Analytical Models for Thick and Modified Cylindrical Shell Vibrations

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Overview

Analytical vibration models for cylindrical shells remain important because they reveal the structural mechanisms behind resonance, mode localization, and radiated noise while also providing rapid prediction tools for design and validation. Existing literature already demonstrates that baseline analytical and semi-analytical treatments are possible for cylindrical shells. In particular, thick-cylinder vibration models based on three-dimensional elasticity and Rayleigh-Ritz-type formulations have been successfully developed, and thin-shell vibration models have also been studied using wave-propagation methods.

For electrical machines, unit-wave-response ideas have been used to characterize how motor and generator structures respond to circumferential force waves. This project asks SIAM participants to extend those foundations to practically important, mathematically challenging cases that appear to remain open or only partially developed.

Motivation and Context

Many electrical machines can be idealized as cylinders or near-cylinders (Fig. 1), but real systems depart from the textbook case in several ways: they may be made of multiple bonded materials, may include rigid endcaps, may include rigid mounting feet, and may be subjected to harmonic wave-like forcing rather than only free vibration. Each of these departures changes the admissible displacement fields, the boundary conditions, the modal structure, and the coupling among radial, circumferential, and axial motion.

The existing literature shows that baseline analytical progress is possible. Thick cylindrical shells have been treated using three-dimensional elasticity with layered or Rayleigh-Ritz-approaches, producing natural frequencies and mode shapes for several boundary conditions (Fig. 2). Janne Roivainen's dissertation further shows how cylindrical machine structures can be characterized by unit-wave responses, especially for circumferential excitation patterns indexed by wave number (r), and highlights the practical importance of the (r=2) response.

The opportunity is therefore not to start from nothing, but to generalize known analytical machinery to more realistic geometries and excitations.

The Problem

We propose a family of related problems as follows.

Problem 1: Two nested cylinders with different material properties

Find an analytical or semi-analytical solution for the free vibrations of two concentric cylinders, optionally of different lengths, that are bonded at their interface and have different elastic material properties. Rather than bonded on the entire surface, the bonding may be assumed to occur along 4 narrow and equally spaced regions only (Fig. 3).

The desired model should:

- allow for transversely isotropic elastic constants for the inner cylinder and distinct elasticity and density for the other cylinder
- enforce displacement and traction continuity at the bonding interface
- produce natural frequencies and mode shapes
- optionally work for cases where the outer cylinder is longer than the inner cylinder as shown in Fig. 3

Problem 2a: Rigid foot-mount foundation

Find an analytical or semi-analytical free-vibration solution for a thick cylinder (or the nested, bonded cylinders from problem 1) with rigid foot-mount foundation (Fig. 4).

The desired model should:

- represent the foot-mount foundation as rigid constraints
- characterize how modal families split when circular symmetry is broken
- determine how the foot-mount foundation alters boundary conditions and frequency spectra

Problem 2b: Rigid endcaps at both ends

Find an analytical or semi-analytical free-vibration solution for a cylinder (or the nested, bonded cylinders from problem 1) with rigid endcaps attached at both ends (Fig. 5).

The desired model should:

- represent the endcaps as rigid constraints or rigid inertial attachments,
- determine how the endcaps alter axial boundary conditions and frequency spectra
- distinguish the effect of rigid end conditions from classical simply supported, free, or clamped conditions
- identify whether the problem can still be reduced to separable axial functions or instead requires coupled expansions.

Problem 3: Rigid foot-mount and endcaps at both ends

Find an analytical method for a thick cylinder that has a foot-mount and endcaps (Fig. 6). This problem moves beyond periodic Fourier circumferential modes into a sector-domain eigenproblem with mixed geometric constraints.

The desired model should:

- handle loss of full circumferential periodicity
- impose clamped conditions on the foot-mount
- impose appropriate conditions for the endcaps
- characterize how modal families split when circular symmetry is broken

Problem 4: Harmonic excitation and unit-wave response

Find a solution method for harmonic excitation by computing the response to a unit wave input, with special emphasis on circumferential wave number ($r = 2$) (Fig. 7). This problem builds directly on the unit-wave-response viewpoint, where specific spatial excitation patterns are more informative than point-force frequency-response functions.

The desired model should:

- formulate the forced response to a unit circumferential wave input,
- derive the frequency response or unit-wave response for ($r = 2$),
- connect the free-vibration eigenstructure to the forced response,
- support extension toward vibration and possibly sound-radiation prediction.

Mathematical Starting Points

Participants need not treat the four problems as isolated. Several existing analytical frameworks suggest natural entry points:

1. **Rayleigh-Ritz and energy methods for thick cylinders.** These appear promising because they accommodate nontrivial boundary conditions and richer displacement fields.
2. **Three-dimensional elasticity formulations with layered discretization through thickness.** These are especially relevant for thick-shell and multi-material extensions.
3. **Modal superposition and unit-wave response.** These connect free vibration and harmonic forcing, especially when the forcing is naturally expressed by circumferential travelling force waves.
4. **Ref. [1], Chapter 4.** The dissertation linked below shows how parts of this approach can be applied to motor housing vibrations, although modifications are needed to address the specific problems.

Why This is Interesting

The appeal of this project is that it sits squarely between rigorous applied mathematics and useful engineering modeling:

- The baseline cylinder already admits analytical structure.
- Each extension introduces mathematically meaningful complications: interface conditions, rigid attachments, broken periodicity, and structured harmonic forcing.
- Even partial success would be very valuable: asymptotic models, reduced-order semi-analytical formulations, or provably accurate approximations would all be useful outcomes.

Workshop Goals

The objective of these workshop problems is to develop analytical or semi-analytical methods for modified cylindrical-shell vibration problems that go beyond the standard single-material full-cylinder case.

Teams should explore questions such as:

- Which of the problems admit exact separation of variables, and which require Ritz methods, transfer, or hybrid expansions?
- How should interface continuity conditions be imposed for bonded concentric cylinders?
- How do rigid endcaps modify the admissible axial functions and eigenvalue conditions?
- How can a unit circumferential wave input be used to derive a forced response formula for ($r = 2$)?
- Can one unified framework cover all four problems, or are fundamentally different methods needed?
- What asymptotic limits recover the known full-cylinder, open-end, or single-material results?

Figures

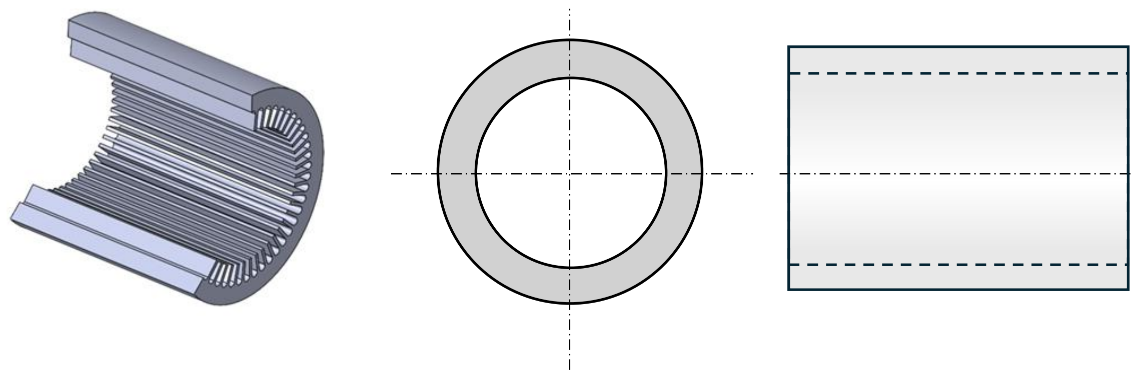

Fig. 1: Prior art: Thick-wall cylinder approximation of the actual part

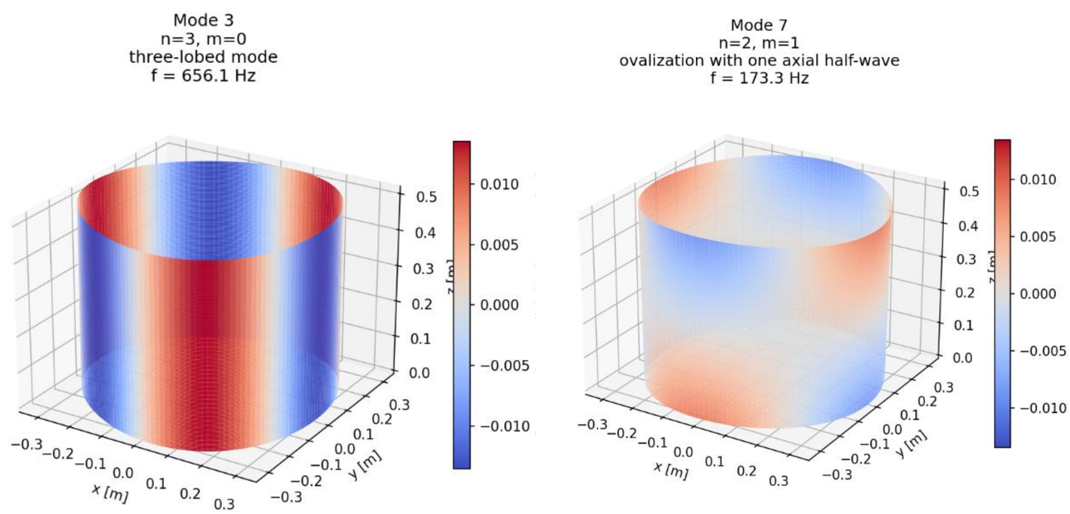

Fig. 2: Prior art: Examples of eigenmode and eigenfrequency solutions for a thick cylinder using the Rayleigh-Ritz approach

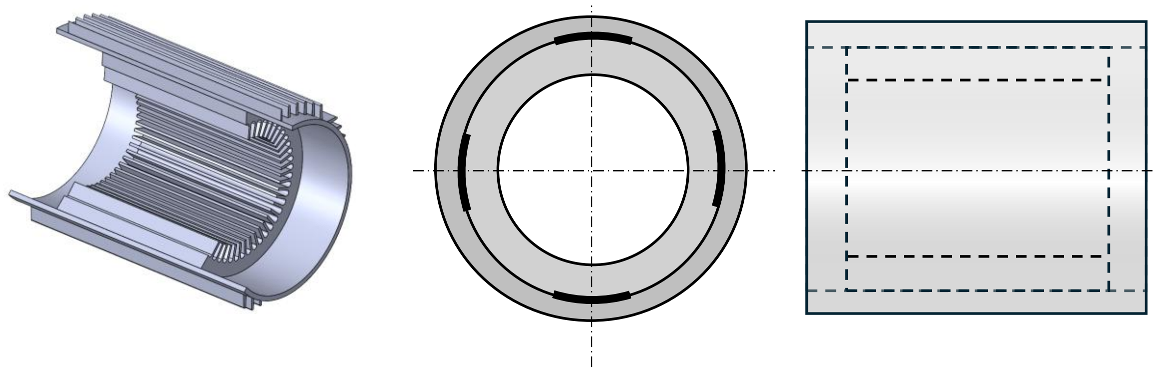

Fig. 3: Problem 1: Nested cylinders that are bonded either on the entire surface or only in 4 narrow equally spaced mating surfaces (bold).

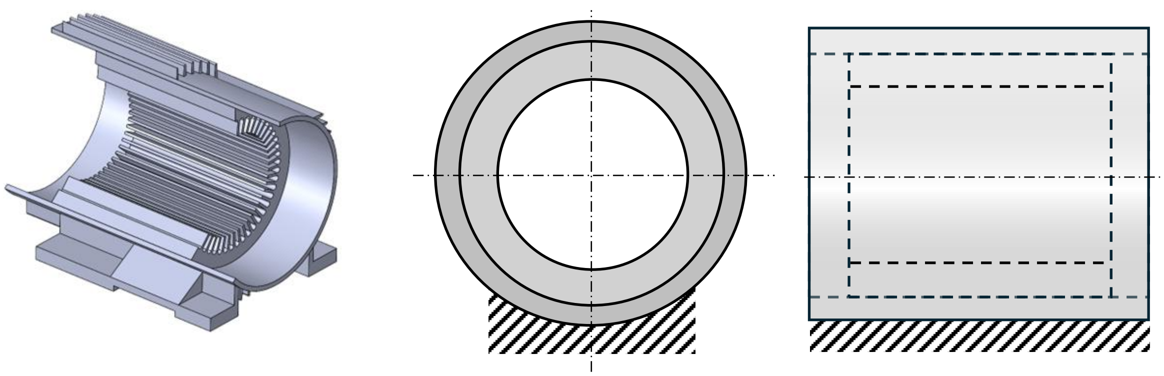

Fig. 4: Problem 2a: Mounting structure rigidly constrains the lower portion of the motor stator.

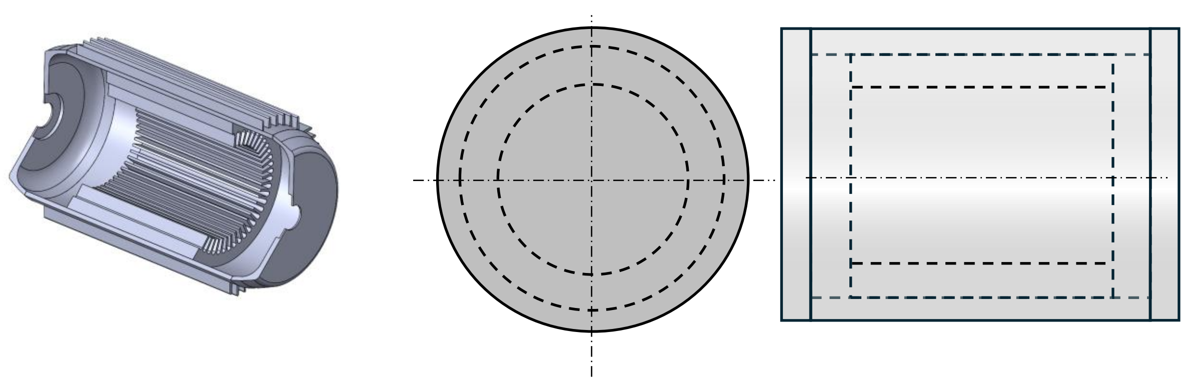

Fig. 5: Problem 2b: Addition of 2 rigid end disks that are rigidly bonded to the cylinders.

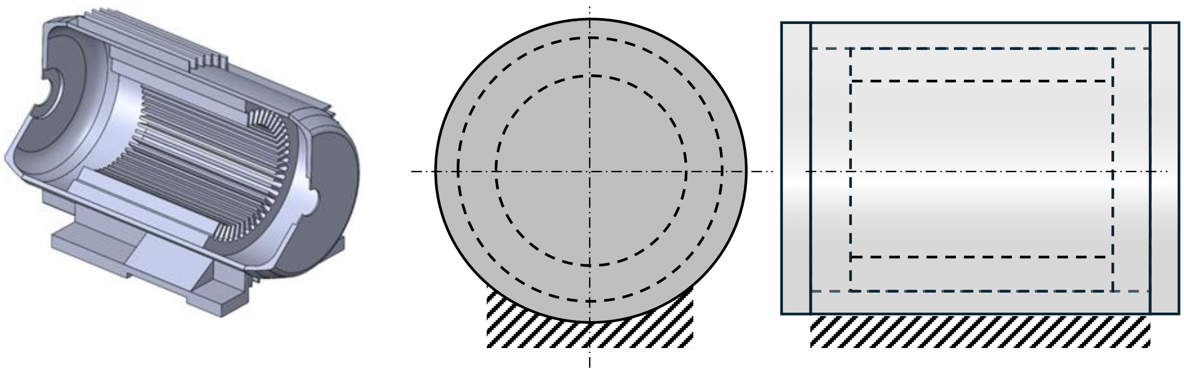

Fig. 6: Problem 3: Final system that represents the motor stator.

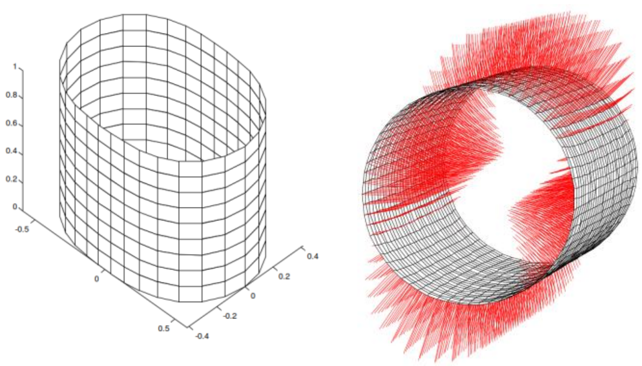

Fig. 7: Problem 4: Rotating force waves for $r = 2$.

References

1. Janne Roivainen, *Unit-Wave Response-Based Modeling of Electromechanical Noise and Vibration of Electrical Machines*, doctoral dissertation, Helsinki University of Technology, 2009. https://aaltodoc.aalto.fi/items/a721c2ba-d89b-4b36-adb7-4cf869cd5400
2. C. T. Loy and K. Y. Lam, "Vibration of Thick Cylindrical Shells on the Basis of Three-Dimensional Theory of Elasticity," *Journal of Sound and Vibration*, 226(4), 719-737, 1999.