Math Matters

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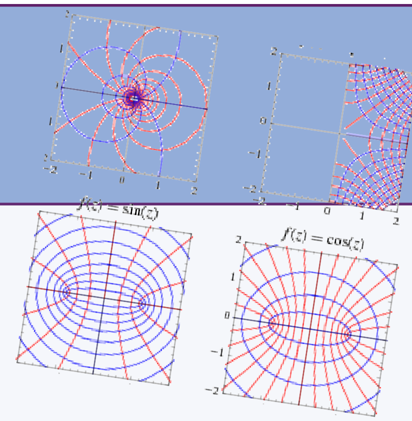
The math behind...

Conformal Maps



Technical terms used:

Conformal map, Riemann mapping theorem, Cauchy-Riemann equation, holomorphic function, (discrete) complex analysis

Uses and applications:

For almost 450 years, conformal world maps have been essential for navigation. Nowadays, they can be found in atlases and are shown by the major online street mapping services. But today's applications of conformal maps go far beyond cartography: Conformal texture mapping is a frequently used technique in computer graphics and many two-dimensional models in statistical physics exhibit properties that are invariant under conformal maps.

How it works:

Before the invention of satellites and GPS, sailors only could use angle measurements for navigation. To correctly determine their position on the open sea, the angles on their nautical charts had to agree with the corresponding angles on the earth. Such maps are called conformal. In 1569, Gerhard Mercator introduced the first conformal world map for use in navigation. Straight line segments on the Mercator map are not shortest paths on the earth (but they are on the gnomonic projection), but a compass needle does not move on such a route.

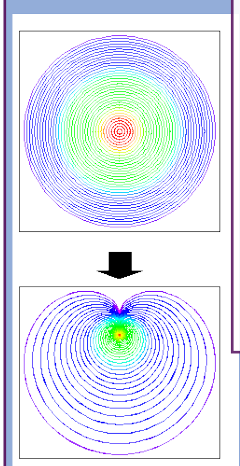
Any conformal map locally scales uniformly in all directions such that small shapes are preserved. For this reason, conformal maps are used in computer graphics to map planar images onto given surfaces. Algorithms computing conformal texture coordinates rely on the Riemann mapping theorem. This theorem states that any closed orientable surface without holes can be conformally mapped to the sphere and any simply-connected closed subset to a disc.

Mathematically, the preservation of angles can be described by the Cauchy-Riemann equations, a set of two real differential equations. Complex functions on subsets of the complex plane that obey these equations are called holomorphic and are studied in complex analysis.

Limited resources of computers and simplifications in models of theoretical physics require discrete theories of complex analysis. The oldest approach is to discretize the Cauchy-Riemann equations, leading to linear theories. Other definitions, leading to nonlinear theories, involve patterns of circles or are based on a discretized notion of conformal equivalence for triangulated surfaces.

Interesting facts:

The linear theory of discrete complex analysis has been instrumental in the study of several discrete models of statistical physics, e.g., in the proof that the Ising model exhibits conformally invariant properties in the thermodynamical limit. For his groundbreaking research on this subject, Stanislav Smirnov was awarded the Fields medal.



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