ABB Group: Models and analysis of thick nested cylinders toward resonance calculations

Manuchehr Aminian, Sam Bradford, Taras Lakoba, Jaaziel de la Luz, Jack Murphy

with contributions from: J. Adriazola, P. Sanaei, B. Tilley

and industrial presenter: Stefan Rakuff of ABB

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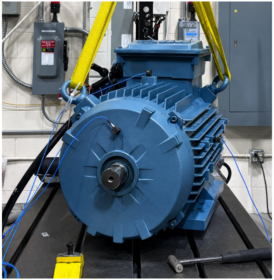
Outline

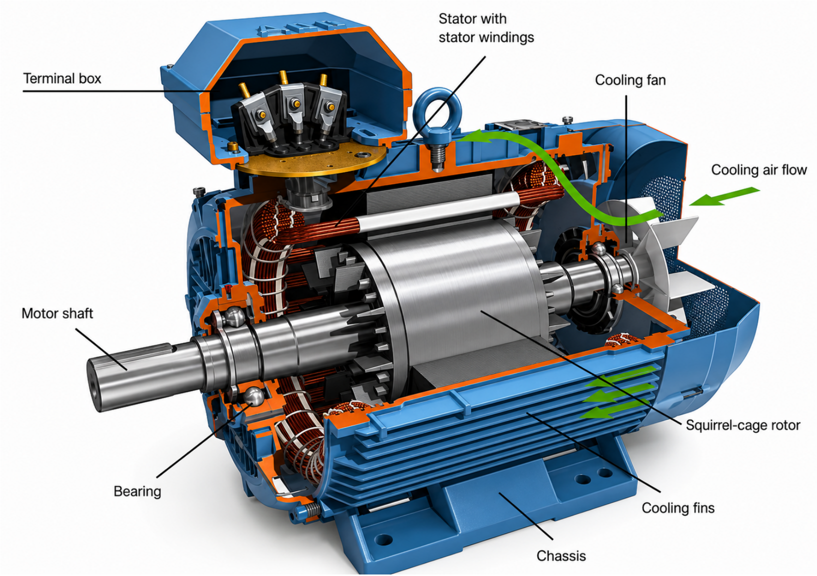
Background

Background

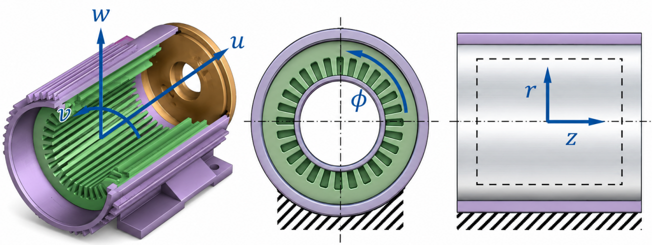
- Applications require the motor to be minimally noisy.
- The noise a motor produces is highly dependent on the geometry.
- **Goal for MPI:** build and analyze mathematical models which will accelerate eventual downstream resonance calculations relative to current practice (full 3D finite-element analysis).



Motor Vibrations



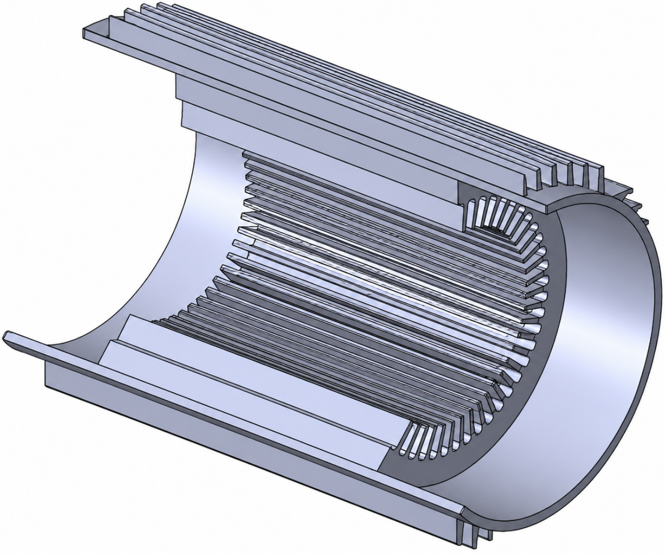
Components to Model



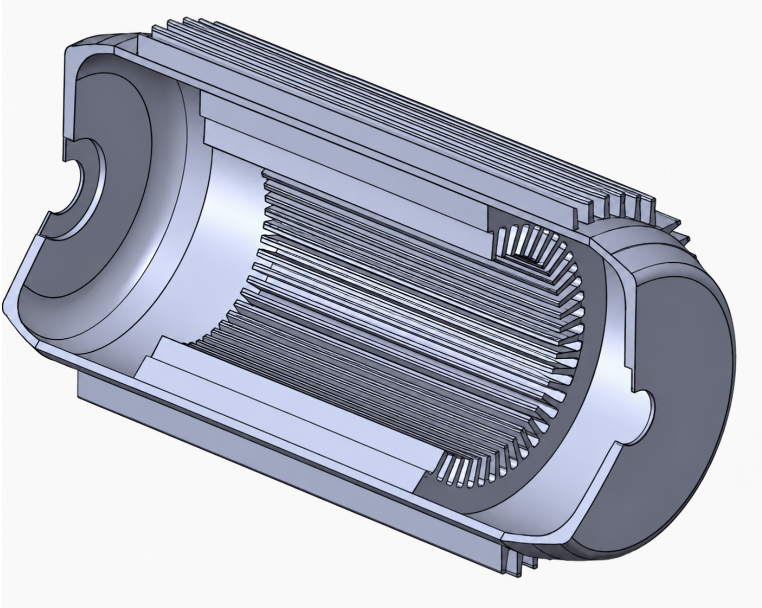
Notations for displacements:

- u: in axial direction z
- v: in circumferential direction ϕ
- w: in radial direction r

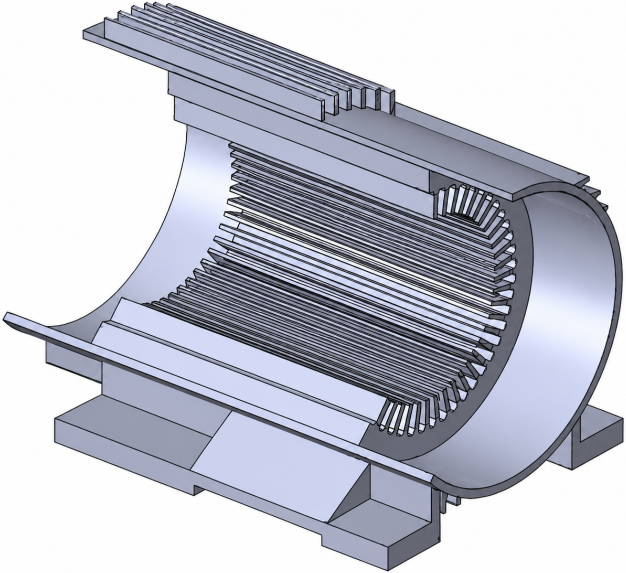
Modeling: Open Ends



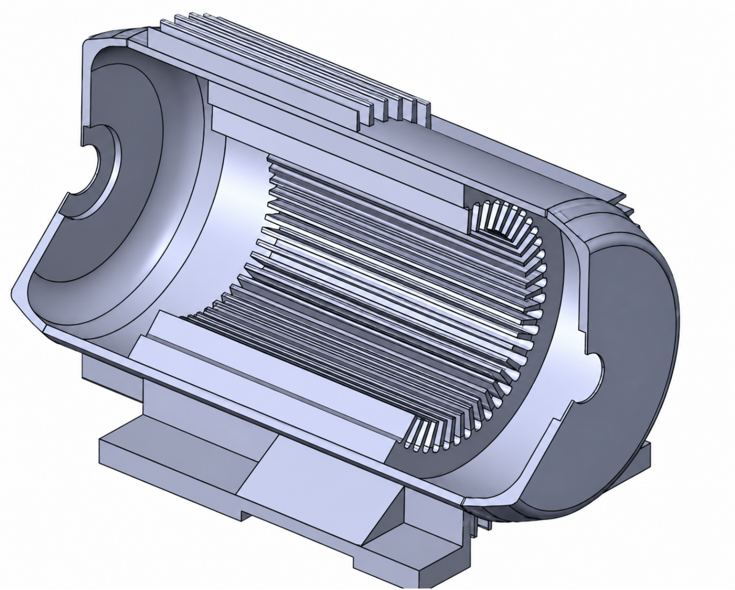
Modeling: With caps



Modeling: With Mounts



Modeling: Combined



Mathematical Formulation

Definitions

In the elasticity theory, one works with stresses and strains.

- Informally, **Stress** is basically force per unit area.
- **Strain** is a measure of deformation related to relative displacements.
- We often have the (linear) stress ($\boldsymbol{\sigma}$) - strain (ε) relations

$$\boldsymbol{\sigma} = \mathbf{C}\,\varepsilon$$

  where **C** is the **elasticity matrix**.

Thick Cylinder (Roivainen 2009)

We consider a thick cylinder of length L, radius R, thickness h, mass density ρ, and elasticity matrix $\mathbf{C}$.
Assume the cylindrical displacements (u, v, w) are separable:

$$
\begin{aligned}
u(r,\phi,z,t) &= u_r(r)u_z(z)\cos(n\phi)e^{i\omega t},\\
v(r,\phi,z,t) &= v_r(r)v_z(z)\sin(n\phi)e^{i\omega t},\\
w(r,\phi,z,t) &= w_r(r)w_z(z)\cos(n\phi)e^{i\omega t},
\end{aligned}
$$

where

$$
\begin{aligned}
u_r(r) &= a_0 + a_1 r + a_2 r^2,\\
v_r(r) &= b_0 + b_1 r + b_2 r^2,\\
w_r(r) &= c_0 + c_1 r + c_2 r^2
\end{aligned}
$$

and u_z, v_z, w_z are derived from elasticity theory
and can be taken as sin & cos for "free" boundary conditions.

Thick Cylinder

We use stress ($\boldsymbol{\sigma}$) - strain (ε) relations

$$\boldsymbol{\sigma} = \mathbf{C}\,\varepsilon$$

where $\mathbf{C}$ is the "elasticity matrix" and
strain vector ε is related to displacement vector $\mathbf{u} = [u\,v\,w]^\top$ by

$$\varepsilon = \mathbf{D}\,\mathbf{u}$$

where $\mathbf{D}$ is some differentiation operator.
Kinetic and potential energies of the shell are:

$$T = \frac{1}{2}\iiint_{L,h,2\pi} \rho\,\|\partial_t \mathbf{u}\|^2 d\phi\, r\, dr\, dz$$

$$\Pi = \frac{1}{2}\iiint \varepsilon^\top \mathbf{C}\,\varepsilon\, d\phi\, r\, dr\, dz = \frac{1}{2}\iiint \mathbf{u}^\top \mathbf{D}^\top \mathbf{C}\,\mathbf{D}\mathbf{u}\, d\phi\, r\, dr\, dz$$

Thick Cylinder

Inserting our ansatz

$$u = (a_0 + a_1 r + a_2 r^2) u_z(z) \cos(n\phi) e^{i\omega t}, \qquad \text{etc}$$

into the Lagrangian

$$\mathcal{L} = T - \Pi$$

and using the Euler-Lagrange formulation, one obtains the equation for the eigenvalue problem

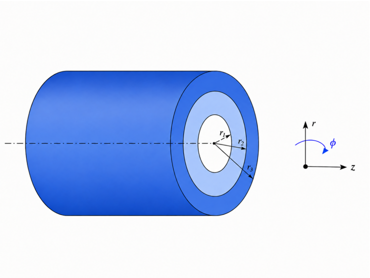
$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{a} = 0$$

for the vector $\mathbf{a} = [a_0, a_1, \dots, c_3]^\top$,
from which one finds modal frequencies ω.

Nested cylinder problem with interface matching

Nested Cylinders Open Ends Model

- The nested cylinder model can be formulated in a similar manner
- We assume that the cylinders are both bonded at their intersection (introduces additional boundary conditions)



Nested Cylinder with Open Ends

- Each cylinder's displacements are modeled as

$$
\begin{aligned}
u^{(j)}(r, \phi, z, t) &= u_r^{(j)}(r) u_z(z) \cos(n\phi) e^{i\omega t} \\
v^{(j)}(r, \phi, z, t) &= v_r^{(j)}(r) v_z(z) \sin(n\phi) e^{i\omega t} \\
w^{(j)}(r, \phi, z, t) &= w_r^{(j)}(r) w_z(z) \cos(n\phi) e^{i\omega t}
\end{aligned}
$$

where $j \in \{1, 2\}$ signifies the inner (1) and outer (2) cylinder.

Problem and Boundary Conditions

- We are interested in extending the existing framework for one hollow cylinder to two nested, perfectly bonded hollow cylinders.
- Continuum mechanics tells us that at the boundary between two media, both the *displacement vector* and 3 components of the *stress tensor* must be continuous.

$$
\begin{aligned}
\lim_{r\to\bar{r}^-} u^{\text{inner}} &= \lim_{r\to\bar{r}^+} u^{\text{outer}}, \qquad & \lim_{r\to\bar{r}^-} \sigma_{rr}^{\text{inner}} &= \lim_{r\to\bar{r}^+} \sigma_{rr}^{\text{outer}} \\
\lim_{r\to\bar{r}^-} v^{\text{inner}} &= \lim_{r\to\bar{r}^+} v^{\text{outer}}, \qquad & \lim_{r\to\bar{r}^-} \sigma_{\theta r}^{\text{inner}} &= \lim_{r\to\bar{r}^+} \sigma_{\theta r}^{\text{outer}} \\
\lim_{r\to\bar{r}^-} w^{\text{inner}} &= \lim_{r\to\bar{r}^+} w^{\text{outer}}, \qquad & \lim_{r\to\bar{r}^-} \sigma_{zr}^{\text{inner}} &= \lim_{r\to\bar{r}^+} \sigma_{zr}^{\text{outer}}
\end{aligned}
$$

- By relating our two systems (the inner and outer cylinders) by these 6 boundary conditions, we can solve the nested cylinder problem by way of Hamilton's principle as shown above.

Ansatz and Coefficient Elimination

- We define the ansatz of our displacement vector $\mathbf{u} = [u \quad v \quad w]^T$ to be

$$u = (a_0 + a_1 r + a_2 r^2)\cos(\lambda_m z)\cos(n\theta)e^{i\omega t},$$
$$v = (b_0 + b_1 r + b_2 r^2)\sin(\lambda_m z)\sin(n\theta)e^{i\omega t},$$
$$w = (c_0 + c_1 r + c_2 r^2)\sin(\lambda_m z)\cos(n\theta)e^{i\omega t}.$$

- Similarly, through the strain-displacement and stress-strain relations

$$\varepsilon = \mathbf{D}\mathbf{u}, \quad \sigma = \mathbf{C}\varepsilon,$$

(where **D** is a matrix differential operator, and **C** is the stiffness matrix)

- Of the n coefficients, 6 are eliminated by interface conditions.
- Hence, one would minimize the Lagrangian $\mathcal{L}$ over the remaining $n - 6$ coefficients.

Radial-only solution

Explanation and motivation

- Simple analytic benchmark for the larger cylindrical motor-vibration model.
- Start with the simplest nontrivial case: the one dimensional radial case for a **single homogeneous hollow cylinder**.
- Purpose of the benchmark:
  - derive an exact solution,
  - code it with company geometry/material data,
  - compare against the existing reduced **Rayleigh–Ritz** model.

Setup and simplifying assumptions

- Hollow cylinder:
$$a < r < b.$$
- Purely radial, angularly uniform, axially uniform motion:
$$\boldsymbol{u}(r,t) = w(r,t)\boldsymbol{e}_r, \qquad \partial_\theta w = 0, \qquad \partial_z w = 0.$$
- Strains:
$$\varepsilon_{rr} = w_r, \qquad \varepsilon_{\theta\theta} = \frac{w}{r}, \qquad \varepsilon_{zz} = 0.$$
- Radial stress:
$$\sigma_{rr} = (\lambda + 2\mu)w_r + \lambda\frac{w}{r}.$$
- Reduced radial PDE:
$$\rho w_{tt} = (\lambda + 2\mu)\left(w_{rr} + \frac{1}{r}w_r - \frac{w}{r^2}\right).$$

Analytic derivation

- Assume harmonic motion:

$$w(r,t) = W(r)e^{i\omega t}.$$

- The spatial problem becomes

$$W'' + \frac{1}{r}W' + \left(k^2 - \frac{1}{r^2}\right)W = 0, \qquad k^2 = \frac{\rho\omega^2}{\lambda + 2\mu}.$$

- This is Bessel's equation of order one:

$$W(r) = AJ_1(kr) + BY_1(kr).$$

- Stress-free condition at the surfaces give

$$\mathcal{T}[W](a) = 0, \qquad \mathcal{T}[W](b) = 0,$$

where

$$\mathcal{T}[W](r) = (\lambda + 2\mu)W'(r) + \lambda\frac{W(r)}{r}.$$

Results: eigenvalue condition and modes

- Applying the two boundary conditions gives a homogeneous system:
$$\begin{pmatrix} \mathcal{T}_J(a;\omega) & \mathcal{T}_Y(a;\omega) \\ \mathcal{T}_J(b;\omega) & \mathcal{T}_Y(b;\omega) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
- Nontrivial modes occur only when the matrix is singular:
$$F(\omega) = \mathcal{T}_J(a;\omega)\mathcal{T}_Y(b;\omega) - \mathcal{T}_Y(a;\omega)\mathcal{T}_J(b;\omega) = 0.$$
- The roots ω_m are radial natural frequencies.
- The null vector gives A_m, B_m, hence the mode shape $W_m(r)$.

Conclusions, validation, and future work

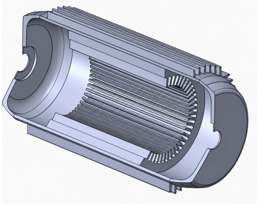
- Implementation plan:
  1. solve $F(\omega) = 0$ for the first several roots,
  2. reconstruct and normalize $W_m(r)$,
  3. compare against the reduced Rayleigh–Ritz eigenproblem

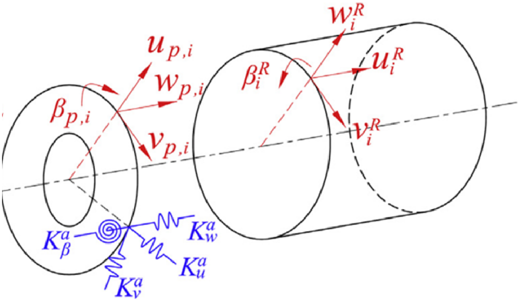
$$\boldsymbol{K}\boldsymbol{q} = \omega^2 \boldsymbol{M}\boldsymbol{q}.$$

- Primary comparison: eigenvalues/natural frequencies.
- Secondary comparisons: normalized mode shapes.
- Next extensions:
  - add a second 1D cylinder with displacement and stress continuity,
  - then restore angular dependence ($n \neq 0$) to obtain quasi-2D case.

Adding cap to housing (aka Plate & Shell)

Plate & Shell — slide 1 of 3





- Along the contact edge of cap and shell, the deformation energy is modeled by fictitious springs.
- Infinite stiffness constant K of a spring yields rigid connection.
- Deformation (potential) and kinetic energies of plate and shell are set up separately by their respective formulas.

Kinetic energies of shell and plate (let $u_p = v_p = 0$ for simplicity):

$$T_s \propto \int \int d\phi dz \, \left((\partial_t u)^2 + \ldots \right), \qquad T_p \propto \int r\, dr\, d\phi \, (\partial_t w_p)^2$$

Potential energies of shell and plate:

$$\Pi_s \propto \int \int d\phi dz \, \left((\partial_{a\phi} u + \partial_z v)^2 + \ldots \right), \qquad \Pi_p \propto \int r\, dr\, d\phi \left((\nabla^2 w_p)^2 + \ldots \right)$$

Potential energy in fictitious springs:

$$\Pi_{\mathrm{edge}} \propto \int d\phi \left(K_{w_p} (w_p - u|_{z=0})^2 + K_\beta (\partial_r w_p + \partial_z w|_{z=0})^2 \right)$$

For the shell, seek solution as:

$$u(r,\phi,z,t) = \sum_m \chi_{r,m}(r)\psi_{r,m}(z)\cos(n\phi)e^{i\omega t}$$

and similarly for v and w, where: m labels "beam functions" $\psi_{r,m}$ and $\chi_{r,m} = a_{0,m} + a_{1,m}r + a_{2,m}r^2$, etc.
For the plate, seek solution as:

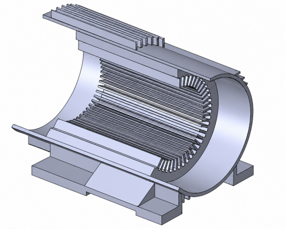
$$w_p = \chi_p(r)\cos(n\phi)e^{i\omega t}$$

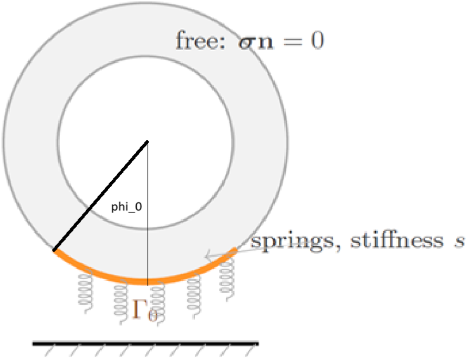
where $\chi_p(r) = p_0 + p_1 r + p_2 r^2 + \ldots$ (or some other basis).
Substitute these *änsatze* into Lagrangian

$$\mathcal{L} = T_s + T_p - \Pi_s - \Pi_p - \Pi_{\mathrm{edge}}$$

and derive Euler-Lagrange eqs for $\{a_{0,m}, a_{1,m}, \ldots, p_0, p_1, p_2, \ldots\}$.

Adding a foot to the assembly





- As for the plate-shell case, the foot can be modeled by springs.
- Infinitely stiff springs imply no displacements at foot's support.
- Seek solution as a series

$$\chi(r)\psi(z)\sum_n a_n \sin\left(\,(\phi-\phi_0)\pi n/(2\pi-2\phi_0)\,\right)$$

with the other steps being the same as for one cylinder.

Young's modulus of materials with axially-periodic materials

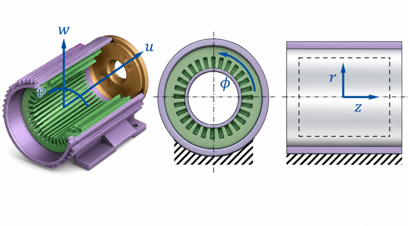
Motivation

- **Young's modulus** roughly refers to the ratio of stress σ to strain if isotropic: ε: $E = \sigma/\varepsilon$.
- For anisotropic materials, E can depend on the direction of force applied and reference plane.
- Roivainen [1] experimentally observed this anisotropy:

Quantity	**Dry core**	Impregnated core	Fully assembled stator
E_ϕ [GPa]	**207**	207	207
E_r [GPa]	**207**	207	207
E_z [GPa]	**1.54**	385	863

Question: Why is E_z (axial Young's modulus) two orders of magnitude smaller?

Motivation



- **Known:** axial construction is a sequence of hundreds of "glued" steel sheets (laminates) with thin resin layers between.
- **Known:** this lamination is in fact done on purpose to improve this behavior in the z-direction.
- **Question:** is this gap in E_r, E_ϕ (steel Young's modulus) compared to E_z explainable via averaging/homogenization? If not, what theory recapitulates observation?

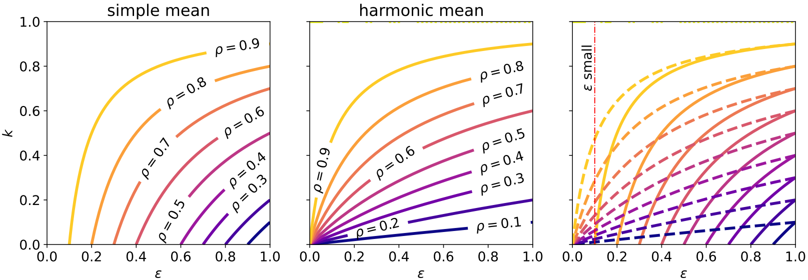
Bulk properties of mixtures

Voigt-Reuss for mixtures (particular for Young's modulus):

- If the **strain** in the constituent materials are equal, then a linear (simple) average represents the bulk Young's modulus.
- If the **stress** in the materials are equal, then a harmonic average is proper.

Taking $E(z)$ periodic with period 1,

$$E(z) = \begin{cases} 1 & : \quad 0 \leq z \leq 1 - \varepsilon \\ k & : \quad 1 - \varepsilon < z < 1 \end{cases} \tag{1}$$



Results

Apparently,

- Steel: Young's modulus $\approx$ 200-210 GPa. (Also E_r, E_ϕ).
- Resins: Young's modulus $\approx$ 1-1.5 GPa.

Substituting using $\varepsilon = 0.01$ *(too much glue, probably)* yields an equivalent multiplier

- $\rho \in [0.99005, 0.99007]$ for simple averaging (equal-strain assumption)
- $\rho \in [0.327, 0.422]$ for harmonic averaging (equal-stress assumption)
- Experimental $\rho = E_z/E_r \approx 0.0074$.

Seeking ε such that $\rho \approx 0.0074$ yields $\varepsilon \approx 0.64$!
Conclusion: something else is going on. Or I just don't know enough to understand why Roivainen thinks this $207 \to 1.54$ is trivial.

Conclusions 1

- We studied constitutive laws a second cylinder bonded to the first one along the radial directions, expecting improved model fidelity when explicitly incorporating these constraints in ABB's coding harness.
- We provided a literature review, including analytic models incorporating caps and mounts to improve model fidelity for future work.

Conclusions 2

- We worked out a "toy problem" in 1D (extendable to azimuthal harmonics $\cos n\phi$ in 2D), providing a benchmark for both single cylinder and two nested ones.
  Extending this toward a hybrid analytic/FEM approach in code **will accelerate computation,** to significantly accelerate parameter/resonance studies at ABB relative to their current approach (full 3D finite-element analysis).
- Revisiting the basics suggests degree of bulk anisotropy of material **isn't explained** by averaging theory. Whether this is a mismatch in the physics or engineering or model formulation is unknown to us.
  Addressing these and validating models against these observations is important to improve model fidelity to reduce eventual computational/simulation burden.

Thank you!

References

[1] Janne Roivainen. "Unit-wave response-based modeling of electromechanical noise and vibration of electrical machines". In: (2009).