other two fixed points $(0, 0)$ (black) and $(k_1, 0)$ (blue) have no change in stability whatsoever. The trace determinant plane and phase portrait analysis ultimately shows an exchange of stability between two fixed points. This is indicative of a trans-critical bifurcation as the $(0, k_2)$ (red) fixed point exists for all real parameter values of c_1, but the stability of it eventually changes.

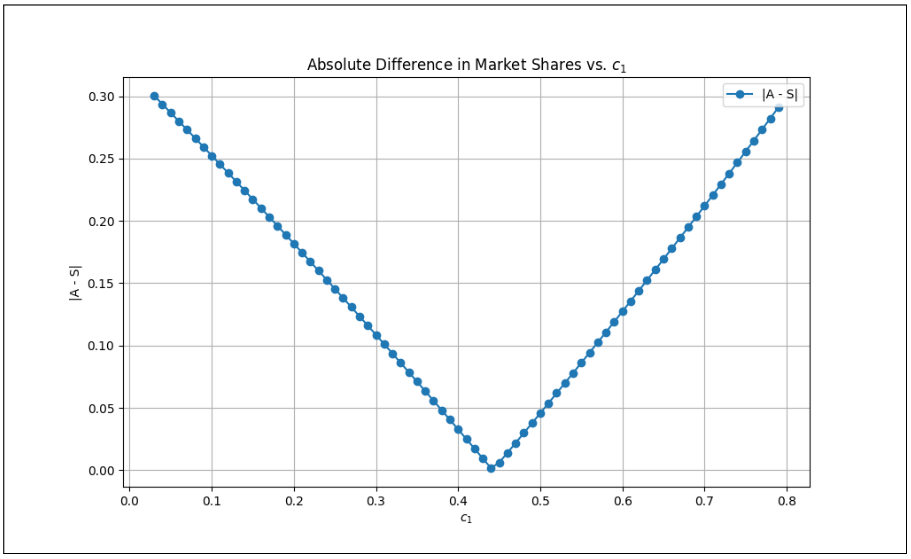


Figure 8: $|A - S|$ vs. c_1 bifurcation diagram.

The long-term implications of this topological shift is that any economically sensible initial condition in quadrant 1 where either $A_0 \neq 0$ or $S_0 \neq 0$ now approach fixed point $(0, k_2)$ (red) in forward time. The bifurcation leads to a scenario where Samsung would dominate the market with Apple having no long-term market presence whatsoever. This constitutes a true monopolistic market that doesn't rely on stable manifolds. Notice that the nontrivial point is still present, and as it is now a saddle, it does possess a stable manifold. However, it is outside quadrant 1 and is therefore not economically sensible as a company cannot possess a negative market share. Another critical competition insight comes from the creation of a bifurcation diagram for the varying c_1 parameter and can be observed in Figure 8. Here, we observe the change in $|A - S|$, the magnitude of the difference between the market share of Apple and Samsung. The bigger the difference, the less competitive Apple and Samsung are with each other as one firm dominates. The smaller the difference, the more competitive Apple and

Samsung are with each other. As c_1 approaches 0.8, it becomes evident that there exists a c_1 value where the market share difference of Apple and Samsung gets closest to 0. In terms of hundredths, this occurs at $c_1 = 0.44$, representing evenly matched competitiveness between Apple and Samsung. However as c_1 increases from that, the difference begins to rise again, showing that one of the firms begins to get the advantage again.

6. Summary. In conclusion, this alternative Lotka-Volterra model that was originally designed to model ecological competition between species does have the ability to reasonably predict how the market shares of Apple and Samsung will behave in the long-run even with imperfect parameters. Graphically and analytically, it has been shown that the equilibrium point $(0.5611, 0.2607)$ is a stable nodal sink. Given any initial condition in quadrant 1 that is not on any of the stable manifolds, it is expected through forward time that Apple converges to control 56.11% of the smartphone market while Samsung would converge to control 26.07% of the smartphone market. This is reasonable because empirical smartphone market share data shows that Apple has stabilized in the 50% to 60% range while Samsung has stabilized in the 20% to 30% range. Furthermore, when the fitted curve is extended an additional 100 years, Apple's forecasted long run market share is 58.02% and Samsung's forecasted long run market share is 27.19%. The results of the two approaches, analytical and numerical, are consistent, thus providing credence to the model.

However, the analytical and graphical analysis of the system also provides additional insights that the numerical approach does not. It is demonstrated that monopolies and a high barrier to entry to the market happens in the long-run if initial conditions possess either $A_0 = 0$ or $S_0 = 0$. If you are a firm that starts with no market power, you stay with none for all forward time. As the second firm, you simply cannot enter the market out of nowhere if one firm already dominates it. All of this highlights that the model can not only be applied successfully in predicting future market share, but can yield additional insights as well.

The practical applications of this model are many. This model could further be modified to be a system of three differential equations (3-dimensional system) to model three firms competing or more (nth-dimensional system). However, with the introduction of 3-dimensional systems comes the possibility of chaos dynamics. More work would need to be done to precisely understand the implications of it. In addition, further research could involve testing this model in other markets besides smartphones to measure its repeatability. One bifurcation scenario was explored, but with six parameters, there are many more bifurcations scenarios that further work could be done on. The incorporation of stochastic processes to the system could also add tremendous value to the understanding of this system in order to analyze and better account for uncertainty.

Another area of potential exploration is defining the more abstract competition coefficient c. The bifurcation analysis in Section 5 gave an example of how a change in c_1 could have major ramifications on the dynamics of the system. It directly affects just how competitive Apple and Samsung are with each other as seen with the bifurcation diagram (Figure 8). The question of how to define the competition coefficient is a difficult one. An extensive amount of work would need to be done in order to find a defensible way to capture just how much policy and other factors would interact and affect the competition parameter.

As for further improvements, one aspect is that the historical data utilized could be more robust. More data points could have led to a better and more realistic fitting. Since the iPhone came out in 2007, only 3 data points could have been added in yearly time. However, it is possible that looking at the system in terms of monthly time could provide the extra data points that could lead to a more granular fitting. This in turn could have led to better parameters and an even better predictive output. On the other hand, perhaps it is the case that the data is sufficient but there could be a more complex version of this model, that when modified, could capture the competition dynamics and predict future market share even better. Nevertheless, this literature shows that this alternative Lotka-Volterra model that contains an explicit carrying capacity can have successful applications in the world of economics and industrial organization.

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Appendix A. Historic annual U.S. smartphone market share data.

Apple		Samsung	
Year	**Market Share**	**Year**	**Market Share**
2023	57.23%	2023	29.08%
2022	56.74%	2022	28.94%
2021	58.58%	2021	25.64%
2020	59.54%	2020	24.72%
2019	55.23%	2019	25.67%
2018	54.82%	2018	24.76%
2017	53.89%	2017	26.62%
2016	53.19%	2016	26.79%
2015	50.85%	2015	26.57%
2014	52.28%	2014	25.34%
2013	52.76%	2013	18.38%
2012	51.65%	2012	12.84%
2011	54.04%	2011	9.12%
2010	46.19%	2010	5.27%

Table 6: Historic annual U.S. smartphone market share data for Apple and Samsung. Source: https://www.bankmycell.com/blog/us-smartphone-market-share

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